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Research Paper

COUPLED COINCIDENCE RESULTS FOR G-COMPATIBLE FUNCTIONS

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ABSTRACT. Introducing some coupled coincidence points for a mixed monotone operator in partially ordered complete **G**-metric spaces is the purpose of this paper. We also demonstrate the existence and uniqueness of coupled common fixed points. The main results of this paper improve the result given by Nashine and Shatanawi [Computers and Mathematics with Applications 62 (2011) 1984-1993]. An example is given to support the usability of our results.

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1. Introduction and Background

Fixed point and coincidence point theorems play a very important role in optimization problems. See for example [6] and the references therein.

In [15], Mustafa and Sims provided a reliable generalized metric known as **G**-metric spaces. Many authors worked on **G**-metric spaces, see for example [3, 9, 15, 16, 17, 18, 19, 20, 25, 29]. There has been recent interest in constructing fixed-point theorems in partially ordered complete metric spaces that fulfill a contraction condition by partial order, see for example [4, 5, 7, 8, 10, 11, 14, 25, 21, 23, 24, 26, 28, 30].

Fixed-point problems were also examined in partly ordered probability metric spaces [11] and partly ordered **G**-metric spaces [8, 25]. In [14], Bhaskar and Lakshmikantham produced the results of coupled fixed points in partially ordered metric spaces. After that, several results coupled point and point of coincidence have emerged in the recent literature. See for example [1, 2, 7, 8, 13, 14, 26, 27, 28, 30, 31, 32].

In the present work, we establish these results for mixed h-monotone mapping in partly ordered **G**-metric spaces. Our findings generalize the very recent findings of Nashine and Shatanawi [22]. At the beginning, we are reminded of some definitions and properties in the **G**-metric spaces used in this document.

Definition 1.1 ([18]). Let $Y \neq \emptyset$ and let $\mathbf{G} : Y \times Y \times Y \to \mathbb{R}^+$ be a mapping satisfying the following conditions:
(A_1) $\mathbf{G}(u, v, w) = 0$ iff $u = v = w$.
(A_2) $0 < \mathbf{G}(u, u, v)\ \forall\ u, v \in Y$ with $u \neq v$.
(A_3) $\mathbf{G}(u, u, v) \leq \mathbf{G}(u, v, w)\ \forall\ u, v, w \in Y$ with $v \neq w$.

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(A_4) $\mathbf{G}(u,v,w)=\mathbf{G}(u,w,v)=\mathbf{G}(v,w,u)=....$
(A_5) $\mathbf{G}(u,v,w)\leq \mathbf{G}(u,a,a)+\mathbf{G}(a,v,w)\ \forall\ u,v,w,a\in X.$
We say that $\mathbf{G}$ is a $\mathbf{G}$-metric on Y, and we say that $(Y,\mathbf{G})$ is a $\mathbf{G}$-metric space.

Definition 1.2 ([18])**.** Let $\{u_n\}$ be be a sequence in $\mathbf{G}$-metric space Y. Then $\{u_n\}$ is $\mathbf{G}$-convergent to $u\in Y$ if $\lim\limits_{n,m\to\infty}\mathbf{G}(u,u_n,u_m)=0$, that is,

$$\forall\epsilon>0\ \ \exists N\in\mathbb{N}\ \ s.t.\ \ \mathbf{G}(u,u_n,u_m)<\epsilon\ \ \forall n,m\geq N.$$

We write $u_n\to u$ or $lim\ u_n=u$.

In [18], Mustafa and Sims have shown the following conditions to be equivalent in $\mathbf{G}-$metric spaces:
(a) $\{u_n\}$ is $\mathbf{G}$-convergent to u.
(b) $\mathbf{G}(u_n,u_n,u)\to 0$ as $n\to+\infty$.
(c) $\mathbf{G}(u_n,u,u)\to 0$ as $n\to+\infty$.
(d) $\mathbf{G}(u_n,u_m,u)\to 0$ as $n,m\to+\infty$.

Also in [18], Mustafa and Sims have defined the $\mathbf{G}$-Cauchy concept for sequences in $\mathbf{G}$-metrics spaces as follows.

Definition 1.3 ([18])**.** In a $\mathbf{G}$-metric space $(Y,\mathbf{G})$ a sequence $\{u_n\}$ is called $\mathbf{G}$-Cauchy if,

$$\forall\epsilon>0\ \ \exists N\in\mathbb{N}\ \ s.t.\ \ \mathbf{G}(u_n,u_m,u_l)<\epsilon\ \ \forall n,m,l\geq N.$$

In [15], they proved that, the concept of $\mathbf{G}$-Cauchy for sequence $\{u_n\}$ in $\mathbf{G}$-metric space Y equivalents with the following condition:

$$\forall\epsilon>0\ \ \exists N\in\mathbb{N}\ \ s.t.\ \ \mathbf{G}(u_n,u_m,u_m)<\epsilon\ \ \forall n,m\geq N.$$

In [18] they showed that, a mapping $h:Y\to Y$ is $\mathbf{G}$-continuous at $u\in Y$ iff it is $\mathbf{G}$-sequentially continuous at u.

Definition 1.4 ([18])**.** If every $\mathbf{G}$-Cauchy sequence is $\mathbf{G}$-convergent, thwn we say that G-metric space $(Y,\mathbf{G})$ is $\mathbf{G}$-complete.

We say that a $\mathbf{G}$-metric space Y is symmetric, if $\mathbf{G}(u,v,v)=\mathbf{G}(v,u,u)$ for all $u,v\in Y$.

Definition 1.5 ([8])**.** In a $\mathbf{G}$-metric space $(Y,\mathbf{G})$, a mapping $T:Y\times Y\to Y$ is continuous if for $\mathbf{G}$-convergent sequences $\{u_n\}$ and $\{v_n\}$ converging to u and v respectively, $\{T(u_n,v_n)\}$ is $\mathbf{G}$-convergent to $T(u,v)$.

In 2006 [5], Bhaskar and Lakshmikantham brought in the concept of a monotonous mixed ownership. Subsequently, in 2009 [14], Lakshmikantham refined this concept as follows:

Definition 1.6 ([14])**.** Let Y be a partially ordered set. For functions $T:Y\times Y\to Y$ and $h:Y\to Y$ on partially ordered set $(Y,\preceq)$, we say that, T hase mixed h-monotone property if, the following conditions for all $u,v\in Y$ are hold:

$$u_1,u_2\in Y, hu_1\preceq hu_2\ implies\ T(u_1,v)\preceq T(u_2,v),$$
$$v_1,v_2\in Y, hv_1\preceq hv_2\ implies\ T(u,v_2)\preceq T(u,v_1).$$

Definition 1.7 ([5])**.** $(u,v)\in Y\times Y$ is a coupled fixed point of function $T:Y\times Y\to Y$ if

$$T(u,v)=u,\ T(v,u)=v.$$

Definition 1.8 ([14]). $(u,v) \in Y \times Y$ is a coupled coincidence point (C-C-point) of the mappings $T : Y \times Y \to Y$ and $h : Y \to Y$ if

$$T(u,v) = hu \text{ and } T(v,u) = hv.$$

In 2011 ([8]), Choudhury and Maity by reconciling the concept of mixed monotone property, established certain linked fixed points.
Recently, Neshin and Shatanawi have developed some coupled fixed point results for mixed these mappings in [22].

In Section 2, we present the main outcome of this article for the coupled coincidence point. In Sections 3, we demonstrate the existence and the uniqueness theorem of a coupled common fixed point. At the end of the article, here is an example that supports the usability of our results.

2. Coupled Coincidence Point

We start this section with the following definition.

Definition 2.1. Let $h : Y \to Y$ and $T : Y \times Y \to Y$ be two functions on **G**-metric space Y. h and T are said to be **G**-compatible if

$$\lim_{n\to\infty} \mathbf{G}(hT(u_n,v_n), T(hu_n,hv_n), T(hu_n,hv_n)) = 0,$$

and

$$\lim_{n\to\infty} \mathbf{G}(hT(v_n,u_n), T(hv_n,hu_n), T(hv_n,hu_n)) = 0,$$

for all sequences $\{u_n\}$ and $\{v_n\}$ with $\lim_{n\to\infty} T(u_n,v_n) = \lim_{n\to\infty} hu_n$ and $\lim_{n\to\infty} T(v_n,u_n) = \lim_{n\to\infty} hv_n$.

NOTE 1: Tow next theorems are the main results of this article. These theorems and their proofs are the same as Theorem 2.1 and Theorem 2.2 of [12], but our contraction condition is different from the contraction condition of that theorems. Also in Theorem 2.3 of this paper, the condition of Theorem 2.2 of [12] is reduced.

Theorem 2.2. *Let* $\boldsymbol{G}$ *be a complete* $\boldsymbol{G}$*-metric on partially ordered set* $(Y,\preceq)$*. Suppose two functions* $T : Y \times Y \to Y$ *and* $h : Y \to Y$ *where* T *has the mixed* h*-monotone property on* Y*. Suppose for* $\alpha,\beta,\gamma,L \geqslant 0$ *with* $\alpha+\beta+\gamma < 1$ *we have*

$$\begin{aligned} &\boldsymbol{G}(T(x,y),T(u,v),T(z,w)) \\ \leq\ &\alpha \min\{\boldsymbol{G}(T(x,y),T(x,y),hx),\, \boldsymbol{G}(T(u,v),hx,hx),\, \boldsymbol{G}(T(z,w),hx,hx)\} \\ &+\beta \min\{\boldsymbol{G}(T(x,y),hu,hu),\, \boldsymbol{G}(T(u,v),T(u,v),hu),\, \boldsymbol{G}(T(z,w),hu,hu)\} \\ &+\gamma \min\{\boldsymbol{G}(T(x,y),hz,hz),\, \boldsymbol{G}(T(u,v),hz,hz),\, \boldsymbol{G}(T(z,w),T(z,w),hz)\} \\ &+L \min\{\boldsymbol{G}(T(z,w),hx,hx),\, \boldsymbol{G}(T(z,w),hu,hu),\, \boldsymbol{G}(T(x,y),hz,hz)\} \end{aligned} \tag{2.1}$$

for all $x,y,u,v,z,w \in Y$ *with* $hx \preceq hu \preceq hz$ *and* $hy \succeq hv \succeq hw$*. Also suppose that* $T(Y \times Y) \subseteq h(Y)$ *and* h *is continuous nondecreasing and* h *and* T *are* $\boldsymbol{G}$*-compatible. Suppose that either*
(a) T *is a continuous function,*
or
(b) Y *applies in the following conditions:*

(i) *if* $\{u_n\}$ *is nondecreasing and* $\{u_n\} \to u$*, then* $u_n \preceq u$*,* $\forall\ n \geq 0$*;*

(ii) *if* $\{v_n\}$ *is nonincreasing and* $\{v_n\} \to v$*, then* $v \preceq v_n$*,* $\forall\, n \geq 0$*.*

If there exist $u_0, v_0 \in Y$ *such that* $hu_0 \preceq T(u_0, v_0)$ *and* $hv_0 \succeq T(v_0, u_0)$*, then* h *and* T *have a C-C-point.*

Proof. Let $u_0, v_0 \in X$ such that $hu_0 \preceq T(u_0, v_0)$ and $hy_0 \succeq T(v_0, u_0)$. Since $T(Y \times Y) \subseteq h(Y)$, we can choose $u_1, v_1 \in Y$ such that $hu_1 = T(u_0, v_0)$ and $hv_1 = T(v_0, u_0)$. Again since $T(Y \times Y) \subseteq h(Y)$, we can choose $u_2, v_2 \in Y$ such that $hu_2 = T(u_1, v_1)$ and $hv_2 = T(v_1, u_1)$. By doing so, we build two sequences $\{u_n\}$ and $\{v_n\}$ in Y so that,

$$hu_{n+1} = T(u_n, v_n), \;\; hv_{n+1} = T(v_n, u_n) \;\; \forall n \geq 0. \tag{2.2}$$

Now we conclude that for all $n \geq 0$

$$hu_n \preceq hu_{n+1}, \tag{2.3}$$

and

$$hv_n \succeq hv_{n+1}. \tag{2.4}$$

Let us begin with mathematical induction. Let $n = 0$. Since $hu_0 \preceq T(u_0, v_0)$ and $hv_0 \succeq T(v_0, u_0)$, in view of $hu_1 = T(u_0, v_0)$ and $hv_1 = T(v_0, u_0)$, we have $h(u_0) \preceq h(u_1)$ and $h(v_0) \succeq h(v_1)$. Suppose the relations (2.3) and (2.4) stand for some n. As T has the mixed h-monotone property and $h(u_n) \preceq h(u_{n+1})$, $h(v_n) \succeq h(v_{n+1})$, from (2.2), we have

$$hu_{n+1} = T(u_n, v_n) \preceq T(u_{n+1}, v_n) \textit{ and } hv_{n+1} = T(v_n, u_n) \succeq T(v_{n+1}, u_n). \tag{2.5}$$

Also for the same reason we have

$$T(u_{n+1}, v_n) \preceq T(u_{n+1}, v_{n+1}) = hu_{n+2} \textit{ and } T(v_{n+1}, u_n) \succeq T(v_{n+1}, u_{n+1}) = hv_{n+2}. \tag{2.6}$$

Then from (2.5) and (2.6), we get $hu_{n+1} \preceq hu_{n+2}$ and $hv_{n+1} \succeq hv_{n+2}$. Through mathematical induction, it follows that (2.3) and (2.4) stand for all $n > 0$. Therefore,

$$hu_0 \preceq hu_1 \preceq hu_2 \preceq \ldots \preceq gx_n \preceq hu_{n+1} \preceq \ldots, \tag{2.7}$$

and

$$hv_0 \succeq hv_1 \succeq hv_2 \succeq \ldots \succeq hv_n \succeq hv_{n+1} \succeq \ldots. \tag{2.8}$$

If for some n, $(hu_{n+1}, hv_{n+1}) = (hu_n, hv_n)$, then $T(u_n, v_n) = hu_n$ and $T(v_n, u_n) = hv_n$, that is, T and h have a coincidence point. So, we assume that $(hu_{n+1}, hv_{n+1}) \neq (hu_n, hv_n)$, for all $n \in \mathbb{N}$, that is, we assume that either $hu_{n+1} = T(u_n, v_n) \neq hu_n$ or $hv_{n+1} = T(v_n, u_n) \neq hv_n$. Since $hu_n \succeq hu_{n-1}$ and $hv_n \preceq hv_{n-1}$, from (2.1) and (2.2), we get

$$\begin{aligned}
& \mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n) = \mathbf{G}(T(u_n, v_n), T(u_n, v_n), T(u_{n-1}, v_{n-1}))) \\
\leq \;& \alpha \min\{\mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n), \mathbf{G}(hu_{n+1}, hu_n, hu_n), \mathbf{G}(hu_n, hu_n, hu_n)\} \\
& +\beta \min\{\mathbf{G}(hu_{n+1}, hu_n, hu_n), \mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n), \mathbf{G}(hu_n, hu_n, hu_n)\} \\
& +\gamma \min\{\mathbf{G}(hu_{n+1}, hu_{n-1}, hu_{n-1}), \mathbf{G}(hu_{n+1}, hu_{n-1}, hu_{n-1}), \mathbf{G}(hu_n, hu_n, hu_{n-1})\} \\
& +L \min\{\mathbf{G}(hu_n, hu_n, hu_n), \mathbf{G}(hu_n, hu_n, hu_n), \mathbf{G}(hu_{n+1}, hu_{n-1}, hu_{n-1})\},
\end{aligned}$$

and hence

$$\mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n) \leq \gamma \mathbf{G}(hu_n, hu_n, hu_{n-1}). \tag{2.9}$$

Similarly, from $hv_{n-1} \succeq hv_n$, $hu_{n-1} \preceq hu_n$, (2.1) and (2.2), we have

$$\begin{aligned} & \mathbf{G}(hv_n, hv_{n+1}, hv_{n+1}) = \mathbf{G}(T(v_{n-1}, u_{n-1}), T(v_n, u_n), T(v_n, u_n)) \\ \leq\ & \alpha \min\{\mathbf{G}(hv_n, hv_n, hv_{n-1}), \mathbf{G}(hv_{n+1}, hv_{n-1}, hv_{n-1}), \mathbf{G}(hv_{n+1}, hv_{n-1}, hv_{n-1})\} \\ & +\beta \min\{\mathbf{G}(hv_n, hv_n, hv_n), \mathbf{G}(hv_{n+1}, hv_{n+1}, hv_n), \mathbf{G}(hv_{n+1}, hv_n, hv_n)\} \\ & +\gamma \min\{\mathbf{G}(hv_n, hv_n, hv_n), \mathbf{G}(hv_{n+1}, hv_n, hv_n), \mathbf{G}(hv_{n+1}, hv_{n+1}, hv_n)\} \\ & +L \min\{\mathbf{G}(hv_{n+1}, hv_{n-1}, hv_{n-1}), \mathbf{G}(hv_{n+1}, hv_n, hv_n), \mathbf{G}(hv_n, hv_n, hv_n)\}, \end{aligned}$$

and hence

$$\mathbf{G}(hv_n, hv_{n+1}, hv_{n+1}) \leq \alpha \mathbf{G}(hv_n, hv_n, hv_{n-1}). \tag{2.10}$$

By adding (2.9) and (2.10), we have

$$\begin{aligned} & \mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n) + \mathbf{G}(hv_n, hv_{n+1}, hv_{n+1}) \\ \leq\ & \gamma \mathbf{G}(hu_n, hu_n, hu_{n-1}) + \alpha \mathbf{G}(hv_n, hv_n, hv_{n-1}) \\ \leq\ & (\alpha+\beta+\gamma)\mathbf{G}(hu_n, hu_n, hu_{n-1}) + (\alpha+\beta+\gamma)\mathbf{G}(hv_n, hv_n, hv_{n-1}) \\ =\ & (\alpha+\beta+\gamma)[\mathbf{G}(hu_n, hu_n, hu_{n-1}) + \mathbf{G}(hv_n, hv_n, hv_{n-1})] \\ <\ & \mathbf{G}(hu_n, hu_n, hu_{n-1}) + \mathbf{G}(hv_n, hv_n, hv_{n-1}). \end{aligned}$$

Set $\rho_n = \mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n)) + \mathbf{G}(hv_n, hv_{n+1}, hv_{n+1})$ and $\delta = \alpha + \beta + \gamma$. Then

$$0 \leq \rho_n \leq \delta \rho_{n-1} \leq \delta^2 \rho_{n-2} \leq ... \leq \delta^n \rho_0.$$

Since $\delta < 1$, then

$$\lim_{n\to\infty} \rho_n = 0.$$

Thus

$$\lim_{n\to\infty} \mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n)) = 0 \textit{ and } \lim_{n\to\infty} \mathbf{G}(hv_n, hv_{n+1}, hv_{n+1}) = 0. \tag{2.11}$$

Now, we prove that both sequences $\{hu_n\}$ and $\{hv_n\}$ are Cauchy. For each $m \geq n$:

$$\begin{aligned} \mathbf{G}(hu_m, hu_m, hu_n) \leq\ & \mathbf{G}(hu_m, hu_m, hu_{m-1}) + \mathbf{G}(hu_{m-1}, hu_{m-1}, hu_{m-2}) \\ & +... + \mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n), \end{aligned}$$

and

$$\begin{aligned} \mathbf{G}(hv_m, hv_m, hv_n) \leq\ & \mathbf{G}(hv_m, hv_m, hv_{m-1}) + \mathbf{G}(hv_{m-1}, hv_{m-1}, hv_{m-2}) \\ & +... + \mathbf{G}(hv_{n+1}, hv_{n+1}, hv_n). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{G}(hu_m, hu_m, hu_n) + \mathbf{G}(hv_m, hv_m, hv_n) &\leq \rho_{m-1} + \rho_{m-2} + ... + \rho_n \\ &= (\delta^{m-1} + \delta^{m-2} + ... + \delta^n)\rho_0 \\ &= \delta^n(1 + \delta + ... + \delta^{m-n-1})\rho_0 \\ &< \frac{\delta^n}{1-\delta}\rho_0, \end{aligned}$$

which implies that

$$\lim_{m,n\to\infty} [\mathbf{G}(hu_m, hu_m, hu_n)) + \mathbf{G}(hv_m, hv_m, hv_n)] = 0. \tag{2.12}$$

Thus two sequences $\{hu_n\}$ and $\{hv_n\}$ are Cauchy. Hence, there are $u, v \in Y$ such that:

$$\lim_{n\to\infty} hu_n = \lim_{n\to\infty} T(u_n, v_n) = u \text{ and } \lim_{n\to\infty} hv_n = \lim_{n\to\infty} T(v_n, u_n) = y. \tag{2.13}$$

$\mathbf{G}$-compatibility of h and T implies that

$$\lim_{n\to\infty} \mathbf{G}(hT(u_n, v_n), T(hu_n, hv_n), T(hu_n, hv_n)) = 0, \tag{2.14}$$

and

$$\lim_{n\to\infty} \mathbf{G}(hT(v_n, u_n), T(hv_n, hu_n), T(hv_n, hu_n)) = 0. \tag{2.15}$$

First, assume that the hypothesis (a) holds. We have

$$\begin{aligned} & \mathbf{G}(hu, T(hu_n, hv_n), T(hu_n, hv_n)) \\ \leq \; & \mathbf{G}(hu, hT(u_n, v_n), hT(u_n, v_n)) \\ & + \mathbf{G}(hT(u_n, v_n), T(hu_n, hv_n), T(hu_n, hv_n)). \end{aligned} \tag{2.16}$$

Taking the limit from (2.16) and using (2.13), (2.14) and continuity of h and T we get that $\mathbf{G}(hu, T(u, v), T(u, v)) = 0$, and hence, $hu = T(u, v)$. Similarly, $T(v, u) = hv$.

Now, suppose that (b) holds. From (2.7) and (2.8), $\{hu_n\}$ and $\{hv_n\}$ are nondecreasing and nonincreasing sequences respectively, and since h is a nondecreasing function, we get $h(hu_n) \preceq hu$ and $h(hv_n) \succeq hv$ hold for all $n \in \mathbb{N}$. On other hand, $\mathbf{G}$-compatibility of h and T implies that $T(hu_n, hv_n) = hT(u_n, v_n) = h(hu_{n+1}) \to hu$ and $T(hv_n, hu_n) = hT(v_n, u_n) = h(hv_{n+1}) \to hv$. So by (2.1), we have

$$\begin{aligned} & \mathbf{G}(h(hu_{n+1}), h(hu_{n+1}), T(u, v)) = \mathbf{G}(T(hu_n, hv_n), T(hu_n, hv_n), T(u, v)) \\ \leq \; & \alpha \min\{\mathbf{G}(h(hu_{n+1}), h(hu_{n+1}), h(hx_n)), \mathbf{G}(h(hu_{n+1}), h(hx_n), h(hx_n)), \\ & \mathbf{G}(T(u, v), h(hx_n), h(hx_n))\} \\ & + \beta \min\{\mathbf{G}(h(hu_{n+1}), h(hx_n), h(hx_n)), \\ & \mathbf{G}(h(hu_{n+1}), h(hu_{n+1}), h(hx_n)), \mathbf{G}(T(u, v), h(hx_n), h(hx_n))\} \\ & + \gamma \min\{\mathbf{G}(h(hu_{n+1}), hu, hu), \\ & \mathbf{G}(h(hu_{n+1}), hu, hu), \mathbf{G}(T(u, v), T(u, v), hu)\} \\ & + L \min\{\mathbf{G}(T(u, v), h(hu_n), h(hu_n)), \\ & \mathbf{G}(T(u, v), h(hu_n), h(hu_n)), \mathbf{G}(h(hu_{n+1}), hu, hu)\}, \end{aligned}$$

Taking the limit from above inequality, we get $\mathbf{G}(hu, hu, T(u, v)) = 0$. Hence $hu = T(u, v)$. In the same way, we can show that $hv = T(v, u)$. So T and h have a C-C-point. □

NOTE 2: In Theorem 2.2 of [12], the authors, consider that the function g is commutes with F. In the next theorem we omit this condition.

Theorem 2.3. *Let $\boldsymbol{G}$ be a complete $\boldsymbol{G}$-metric on partially ordered set $(Y, \preceq)$. Suppose $T : Y \times Y \to Y$ and $h : Y \to Y$ are two functions where T has the mixed h-monotone property on Y. Suppose for $\alpha, \beta, \gamma, L \geqslant 0$ with $\alpha + \beta + \gamma < 1$ we have*

$$\begin{aligned} & \boldsymbol{G}(T(x, y), T(u, v), T(z, w)) \\ \leq \; & \alpha \min\{\boldsymbol{G}(T(x, y), T(x, y), hx), \boldsymbol{G}(T(u, v), hx, hx), \boldsymbol{G}(T(z, w), hx, hx)\} \\ & + \beta \min\{\boldsymbol{G}(T(x, y), hu, hu), \boldsymbol{G}(T(u, v), T(u, v), hu), \boldsymbol{G}(T(z, w), hu, hu)\} \\ & + \gamma \min\{\boldsymbol{G}(T(x, y), hz, hz), \boldsymbol{G}(T(u, v), hz, hz), \boldsymbol{G}(T(z, w), T(z, w), hz)\} \\ & + L \min\{\boldsymbol{G}(T(z, w), hx, hx), \boldsymbol{G}(T(z, w), hu, hu), \boldsymbol{G}(T(x, y), hz, hz)\} \end{aligned} \tag{2.17}$$

for all $x, y, u, v, z, w \in Y$ with $hx \preceq hu \preceq hz$ and $hy \succeq hv \succeq hw$. Also, suppose that:

(i) *if $\{u_n\}$ is nondecreasing and $\{u_n\} \to u$, then $u_n \preceq u$, $\forall\ n \geq 0$;*
(ii) *if $\{v_n\}$ is nonincreasing and $\{v_n\} \to v$, then $v \preceq v_n$, $\forall\ n \geq 0$.*

If there exist $u_0, v_0 \in Y$ such that $hu_0 \preceq T(u_0, v_0)$ and $hv_0 \succeq T(v_0, u_0)$, then h and T have a C-C-point.

Proof. Further to the proof of Theorem 2.2, we can conclude that $\{hu_n\}$ and $\{hv_n\}$ are Cauchy sequences in $h(Y)$. From the completeness of $h(Y)$, for some $u, v \in Y$ we have:

$$\lim_{n\to\infty} hu_n = \lim_{n\to\infty} T(u_n, v_n) = hu \text{ and } \lim_{n\to\infty} hv_n = \lim_{n\to\infty} T(v_n, u_n) = gy. \tag{2.18}$$

From (2.7) and (2.8), $\{hu_n\}$ and $\{hv_n\}$ are nondecreasing and nonincreasing respectively. So by assumption we have $hu_n \preceq hu$ and $hv_n \succeq hv$ for all n, we get

$$\begin{aligned}
&\mathbf{G}(hu_{n+1}, hu_{n+1}, T(u,v)) = \mathbf{G}(T(u_n, v_n), T(u_n, v_n), T(u,v))\\
\leq\ & \alpha \min\{\mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n), \mathbf{G}(hu_{n+1}, hu_n, hu_n), \mathbf{G}(T(u,v), hu_n, hu_n)\}\\
&+\beta \min\{\mathbf{G}(hu_{n+1}, hu_n, hu_n), \mathbf{G}(hu_{n+1}, hu_{n+1}, hu_n), \mathbf{G}(T(u,v), hu_n, hu_n)\}\\
&+\gamma \min\{G(gx_{n+1}, gx, gx), G(gx_{n+1}, gx, hu), \mathbf{G}(T(u,v), T(u,v), hu)\}\\
&+L \min\{\mathbf{G}(T(u,v), hu_n, hu_n), \mathbf{G}(T(u,v), hu_n, hu_n), \mathbf{G}(hu_{n+1}, hu, hu)\},
\end{aligned}$$

Now by taking the limit, we obtain $\mathbf{G}(hu, hu, T(u,v)) = 0$. Hence $hu = T(u,v)$. Similarly, one can show that $hv = T(v,u)$. Thus, T and h have a C-C-point. □

Theorem 2.4. *Let $\boldsymbol{G}$ be a complete $\boldsymbol{G}$-metric on partially ordered set $(Y, \preceq)$. Suppose $T : Y \times Y \to Y$ and $h : Y \to Y$ are two functions where T has the mixed h-monotone property on Y. Suppose for $\alpha, \beta, \gamma, L \geqslant 0$ with $\alpha + \beta + \gamma < 1$ we have*

$$\begin{aligned}
&\boldsymbol{G}(T(x,y), T(x,y), T(u,v))\\
\leq\ & \alpha \min\{\boldsymbol{G}(T(x,y), T(x,y), hx), \boldsymbol{G}(T(x,y), hx, hx), \boldsymbol{G}(T(u,v), hx, hx)\}\\
&+\beta \min\{\boldsymbol{G}(T(x,y), hu, hu), \boldsymbol{G}(T(x,y), hu, hu), \boldsymbol{G}(T(u,v), T(u,v), hu)\}\\
&+L \min\{\boldsymbol{G}(T(u,v), hx, hx), \boldsymbol{G}(T(x,y), hu, hu)\}
\end{aligned}\tag{2.19}$$

for all $x, y, u, v \in Y$ with $hx \preceq hu$ and $hy \succeq hv$. Suppose also that $T(Y \times Y) \subseteq h(Y)$ and $h(Y)$ is complete. Also, suppose that:

(i) *if $\{u_n\}$ is nondecreasing and $\{u_n\} \to u$, then $u_n \preceq u$, $\forall\ n \geq 0$;*
(ii) *if $\{v_n\}$ is nonincreasing and $\{v_n\} \to v$, then $v \preceq v_n$, $\forall\ n \geq 0$.*

If there exist $u_0, v_0 \in Y$ such that $hu_0 \preceq T(u_0, v_0)$ and $hv_0 \succeq T(v_0, u_0)$, then h and T have a C-C-point.

Proof. The proof is similar to that provided by Theorem 2.3. □

Remark 2.5. By taking $G(x,y,z) = \frac{1}{2}\Big(d(x,y) + d(y,z) + d(z,x)\Big)$, in Theorem 2.4, we can conclude Theorem 2.1 in [22].

The next theorem is an immediate consequence of Theorem 2.2.

Theorem 2.6. *Let $\boldsymbol{G}$ be a complete $\boldsymbol{G}$-metric on partially ordered set $(Y, \preceq)$. Suppose $T : Y \times Y \to Y$ and $h : Y \to Y$ are two mappings such that T has the mixed h-monotone* *property on Y. Suppose for $\alpha, \beta, \gamma, L \geqslant 0$ with $\alpha+\beta+\gamma<1$ we have*

$$\begin{aligned} &\boldsymbol{G}(T(x,y),T(u,v),T(z,w)) \\ \leq\ &\alpha \min\{\boldsymbol{G}(T(x,y),T(x,y),x),\, \boldsymbol{G}(T(u,v),x,x),\, \boldsymbol{G}(T(z,w),x,x)\} \\ &+\beta \min\{\boldsymbol{G}(T(x,y),u,u),\, \boldsymbol{G}(T(u,v),T(u,v),u),\, \boldsymbol{G}(T(z,w),u,u)\} \\ &+\gamma \min\{\boldsymbol{G}(T(x,y),z,z),\, \boldsymbol{G}(T(u,v),z,z),\, \boldsymbol{G}(T(z,w),T(z,w),z)\} \\ &+L \min\{\boldsymbol{G}(T(z,w),x,x),\, \boldsymbol{G}(T(z,w),u,u),\, \boldsymbol{G}(T(x,y),z,z)\} \end{aligned} \tag{2.20}$$

for all $x,y,u,v,z,w \in Y$ with $x \preceq u \preceq z$ and $y \succeq v \succeq w$. Also suppose that either
(a) T is a a continuous function,
or
(b) Y applies in the following conditions:

(i) *if $\{u_n\}$ is nondecreasing and $\{u_n\} \to u$, then $u_n \preceq u$, $\forall\ n \geq 0$;*
(ii) *if $\{v_n\}$ is nonincreasing and $\{v_n\} \to v$, then $v \preceq v_n$, $\forall\ n \geq 0$.*

If there exist $u_0, v_0 \in Y$ such that $u_0 \preceq T(u_0,v_0)$ and $v_0 \succeq T(v_0,u_0)$, then there exist $u, v \in Y$ such that $u = T(u,v)$ and $v = T(v,u)$.

Proof. Let $g = I_X$ and apply Theorem 2.2. □

By letting $\alpha=\beta=\gamma=0$ in Theorem 2.2, we conclude the next result.

Corollary 2.7. *Let $\boldsymbol{G}$ be a complete $\boldsymbol{G}$-metric on partially ordered set $(Y,\preceq)$. Suppose $T: Y\times Y \to Y$ and $h: Y \to Y$ are two functions where T has the mixed h-monotone property on Y. Suppose for some $L \geq 0$:*

$$\begin{aligned} &\boldsymbol{G}(T(x,y),T(u,v),T(z,w)) \\ \leq\ &L \min\{\boldsymbol{G}(T(z,w),hx,hx),\, \boldsymbol{G}(T(z,w),hu,hu),\, \boldsymbol{G}(T(x,y),hz,hz)\} \end{aligned} \tag{2.21}$$

for all $x,y,u,v,z,w \in Y$ with $hx \preceq hu \preceq hz$ and $hy \succeq hv \succeq hw$. Also suppose that $T(Y\times Y) \subseteq h(Y)$ and h is continuous nondecreasing and h and T are $\boldsymbol{G}$-compatible. Suppose that either
(a) T is a continuous function,
or
(b) Y applies in the following conditions:

(i) *if $\{u_n\}$ is nondecreasing and $\{u_n\} \to u$, then $u_n \preceq u$, $\forall\ n \geq 0$;*
(ii) *if $\{v_n\}$ is nonincreasing and $\{v_n\} \to v$, then $v \preceq v_n$, $\forall\ n \geq 0$.*

If there exist $u_0, v_0 \in Y$ such that $hu_0 \preceq T(u_0,v_0)$ and $hv_0 \succeq T(v_0,u_0)$, then h and T have a C-C-point.

NOTE 3: In Corollary 2.3 of [12], the authors, consider that the function g is commutes with F. In the next corollary (results directly from the main theorem) this condition is replaced by **G**-compatibility, which is simpler. Also, our contraction condition is deference from that one.

Corollary 2.8. *Let $\boldsymbol{G}$ be a complete $\boldsymbol{G}$-metric on partially ordered set $(Y,\preceq)$. Suppose $T: Y\times Y \to Y$ and $h: Y \to Y$ are two functions where T has the mixed h-monotone property* *on Y. Suppose for some $K, L \geq 0$ with $K < 1$:*

$$\begin{aligned}
&\boldsymbol{G}(T(x,y),T(u,v),T(z,w))\\
\leq\ &(K)\min\left\{\begin{array}{l}\boldsymbol{G}(\mathrm{T(x,y),T(x,y),hx}),\ \boldsymbol{G}(\mathrm{T(u,v),hx,hx}),\ \boldsymbol{G}(\mathrm{T(z,w),hx,hx})\\ \boldsymbol{G}(\mathrm{T(x,y),hu,hu}),\ \boldsymbol{G}(\mathrm{T(u,v),T(u,v),hu}),\ \boldsymbol{G}(\mathrm{T(z,w),hu,hu})\\ \boldsymbol{G}(\mathrm{T(x,y),hz,hz}),\ \boldsymbol{G}(\mathrm{T(u,v),hz,hz}),\ \boldsymbol{G}(\mathrm{T(z,w),F(z,w),hz})\end{array}\right\}\\
&+L\min\{\boldsymbol{G}(T(z,w),hx,hx),\ \boldsymbol{G}(T(z,w),hu,hu),\ \boldsymbol{G}(T(x,y),hz,hz)\}
\end{aligned}$$

for all $x, y, u, v, z, w \in Y$ with $hx \preceq hu \preceq hz$ and $hy \succeq hv \succeq hw$. Also suppose that $T(Y \times Y) \subseteq h(Y)$ and h is continuous nondecreasing and h and T are $\boldsymbol{G}$-compatible. Suppose that either
(a) T is a continuous function,
or
(b) Y applies in the following conditions:

(i) *if $\{u_n\}$ is nondecreasing and $\{u_n\} \to u$, then $u_n \preceq u$, $\forall\ n \geq 0$;*
(ii) *if $\{v_n\}$ is nonincreasing and $\{v_n\} \to v$, then $v \preceq v_n$, $\forall\ n \geq 0$.*

If there exist $u_0, v_0 \in Y$ such that $hu_0 \preceq T(u_0, v_0)$ and $hv_0 \succeq T(v_0, u_0)$, then h and T have a C-C-point.

Proof. By taking $\alpha = \beta = \gamma = \dfrac{K}{3}$ we have α, β and γ are non-negative real numbers, and

$$\begin{aligned}
&\mathbf{G}(T(x,y),T(u,v),T(z,w))\\
\leq\ &(\alpha+\beta+\gamma)\min\left\{\begin{array}{l}\mathbf{G}(\mathrm{T(x,y),T(x,y),hx}),\mathbf{G}(\mathrm{T(u,v),hx,hx}),\mathbf{G}(\mathrm{T(z,w),hx,hx})\\ \mathbf{G}(\mathrm{T(x,y),hu,hu}),\mathbf{G}(\mathrm{T(u,v),T(u,v),hu}),\mathbf{G}(\mathrm{T(z,w),hu,hu})\\ \mathbf{G}(\mathrm{T(x,y),hz,hz}),\mathbf{G}(\mathrm{T(u,v),hz,hz}),\mathbf{G}(\mathrm{T(z,w),T(z,w),hz})\end{array}\right\}\\
&+L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}\\
\leq\ &\alpha\min\{\mathbf{G}(T(x,y),T(x,y),hx),\mathbf{G}(T(u,v),hx,hx),\mathbf{G}(T(z,w),hx,hx)\}\\
&+\beta\min\{\mathbf{G}(T(x,y),hu,hu),\mathbf{G}(T(u,v),T(u,v),hu),\mathbf{G}(T(z,w),hu,hu)\}\\
&+\gamma\min\{\mathbf{G}(T(x,y),hz,hz),\mathbf{G}(T(u,v),hz,hz),\mathbf{G}(T(z,w),T(z,w),hz)\}\\
&+L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}.
\end{aligned}$$

Now, by applying Theorem 2.2 we get the result. □

3. Couple Fixed Point

This section talks about the existence and uniqueness of coupled common fixed points, which are the results of the previous section.

If $(Y, \preceq)$ is a partially ordered set, we assign the $Y \times Y$ product set to the partial order $\preceq$ defined by

$$(x,y) \preceq (u,v) \Leftrightarrow x \preceq u \text{ and } y \succeq v.$$

Theorem 3.1. *Suppose that $L = 0$ in the addition to the hypotheses of Theorem 2.3, g and T are $\boldsymbol{G}$-compatible and*
(c) for every $(x,y),(u,v) \in Y \times Y$, there exists $(w,z) \in Y \times Y$ such that $(T(w,z), T(z,w))$ is comparable to $(T(x,y), T(y,x))$ and $(T(u,v), T(v,u))$.
Then, there exists a unique $(p,q) \in Y \times Y$ where $p = hp = T(p,q)$ and $q = hq = T(q,p)$.

Proof. According to Theorem 2.3, the set of C-C-points is not empty. We will demonstrate whether if $hx = T(x,y)$, $hy = T(y,x)$, $hu = T(u,v)$ and $hv = T(v,u)$, for (x,y) and (u,v), then

$$hx = hu \text{ and } hy = hv. \tag{3.1}$$

By using assumption, for some $(w,z) \in Y \times Y$ such that $(T(w,z), T(z,w))$ is comparable to $(T(x,y), T(y,x))$ and $(T(u,v), T(v,u))$. Without limiting the generality, it is assumed that

$$\begin{aligned} (T(x,y), T(y,x)) &\preceq (T(w,z), T(z,w)) \\ (T(u,v), T(v,u)) &\preceq (T(w,z), T(z,w)). \end{aligned}$$

Put $w_0 = w$, $z_0 = z$ and choose $w_1, z_1 \in Y$ such that $hw_1 = T(w_0, z_0)$ and $hz_1 = T(z_0, w_0)$. According to the proof of Theorem 2.3, we can define tow sequences $\{hw_n\}$ and $\{hz_n\}$ as follows:

$$hw_{n+1} = T(w_n, z_n) \text{ and } hz_{n+1} = T(z_n, w_n), \tag{3.2}$$

for $n \in \mathbb{N}$. By taking $x_0 = x_1 = x_2 = ... = x_n = x$, $y_0 = y_1 = y_2 = ... = y_n = y$, $u_0 = u_1 = u_2 = ... = u_n = u$ and $v_0 = v_1 = v_2 = ... = v_n = v$, for all $n \in \mathbb{N}$, we have

$$hx_n = T(x,y),\ hy_n = T(y,x) \text{ and } hu_n = T(u,v),\ hv_n = T(v,u).$$

Since

$$(T(x,y), T(y,x)) = (hx_1, hy_1) = (hx, hy) \preceq (T(w,z), T(z,w)) = (hw_1, hz_1),$$

then $hx \preceq hw_1$ and $hy \succeq hz_1$. Since T is mixed h-monotone, we can show that $hx \preceq hw_n$ and $hy \succeq hz_n$ for all $n \geq 1$. Hence, from (2.1), we get

$$\begin{aligned} & \mathbf{G}(hw_{n+1}, hx, hx) = \mathbf{G}(T(w_n, z_n), T(x,y), T(x,y)) \\ \leq\ & \alpha \min\{\mathbf{G}(T(w_n, z_n), T(w_n, z_n), hw_n), \mathbf{G}(T(x,y), hw_n, hw_n), \mathbf{G}(T(x,y), hw_n, hw_n)\} \\ & +\beta \min\{\mathbf{G}(T(w_n, z_n), hx, hx), \mathbf{G}(T(x,y), T(x,y), hx), \mathbf{G}(T(x,y), hx, hx)\} \\ & +\gamma \min\{\mathbf{G}(T(w_n, z_n), hx, hx), \mathbf{G}(T(x,y), hx, hx), \mathbf{G}(T(x,y), T(x,y), hx)\}. \end{aligned}$$

Since $T(x,y) = h(x)$, we have

$$\begin{aligned} \mathbf{G}(hw_{n+1}, hx, hx) \leq\ & \alpha \min\{\mathbf{G}(T(w_n, z_n), T(w_n, z_n), hw_n), \\ & \mathbf{G}(hx, hw_n, hw_n), \mathbf{G}(hx, hw_n, hw_n)\}. \end{aligned}$$

Hence

$$\mathbf{G}(hw_{n+1}, hx, hx) \leq \alpha\ \mathbf{G}(hx, hw_n, hw_n). \tag{3.3}$$

Again from (2.1), we have

$$\begin{aligned} & \mathbf{G}(hy, hy, hz_{n+1}) = \mathbf{G}(T(y,x), T(y,x), T(z_n, w_n)) \\ \leq\ & \alpha \min\{\mathbf{G}(T(y,x), T(y,x), hy), \mathbf{G}(T(y,x), hy, hy), \mathbf{G}(T(z_n, w_n), hy, hy)\} \\ & +\beta \min\{\mathbf{G}(T(y,x), hy, hy), \mathbf{G}(T(y,x), T(y,x), hy), \mathbf{G}(T(z_n, w_n), hy, hy)\} \\ & +\gamma \min\{\mathbf{G}(T(y,x), hz_n, hz_n), \mathbf{G}(T(y,x), hz_n, hz_n), \mathbf{G}(T(z_n, w_n), T(z_n, w_n), hz_n)\}. \end{aligned}$$

Since $T(y,x) = hy$, we have

$$\begin{aligned} \mathbf{G}(hy, hy, hz_{n+1}) \leq\ & \gamma \min\{\mathbf{G}(hy, hz_n, hz_n), \\ & \mathbf{G}(hy, hz_n, hz_n), \mathbf{G}(T(z_n, w_n), T(z_n, w_n), hz_n)\}. \end{aligned}$$

Hence

$$\mathbf{G}(hw_{n+1}, hx, hx) \leq \gamma\ \mathbf{G}(hy, hz_n, hz_n). \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$\begin{aligned}
& \mathbf{G}(hw_{n+1}, hx, hx) + \mathbf{G}(hy, hy, hz_{n+1}) \\
\leq\ & \alpha\ \mathbf{G}(hx, hw_n, hw_n) + \gamma\ \mathbf{G}(hy, hz_n, hz_n) \\
\leq\ & (\alpha+\beta+\gamma)[\mathbf{G}(hx, hw_n, hw_n) + \mathbf{G}(hy, hz_n, hz_n)] \\
\leq\ & (\alpha+\beta+\gamma)^2[\mathbf{G}(hx, hw_{n-1}, hw_{n-1}) + \mathbf{G}(hy, hz_{n-1}, hz_{n-1})] \\
\leq\ & \ldots \\
\leq\ & (\alpha+\beta+\gamma)^{n+1}[\mathbf{G}(hx, hw_0, hw_0) + \mathbf{G}(hy, hz_0, hz_0)]
\end{aligned}$$

Since $\alpha+\beta+\gamma < 1$, we have

$$\lim_{n\to\infty}[\mathbf{G}(hw_n, hx, hx) + \mathbf{G}(hy, hy, hz_n)] = 0.$$

So

$$\lim_{n\to\infty} \mathbf{G}(hw_n, hx, hx) = \lim_{n\to\infty} \mathbf{G}(hy, hy, hz_n) = 0. \tag{3.5}$$

Similarly

$$\lim_{n\to\infty} \mathbf{G}(hw_n, hu, hu) = \lim_{n\to\infty} \mathbf{G}(hz_n, hv, hv) = 0. \tag{3.6}$$

Therefore, from (3.5) and (3.6) we get $hx = hu$ and $hy = hv$. So (3.1) holds.

Now by putting $hx = p$ and $hy = q$, we get

$$hp = h(hx) = hT(x, y) \textit{ and } hq = h(hy) = hT(y, x). \tag{3.7}$$

Obviously,

$$hx_n = T(x, y) = T(x_{n-1}, y_{n-1}) \textit{ and } hy_n = T(y, x) = T(y_{n-1}, x_{n-1}),$$

and so

$$T(x_{n-1}, y_{n-1}) \to T(x, y) \textit{ and } hx_n \to T(x, y),$$

as well as

$$T(y_{n-1}, x_{n-1}) \to T(y, x) \textit{ and } hy_n \to T(y, x).$$

From the compatibility property of h and T we get:

$$\mathbf{G}(hT(x_n, y_n), T(hx_n, hy_n), TF(hx_n, hy_n)) \to 0,\ n \to \infty.$$

So $hT(x, y) = T(hx, hy)$. From (3.7) we get that

$$hp = h(hx) = hT(x, y) = T(hx, hy) = T(p, q). \tag{3.8}$$

In a similar way,

$$hq = h(hy) = hT(y, x) = T(hy, hx) = T(q, p). \tag{3.9}$$

Hence $hp = T(p, q)$ and $hq = T(q, p)$. Thus, (p, q) is a coincidence point. Then, from (3.1) with $u = p$ and $v = q$, we have $hx = hp = p$ and $hy = hq = q$, that is,

$$hp = p \textit{ and } hq = q. \tag{3.10}$$

From (3.8), (3.9) and (3.10), we get

$$p = hp = T(p,q) \text{ and } q = hq = T(q,p),$$

and (p,q) is a common coupled fixed point of h and T. To demonstrate uniqueness, if (x_1,x_2) is another coupled common fixed, then by (3.1), we have $x_1 = hx_1 = hp = p$ and $x_2 = hx_2 = hq = q$. □

Theorem 3.2. *Suppose that $L = 0$ in the addition to the hypotheses of Theorem 2.2 and for every $(x,y),(u,v) \in Y \times Y$, there exists $(w,z) \in Y \times Y$ such that $(T(w,z),T(z,w))$ is comparable to $(T(x,y),T(y,x))$ and $(T(u,v),T(v,u))$. Then T and h have a unique common coupled fixed point.*

Proof. The proof is similar to the proof of the previous theorem. □

Theorem 3.3. *Suppose that $L = 0$ in the addition to the hypotheses of Theorem 2.6 and for every $(x,y),(u,v) \in Y \times Y$, there exists $(w,z) \in Y \times Y$ such that $(T(w,z),T(z,w))$ is comparable to $(T(x,y),T(y,x))$ and $(T(u,v),T(v,u))$. Then T has a unique common coupled fixed point.*

Proof. Let $h = I_X$ and apply Theorem 3.1. □

Theorem 3.4. *If hu_0 and hv_0 are comparable and $L = 0$ in addition to the hypotheses of Theorem 2.3, then T and h have a couple coincidence point of the form (u,u).*

Proof. We construct two sequences $\{u_n\}$ and $\{v_n\}$ in Y, by Theorem 2.3, such that $hu_n \to hu$ and $hv_n \to hv$, where (u,v) is a coincidence point of T and h. Since hu_0 and hv_0 are comparable, we may assume that $hu_0 \preceq hv_0$. By using the induction and mixed monotone property of T:

$$hu_n \preceq hv_n \ \forall \ n \geq 0.$$

where $hu_{n+1} = T(u_n,v_n)$ and $hv_{n+1} = T(v_n,u_n)$, $n = 0,1,....$
Thus, by (2.1) we have

$$\begin{aligned}
&\mathbf{G}(hu_{n+1},hv_{n+1},hv_{n+1}) = \mathbf{G}(T(u_n,v_n),T(v_n,u_n),T(v_n,u_n))\\
\leq\ &\alpha \min\{\mathbf{G}(T(u_n,v_n),T(u_n,v_n),hu_n),\mathbf{G}(T(v_n,u_n),hu_n,hu_n),\mathbf{G}(T(v_n,u_n),hu_n,hu_n)\}\\
&+\beta \min\{\mathbf{G}(T(u_n,v_n),hv_n,hv_n),\mathbf{G}(T(v_n,u_n),T(v_n,u_n),hv_n),\mathbf{G}(T(v_n,u_n),hv_n,hv_n)\}\\
&+\gamma \min\{\mathbf{G}(T(u_n,v_n),hv_n,hv_n),\mathbf{G}(T(v_n,u_n),hv_n,hv_n),\mathbf{G}(T(v_n,u_n),T(v_n,u_n),hv_n)\}.
\end{aligned}$$

Now by letting $n \to \infty$, we get $\mathbf{G}(hu,hv,hv) = 0$. Hence $hu = T(u,v) = T(v,u) = hv$. □

Theorem 3.5. *If hu_0 and hv_0 are comparable and $L = 0$ in Theorem 2.2, then T and h have a couple coincidence point of the form (u,u).*

Proof. By Theorem 2.2, T and h have a C-C-point. We show that $u = v$. By Theorem 2.2 we construct two sequences $\{x_n\}$ and $\{v_n\}$ in Y such that $hu_n \to u$ and $hv_n \to v$, where (u,v) is a coincidence point of T and h. Since hu_0 and hv_0 are comparable, we may assume that $hu_0 \preceq hv_0$. By using the induction and mixed monotone property of T:

$$hu_n \preceq hv_n \ \forall \ n \geq 0.$$

where $hu_{n+1} = T(u_n, v_n)$ and $hv_{n+1} = T(v_n, u_n)$, $n = 0, 1, ...$
By using the triangle inequality:

$$\begin{aligned}
& \mathbf{G}(u,u,v) \le \mathbf{G}(u,u,hu_{n+1}) + \mathbf{G}(hu_{n+1},hu_{n+1},v) \\
\le\; & \mathbf{G}(u,u,hu_{n+1}) + \mathbf{G}(hu_{n+1},hu_{n+1},hv_{n+1}) + \mathbf{G}(hv_{n+1},hv_{n+1},v) \\
=\; & \mathbf{G}(u,u,hu_{n+1}) + \mathbf{G}(T(u_n,v_n),T(u_n,v_n),T(v_n,u_n)) + \mathbf{G}(hv_{n+1},hv_{n+1},v) \\
\le\; & \mathbf{G}(u,u,hu_{n+1}) + \mathbf{G}(hv_{n+1},hv_{n+1},v) \\
& +\alpha \min\{\mathbf{G}(T(u_n,v_n),T(u_n,v_n),hu_n),\mathbf{G}(T(u_n,v_n),hu_n,hu_n),\mathbf{G}(T(v_n,u_n),hu_n,hu_n)\} \\
& +\beta \min\{\mathbf{G}(T(u_n,v_n),hu_n,hu_n),\mathbf{G}(T(u_n,v_n),T(u_n,v_n),hu_n),\mathbf{G}(T(v_n,u_n),hu_n,hu_n)\} \\
& +\gamma \min\{\mathbf{G}(T(u_n,v_n),hv_n,hv_n),\mathbf{G}(T(u_n,v_n),hv_n,hv_n),\mathbf{G}(T(v_n,u_n),T(v_n,u_n),hv_n)\}.
\end{aligned}$$

Since $hu_n \to u$ and $hv_n \to v$, by taking the limit from the above inequality, we get that $\mathbf{G}(y, y, v) \le 0$ and so $\mathbf{G}(u, u, v) = 0$ and hence $u = v$. □

Remark 3.6. In addition to the hypotheses of Theorem 2.3, if u_0 and v_0 are comparable and $L = 0$ and $h = I_X$, then there exists $u \in Y$ such that $u = T(u, u)$.

4. Example

In this section, we examine an example with the help of Theorem 2.2.

Example 4.1. We consider $Y = [0, 1]$ with the natural ordering of real numbers. Let $\mathbf{G}$ be the $\mathbf{G}$- metric on $Y \times Y \times Y$ defined as follows:

$$\mathbf{G}(x,y,z) = max\{|x-y|,|y-z|,|z-x|\}\ \forall\ x,y,z \in X.$$

Define $h : Y \to Y$ by $h(x) = x^2$, for all $x \in Y$. Let $T : Y \times Y :\to Y$ be defined as follows

$$T(x,y) = \begin{cases} \frac{x^2-y^2}{8} & ; x \ge y \\ 0 & ; x < y. \end{cases}$$

Obviously, $(Y, \mathbf{G})$ is $\mathbf{G}$-metric, T obeys the mixed h-monotone property, $F(Y \times Y) \subseteq h(Y)$, $g(Y)$ is complete and Y satisfies the conditions (i) and (ii) of Theorem 2.3.

Let $u_0 = 0$ and $v_0 = c > 0$. Then

$$h(u_0) = h(0) = 0 = T(0,c) = T(u_0,v_0)\ and$$
$$h(v_0) = h(c) = c^2 \ge \frac{c^2}{8} = T(c,0) = T(v_0,u_0).$$

Consequently, $h(u_0) \preceq T(u_0, v_0)$ and $h(v_0) \succeq T(v_0, u_0)$.

We next verify inequality 2.17 of Theorem 2.3 for $\alpha = \beta = \gamma = \frac{1}{4}$ and $L \ge 0$ holds, that is,

$$\begin{aligned}
& \mathbf{G}(T(x,y),T(u,v),T(z,w)) \\
\le\; & \frac{1}{4}\min\{\mathbf{G}(T(x,y),T(x,y),hx),\mathbf{G}(T(u,v),hx,hx),\mathbf{G}(T(z,w),hx,hx)\} \\
& +\frac{1}{4}\min\{\mathbf{G}(T(x,y),hu,hu),\mathbf{G}(T(u,v),T(u,v),hu),\mathbf{G}(T(z,w),hu,hu)\} \\
& +\frac{1}{4}\min\{\mathbf{G}(T(x,y),hz,hz),\mathbf{G}(T(u,v),hz,hz),\mathbf{G}(T(z,w),T(z,w),hz)\} \\
& +L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\},
\end{aligned}$$

for $x, y, u, v, z, w \in X$, such that $hx \preceq hu \preceq hz$ and $hy \succeq hv \succeq hw$, that is, $x^2 \preceq u^2 \preceq z^2$ and $y^2 \succeq v^2 \succeq w^2$. There is eight possible cases.

Case 1. Suppose, $x < y$, $u < v$, and $z < w$. So

$$\begin{aligned}
& \mathbf{G}(T(x,y),T(u,v),T(z,w)) = \mathbf{G}(0,0,0) = 0 \\
\leq\ & \frac{1}{4}\min\{\mathbf{G}(0,0,x^2),\mathbf{G}(0,x^2,x^2),\mathbf{G}(0,x^2,x^2)\} \\
& +\frac{1}{4}\min\{\mathbf{G}(0,u^2,u^2),\mathbf{G}(0,0,u^2),\mathbf{G}(0,u^2,u^2)\} \\
& +\frac{1}{4}\min\{\mathbf{G}(0,z^2,z^2),\mathbf{G}(0,z^2,z^2),\mathbf{G}(0,0,z^2)\} \\
& +L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}
\end{aligned}$$

Case 2. Suppose, $x \geq y$, $u \geq v$, and $z \geq w$. So $w \leq v \leq y \leq x \leq u \leq z$. Hence

$$\begin{aligned}
& \mathbf{G}(T(x,y),T(u,v),T(z,w)) = \mathbf{G}(\frac{x^2-y^2}{8},\frac{u^2-v^2}{8},\frac{z^2-w^2}{8}) \\
=\ & max\{|\frac{x^2-y^2}{8}-\frac{u^2-v^2}{8}|,|\frac{u^2-v^2}{8}-\frac{z^2-w^2}{8}|,|\frac{z^2-w^2}{8}-\frac{x^2-y^2}{8}|\} \\
=\ & |\frac{z^2-w^2}{8}-\frac{x^2-y^2}{8}| = |\frac{z^2-x^2}{8}+\frac{y^2-w^2}{8}| = \frac{1}{8}(z^2-x^2+y^2-w^2) \\
\leq\ & \frac{1}{8}z^2 \ (since\ y \leq x) \\
\leq\ & \min\{\frac{8z^2-x^2+y^2}{32},\frac{7z^2+w^2}{32},\frac{6z^2-u^2+v^2}{32}\} \\
=\ & \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
\leq\ & \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),G(\frac{z^2-w^2}{8},x^2,x^2) \\
& +L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}.
\end{aligned}$$

Case 3. Suppose, $x \geq y$, $u \geq v$, and $z < w$. Hence,

$$
\begin{aligned}
& \mathbf{G}(T(x,y),T(u,v),T(z,w)) = \mathbf{G}(\frac{x^2-y^2}{8},\frac{u^2-v^2}{8},0) \\
= \;& max\{|\frac{x^2-y^2}{8}-\frac{u^2-v^2}{8}|,|\frac{u^2-v^2}{8}-0|,|\frac{x^2-y^2}{8}-0|\} \\
\leq \;& |\frac{u^2-v^2}{8}|+|\frac{x^2-y^2}{8}| = \frac{u^2-v^2}{8}+\frac{x^2-y^2}{8} \\
\leq \;& \frac{1}{8}u^2+\frac{1}{8}x^2 \\
\leq \;& \min\{\frac{8u^2-x^2+y^2}{32},\frac{7u^2+v^2}{32},\frac{8u^2-z^2+w^2}{32}\} \\
& +\min\{\frac{7x^2+y^2}{32},\frac{8x^2-u^2+v^2}{32},\frac{8x^2-z^2+w^2}{32}\} \\
= \;& \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),\mathbf{G}(\frac{z^2-w^2}{8},x^2,x^2) \\
\leq \;& \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),\mathbf{G}(\frac{z^2-w^2}{8},x^2,x^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
& +L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}.
\end{aligned}
$$

Case 4. Suppose, $x < y$, $u \geq v$ and $z \geq w$. Hence,

$$
\begin{aligned}
& \mathbf{G}(T(x,y),T(u,v),T(z,w)) = \mathbf{G}(0,\frac{u^2-v^2}{8},\frac{z^2-w^2}{8}) \\
= \;& max\{|\frac{u^2-v^2}{8}-0|,|\frac{u^2-v^2}{8}-\frac{z^2-w^2}{8}|,|\frac{z^2-w^2}{8}-0|\} \\
\leq \;& |\frac{u^2-v^2}{8}|+|\frac{z^2-w^2}{8}| = \frac{u^2-v^2}{8}+\frac{z^2-w^2}{8} \\
\leq \;& \frac{1}{8}u^2+\frac{1}{8}z^2 \\
\leq \;& \min\{\frac{8u^2-x^2+y^2}{32},\frac{7u^2+v^2}{32},\frac{8u^2-z^2+w^2}{32}\} \\
& +\min\{\frac{8z^2-x^2+y^2}{32},\frac{7z^2+w^2}{32},\frac{8z^2-u^2+v^2}{32}\} \\
\leq \;& \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),\mathbf{G}(\frac{z^2-w^2}{8},x^2,x^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
& +L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}.
\end{aligned}
$$

Case 5. Suppose, $x < y$, $u < v$ and $z \geq w$. So,

$$\begin{aligned}
&\mathbf{G}(T(x,y),T(u,v),T(z,w)) = \mathbf{G}(0,0,\frac{z^2-w^2}{8}) \\
= \;& max\{|0|,|\frac{z^2-w^2}{8}|,|\frac{z^2-w^2}{8}|\} = \frac{z^2-w^2}{8} \\
\leq \;& \frac{1}{8}z^2 \\
\leq \;& \min\{\frac{8z^2-x^2+y^2}{32},\frac{7z^2+w^2}{32},\frac{8z^2-u^2+v^2}{32}\} \\
= \;& \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
\leq \;& \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
&+\frac{1}{4}\min\{G(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
&+\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),\mathbf{G}(\frac{z^2-w^2}{8},x^2,x^2) \\
&+L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}.
\end{aligned}$$

Case 6. Suppose, $x < y$, $u \geq v$ and $z < w$. So,

$$\begin{aligned}
&\mathbf{G}(T(x,y),T(u,v),T(z,w)) = \mathbf{G}(0,\frac{u^2-v^2}{8},0) \\
= \;& max\{|0|,|\frac{u^2-v^2}{8}|,|\frac{u^2-v^2}{8}|\} = \frac{u^2-v^2}{8} \\
\leq \;& \frac{1}{8}u^2 \\
\leq \;& \min\{\frac{8u^2-x^2+y^2}{32},\frac{7u^2+v^2}{32},\frac{8u^2-z^2+w^2}{32}\} \\
= \;& \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
\leq \;& \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
&+\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
&+\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),\mathbf{G}(\frac{z^2-w^2}{8},x^2,x^2) \\
&+L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}.
\end{aligned}$$

Case 7. Suppose, $x \geq y$, $u < v$ and $z < w$. Hence,

$$\begin{aligned}
& \mathbf{G}(T(x,y),T(u,v),T(z,w)) = \mathbf{G}(\frac{x^2-y^2}{8},0,0) \\
= \; & max\{|0|,|\frac{x^2-y^2}{8}|,|\frac{x^2-y^2}{8}|\} = \frac{x^2-y^2}{8} \\
\leq \; & \frac{1}{8}x^2 \\
\leq \; & \min\{\frac{7x^2+y^2}{32},\frac{8x^2-u^2+v^2}{32},\frac{8x^2-z^2+w^2}{32}\} \\
= \; & \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),\mathbf{G}(\frac{z^2-w^2}{8},x^2,x^2) \\
\leq \; & \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),G(\frac{z^2-w^2}{8},x^2,x^2) \\
& +L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}.
\end{aligned}$$

Case 8. Suppose, $x \geq y$, $u < v$ and $z \geq w$. Hence,

$$\begin{aligned}
& \mathbf{G}(T(x,y),T(u,v),T(z,w)) = \mathbf{G}(\frac{x^2-y^2}{8},0,\frac{z^2-w^2}{8}) \\
= \; & max\{|\frac{x^2-y^2}{8}|,|\frac{z^2-w^2}{8}|,|\frac{z^2-w^2}{8}-\frac{x^2-y^2}{6}|\} \\
\leq \; & |\frac{z^2-w^2}{8}|+|\frac{x^2-y^2}{6}| = \frac{z^2-w^2}{8}+\frac{x^2-y^2}{8} \\
\leq \; & \frac{1}{8}z^2+\frac{1}{8}x^2 \\
\leq \; & \min\{\frac{8z^2-x^2+y^2}{32},\frac{7z^2+w^2}{32},\frac{8z^2-u^2+v^2}{32}\} \\
& +\min\{\frac{7x^2+y^2}{32},\frac{8x^2-u^2+v^2}{32},\frac{8x^2-z^2+w^2}{32}\} \\
= \; & \frac{1}{4}\min\{G(\frac{x^2-y^2}{8},z^2,z^2),G(\frac{u^2-v^2}{8},z^2,z^2),G(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),G(\frac{z^2-w^2}{8},x^2,x^2) \\
\leq \; & \frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},z^2,z^2),\mathbf{G}(\frac{u^2-v^2}{8},z^2,z^2),\mathbf{G}(\frac{z^2-w^2}{8},\frac{z^2-w^2}{8},z^2) \\
& +\frac{1}{4}\min\{G(\frac{x^2-y^2}{8},u^2,u^2),\mathbf{G}(\frac{u^2-v^2}{8},\frac{u^2-v^2}{8},u^2),\mathbf{G}(\frac{z^2-w^2}{8},u^2,u^2) \\
& +\frac{1}{4}\min\{\mathbf{G}(\frac{x^2-y^2}{8},\frac{x^2-y^2}{8},x^2),\mathbf{G}(\frac{u^2-v^2}{8},x^2,x^2),G(\frac{z^2-w^2}{8},x^2,x^2) \\
& +L\min\{\mathbf{G}(T(z,w),hx,hx),\mathbf{G}(T(z,w),hu,hu),\mathbf{G}(T(x,y),hz,hz)\}.
\end{aligned}$$

Hence the required condition of Theorem 2.3 are satisfied and there exists C-C-point $(0,0)$ of the mappings h and T.

References

[1] M. Abbas, M.A. Khan, S. Radenoviæ, Common coupled fixed point theorem in cone metric space for w-compatible mappings, *Appl. Math. Comput.* **2010** *217* 195–202.

[2] M. Abbas, A.R. Khan, T. Nazir, Coupled common fixed point results in two generalized metric spaces, *Appl. Math. Comput.* (**2011**)

[3] M. Abbas, B.E. Rhoades, Common fixed point results for non-commuting mappings without continuity in generalized metric spaces, *Appl. Math. Computation.* **2009** *215* 797–801.

[4] R.P. Agarwal, M.A. El-Gebeily, D. Oregan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.* **2008** *87* 109–116.

[5] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.* **2006** *65* 1379–1393.

[6] S. S. Chang, Y. J. Cho, B. S, Lee, J. S. Jung, S. M. Kang, Coincidence point theorems and minimization theorems in fuzzy metric spaces, *Fuzzy Sets and Systems* **1997** *88(1)*, 119–127.

[7] B.S. Choudhury, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, *Nonlinear Anal.* **2010** *73* –2531.

[8] B.S. Choudhury, P. Maity, Coupled fixed point results in generalized metric spaces, *Math. Comput. Modelling* **2011** *54(1-2)* 73–79.

[9] R. Chugh, T. Kadian, A. Rani, B.E. Rhoades, Property P in G-metric spaces, *Fixed Point Theory Appl.* (**2010**) 12 pages, DOI:10.1155/2010/401684.

[10] Lj. Ćirić, N. Cakić, M. Rajović, J.S. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, *Fixed Point Theory Appl.*, **2008** 11 pages. Article ID 131294.

[11] Lj. Ćirić, D. Mihet, R. Saadati, Monotone generalized contractions in partially ordered probabilistic metric spaces, *Topology Appl.* **2009** *156(17)* 2838–2844.

[12] F. Gu and S. Zhou, Coupled common fixed point theorems for a pair of commuting mappings in partially ordered G-metric spaces ,*Fixed Point Theory Appl.* **2013** doi:10.1186/1687-1812-2013-64.

[13] K. Kalyani, N. Seshagiri Rao, Coincidence point results of nonlinear contractive mappings in partially ordered metric spaces ,*CUBO, A Mathematical Journal* **2021** *23(02)* 207–224.

[14] V. Lakshmikantham, Lj. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.* **2009** *70* 4341–4349.

[15] Z. Mustafa, B. Sims, Some remarks concerning D-metric spaces, *in: Proceedings of the International Conference on Fixed Point Theory and Applications*, Yokohama, Japan, **2004**, 189–198.

[16] Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point theory, *Ph.D. Thesis, The University of Newcastle, Callaghan, Australia*, **2005**.

[17] Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorem for mapping on complete metric spaces, *Fixed Point Theory Appl.* **2008** 12 pages. Article ID 189870.

[18] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.* **2006** *7(2)* 289–297.

[19] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, *Fixed Point Theory Appl.*, **2009** 10 pages. Article ID 917175.

[20] Z. Mustafa, W. Shatanawi, M. Bataineh, Existence of fixed point results in G-metric spaces, *Int. J. Math. Math. Sci.* **2009** 10 pages. Article ID 283028.

[21] H.K. Nashine, B. Samet, Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces, *Nonlinear Analysis*, **2011** *74* 2201–2220.

[22] H. K. Nashine, W. Shatanawi, Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces, *Computers and Mathematics with Applications*, **2011** *62* 1984–1993.

[23] J.J. Nieto, R. Rodriguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sin. (Engl. Ser.)*, **2007** *23* 2205–2212.

[24] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Pror. Amer. Math. Soc.*, **2004** *132* 1435–1443.

[25] R. Saadati, S.M. Vaezpour, P. Vetro, B.E. Rhoades, Fixed point theorems in generalized partially ordered G- metric spaces, *Math. Comput. Modelling.*, **2010** *52* 797–801.

[26] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, *Nonlinear Anal.* 72 **2010** *72* 4508–4517.

[27] B. Samet, H. Yazidi, Coupled fixed point theorems in partially ordered ϵ-chainable metric spaces, *J. Math. Comput. Sci.*, **2010** *1(3)* 142–151.
[28] B. Samet, C. Vetro, Coupled fixed point, F-invariant set and fixed point of N-order, *Ann. Funct. Anal.*, **2010** *1(2)* 46–56.
[29] W. Shatanawi, Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces, *Fixed Point Theory Appl.*, **2010** 9 pages. Article ID 181650.
[30] W. Shatanawi, Partially ordered cone metric spaces and coupled fixed point results, *Comput. Math. Appl.* **2010** *60* 2508–2515.
[31] P. Semwal, Coincidence and common fixed points in metric space over Banach algebra, *Int. J. Nonlinear Anal. Appl.*, **2022** *13* 479–484.
[32] M. Tesfaye, K. Koyas, S. Gebregiorgis, Some coupled coincidence and coupled common fixed point result in dislocated quasi b-metric spaces for rational type contraction mappings, *Int. J. Nonlinear Anal. Appl.*, **2022** *13* 573–582.

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