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Research Paper

SCHUR-CONVEXITY OF INTEGRAL ARITHMETIC MEANS OF CO-ORDINATED CONVEX FUNCTIONS IN $\mathbb{R}^3$

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Abstract. In this paper, we investigate Schur-convexity of some functions which are obtained from the co-ordinated convex functions on a rectangular box in $\mathbb{R}^3$

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1. Introduction

The first study of Schur-convexity was done by Issai Schur in 1923. Since then numerous articles have been written about it, see for exmple [3, 4, 9, 10]. Schur-convexity has many important applications in analytic and geometric inequality, combinatorial analysis, combinatorial optimization, matrix theory, information theory, and other fields . We recall some definitions as follows:

Definition 1.1. [1] Suppose that $x = (x_1, x_2, \dots, x_n),\ y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. x is said to be majorized by y (in symbols $x \prec y$) if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad k = 1, 2, \dots, n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$, denotes the i-th largest component in x.

Definition 1.2. [1] Let $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}$ is said to be Schur-convex function on E if $x \prec y$ on E implies $f(x) \le f(y)$. f is said to be Schur-concave if and only if $-f$ is Schur-convex.

Definition 1.3. [1, 8] (i) A set $E \subset \mathbb{R}^n$ is called symmetric, if $x \in E$ implies $Px \in E$ for every $n \times n$ permutation matrix P.
(ii) A function $f : E \to \mathbb{R}$ is is said to be a symmetric function if $f(Px) = f(x)$ for every permutation matrix P, and for every $x \in E$.

Recall that a $n \times n$ square matrix P is said to be a permutation matrix if each row and column has a single unite entry, and all other entries are zero. The following theorem called the schur's condition, is very useful for specifying Schur-convexity or Schur-concavity of functions.

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Theorem 1.4. [1] *Let $E \subset \mathbb{R}^n$ be a symmetric convex set with nonempty interior (E° is the interior of E), and $f : E \to \mathbb{R}$ is a symmetric continuous function on E. If f is differentiable on E°, then f is Schur-convex (Schur-concave) on E° if and only if*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geq 0 \quad (\leq 0),$$

for every $x = (x_1, x_2, \ldots, x_n) \in E^\circ$.

In [5], S.S. Dragomir defined convex function on the co-ordinates (or co-ordinated convex functions) on the set $[a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ as follows:

Definition 1.5. A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is said to be convex on the co-ordinates on $[a, b] \times [c, d]$ if for every $y \in [c, d]$ and $x \in [a, b]$, the partial mappings,

$$f_y\colon [a, b] \to \mathbb{R}, \quad f_y(u) = f(u, y),$$

and

$$f_x\colon [c, d] \to \mathbb{R}, \quad f_x(v) = f(x, v),$$

are convex. This means that for every $(x, y), (z, w) \in [a, b] \times [c, d]$ and $t, s \in [0, 1]$,

$$\begin{aligned} f\left(tx + (1-t)z, sy + (1-s)w\right) \leq{}& tsf(x, y) + s(1-t)f(z, y) \\ &+ t(1-s)f(x, w) + (1-t)(1-s)f(z, w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex functions which are not convex. The following Hermite-Hadamard type inequality for co-ordinated convex functions was also proved in [5].

Theorem 1.6. *Suppose that $f : [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates on $[a, b] \times [c, d]$. Then,*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)dydx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x, c)dx + \frac{1}{b-a}\int_a^b f(x, d)dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a, y)dy + \frac{1}{d-c}\int_c^d f(b, y)dy\right] \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned}$$

The above inequalities are sharp.

In [7], M.E. Özdemir defined convex function on a rectangular box $\Omega = [a, b] \times [c, d] \times [e, f]$ in $\mathbb{R}^3$ as follows: A function $f : \Omega \to \mathbb{R}$ is said to be convex on the co-ordinates on Ω if for every $(x, y) \in [a, b] \times [c, d], (x, z) \in [a, b] \times [e, f]$ and $(y, z) \in [c, d] \times [e, f]$, the partial mappings,

$$\begin{aligned} &f_z : [a, b] \times [c, d] \to \mathbb{R}, \quad f_z(u, v) = f(u, v, z), \quad z \in [e, f], \\ &f_y : [a, b] \times [e, f] \to \mathbb{R}, \quad f_y(u, w) = f(u, y, w), \quad y \in [c, d], \\ &f_x : [c, d] \times [e, f] \to \mathbb{R}, \quad f_x(v, w) = f(x, v, w), \quad x \in [a, b], \end{aligned}$$

are convex. The following theorem is given in [7].

Theorem 1.7. *Suppose that $f : \Omega = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$ is convex on the co-ordinates on Ω. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}, \frac{e+f}{2}\right) &\le \frac{1}{(b-a)(d-c)(f-e)} \int\int\int_{\Omega} f(x, y, z)dydzdx \\ &\le \frac{1}{6}\Bigg[\frac{1}{(b-a)(d-c)} \int\int_{\Delta_1} f(x, y, e)dydx \\ &\quad + \frac{1}{(b-a)(d-c)} \int\int_{\Delta_1} f(x, y, f)dydx \\ &\quad + \frac{1}{(b-a)(f-e)} \int\int_{\Delta_2} f(x, c, z)dzdx \\ &\quad + \frac{1}{(b-a)(f-e)} \int\int_{\Delta_2} f(x, d, z)dzdx \\ &\quad + \frac{1}{(d-c)(f-e)} \int\int_{\Delta_3} f(a, y, z)dzdy \\ &\quad + \frac{1}{(d-c)(f-e)} \int\int_{\Delta_3} f(b, y, z)dzdy\Bigg] \end{aligned}$$

where $\Delta_1 = [a, b] \times [c, d]$, $\Delta_2 = [a, b] \times [e, f]$ and $\Delta_3 = [c, d] \times [e, f]$.

In [6] Elezović and Pečarić investigated the Schur-convexity on the upper and the lower limit of the integral for the mean of convex function and proved the following important result by using the Hermite-Hadamard inequality.

Theorem 1.8. *Let f be a continuous function on an interval I, and*

$$F(x, y) = \begin{cases} \frac{1}{y-x} \int_x^y f(t)dt, & x, y \in I, \ x \neq y, \\ f(x), & x = y \in I. \end{cases}$$

Then $F(x, y)$ is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Let $I \subset \mathbb{R}$ be an open interval and $f \in C^2(I)$. In [3] Y. Chu et al. proved the following theorem.

Theorem 1.9. *Let $f : I \to \mathbb{R}$ be a continuous function. The function*

$$F(x, y) = \begin{cases} \frac{1}{y-x} \int_x^y f(t)dt - f(\frac{x+y}{2}), & x, y \in I, \ x \neq y, \\ 0, & x = y \in I, \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

We recall the following lemma from [2], which is known as Leibniz's Formula.

Lemma 1.10. *Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ and $\frac{\partial f}{\partial t} : [a, b] \times [c, d] \to \mathbb{R}$ are continuous and $\alpha_1, \alpha_2 : [c, d] \to [a, b]$ are differentiable functions. Then, the function $\varphi : [c, d] \to \mathbb{R}$ defined by*

$$\varphi(t) = \int_{\alpha_1(t)}^{\alpha_2(t)} f(x, t)dx,$$

has a derivative for each $t \in [c,d]$, which is given by

$$\varphi'(t) = f(\alpha_2(t),t)\alpha_2'(t) - f(\alpha_1(t),t)\alpha_1'(t) + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{\partial f}{\partial t}(x,t)dx.$$

Moreover, we use the following lemma which will be useful in the sequal. A version of the following lemma proved in [9].

Lemma 1.11. *Let $F(u,v) = \int_u^v \int_u^v \int_u^v f(x,y,z)dxdydz$, where*

$$f(x,y,z), \quad \frac{\partial}{\partial b}\int_u^v f(x,y,z)dx,$$

and

$$\frac{\partial}{\partial b}\int_u^v \int_u^v f(x,y,z)dxdy$$

are continuous on the cube $\Omega = [a,p] \times [a,p] \times [a,p]$, $u = u(b)$ and $v = v(b)$ are differentiable with $a \le u(b) \le p$ and $a \le v(b) \le p$. Then,

$$\begin{aligned} \frac{\partial F}{\partial b} = &\Big(\int_u^v \int_u^v f(x,y,v)dxdy + \int_u^v \int_u^v f(x,v,z)dxdz \\ &+ \int_u^v \int_u^v f(v,y,z)dydz\Big)v'(b) - \Big(\int_u^v \int_u^v f(x,y,u)dxdy \\ &+ \int_u^v \int_u^v f(x,u,z)dxdz + \int_u^v \int_u^v f(u,y,z)dydz\Big)u'(b). \end{aligned} \tag{1.1}$$

Proof. If $G(u,v,y,z) = \int_u^v f(x,y,z)dx$ and $H(u,v,z) = \int_u^v G(u,v,y,z)dy$ then $F(u,v) = \int_u^v H(u,v,z)dz$. Therefore by Lemma 1.10, we have

$$\frac{\partial F}{\partial b} = H(u,v,v)v'(b) - H(u,v,u)u'(b) + \int_u^v \frac{\partial H(u,v,z)}{\partial b}dz, \tag{1.2}$$

$$\begin{aligned} \frac{\partial H(u,v,z)}{\partial b} =& G(u,v,v,z)v'(b) - G(u,v,u,z)u'(b) \\ &+ \int_u^v \frac{\partial G(u,v,y,z)}{\partial b}dy, \end{aligned} \tag{1.3}$$

$$\frac{\partial G(u,v,y,z)}{\partial b} = f(v,y,z)v'(b) - f(u,y,z)u'(b). \tag{1.4}$$

By replacing (1.3) and (1.4) in (1.2) we obtained required result in (1.1). □

2. Main Results

In this section we prove new theorems like those Theorem 1.8 and Theorem 1.9 for co-ordinated convex functions.

To reach our main results, we need the following two lemmas.

Lemma 2.1. *Let $\Omega := [a_1,b_1] \times [a_1,b_1] \times [a_1,b_1]$ be a cube in $\mathbb{R}^3$ with $a_1 < b_1$, and the function $f : \Omega \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on Ω°(*

the interior of Ω *). Choose* $a, b \in (a_1, b_1)$*, with* $a < b$*, and let* $D := [a, b] \times [a, b]$*. Suppose that the function* $F : D \to \mathbb{R}$ *is defined by*

$$F(x,y) := \begin{cases} \frac{1}{(y-x)^3}\int_x^y \int_x^y \int_x^y f(r,s,t)drdsdt, & x \neq y, \quad x,y \in [a,b], \\ f(x,x,x), & x = y, \quad x,y \in [a,b]. \end{cases}$$

Then,

$$\begin{aligned} \frac{\partial F}{\partial x}\Big|_{(t_0,t_0)} =& \frac{\partial F}{\partial y}\Big|_{(t_0,t_0)} \\ =& \frac{1}{24}\bigg[6\frac{\partial f}{\partial t}(t,t,t)\mid_{t_0} + 2\Big(g_1(t_0,t_0,t_0) + g_2(t_0,t_0,t_0) + g_3(t_0,t_0,t_0) \\ & + f_1(t_0,t_0,t_0) + f_2(t_0,t_0,t_0) + f_3(t_0,t_0,t_0)\Big)\bigg], \end{aligned} \tag{2.1}$$

for all $t_0 \in [a, b]$ *,where*

$$\begin{aligned} f_1(u,v,t_0+t) &= \frac{\partial f}{\partial t}(u,v,t_0+t), \\ f_2(u,t_0+t,w) &= \frac{\partial f}{\partial t}(u,t_0+t,w), \\ f_3(t_0+t,v,w) &= \frac{\partial f}{\partial t}(t_0+t,v,w), \end{aligned}$$

and

$$\begin{aligned} g_1(u,t_0+t,t_0+t) &= \frac{\partial f}{\partial t}(u,t_0+t,t_0+t), \\ g_2(t_0+t,v,t_0+t) &= \frac{\partial f}{\partial t}(t_0+t,v,t_0+t), \\ g_3(t_0+t,t_0+t,w) &= \frac{\partial f}{\partial t}(t_0+t,t_0+t,w). \end{aligned}$$

Proof. Fix $t_0 \in [a, b]$. We put

$$\begin{aligned} h_1(u,v,t_0+t) &= \frac{\partial f_1}{\partial t}(u,v,t_0+t), \\ h_2(u,t_0+t,w) &= \frac{\partial f_2}{\partial t}(u,t_0+t,w), \\ h_3(t_0+t,v,w) &= \frac{\partial f_3}{\partial t}(t_0+t,v,w). \end{aligned}$$

By using the L'Hopital's rule, and Lemmas 1.10, 1.11 we see that

$$\begin{aligned} \frac{\partial F}{\partial x}\Big|_{(t_0,t_0)} &= \lim_{t\to 0}\frac{F(t_0+t,t_0) - F(t_0,t_0)}{t} \\ &= \lim_{t\to 0}\frac{1}{t^4}\left[\int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t} f(u,v,w)dudvdw - t^3 f(t_0,t_0)\right] \\ &= \lim_{t\to 0}\frac{1}{4t^3}\bigg[\int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t} f(u,v,t_0+t)dudv + \int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t} f(u,t_0+t,w)dudw \\ &\quad + \int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t} f(t_0+t,v,w)dvdw - 3t^2 f(t_0,t_0,t_0)\bigg] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0}\frac{1}{12t^2}\Big[\int_{t_0}^{t_0+t} f(u,t_0+t,t_0+t)du + \int_{t_0}^{t_0+t} f(t_0+t,v,t_0+t)dv \\
&\quad + \int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t}\frac{\partial f}{\partial t}(u,v,t_0+t)dudv + \int_{t_0}^{t_0+t} f(u,t_0+t,t_0+t)du \\
&\quad + \int_{t_0}^{t_0+t} f(t_0+t,t_0+t,w)dw + \int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t}\frac{\partial f}{\partial t}(u,t_0+t,w)dudw \\
&\quad + \int_{t_0}^{t_0+t} f(t_0+t,v,t_0+t)dv + \int_{t_0}^{t_0+t} f(t_0+t,t_0+t,w)dw \\
&\quad + \int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t}\frac{\partial f}{\partial t}(t_0+t,v,w)dvdw - 6tf(t_0,t_0,t_0)\Big] \\
&= \lim_{t\to 0}\frac{1}{24t}\Big[6f(t_0+t,t_0+t,t_0+t) + 2\int_{t_0}^{t_0+t}\frac{\partial f}{\partial t}(u,t_0+t,t_0+t)du \\
&\quad + 2\int_{t_0}^{t_0+t}\frac{\partial f}{\partial t}(t_0+t,v,t_0+t)dv + 2\int_{t_0}^{t_0+t}\frac{\partial f}{\partial t}(t_0+t,t_0+t,w)dw \\
&\quad + \int_{t_0}^{t_0+t} f_1(u,t_0+t,t_0+t)du + \int_{t_0}^{t_0+t} f_1(t_0+t,v,t_0+t)dv \\
&\quad + \int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t} h_1(u,v,t_0+t)dudv + \int_{t_0}^{t_0+t} f_2(u,t_0+t,t_0+t)du \\
&\quad + \int_{t_0}^{t_0+t} f_2(t_0+t,t_0+t,w)dw + \int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t} h_2(u,t_0+t,w)dudw \\
&\quad + \int_{t_0}^{t_0+t} f_3(t_0+t,v,t_0+t)dv + \int_{t_0}^{t_0+t} f_3(t_0+t,t_0+t,w)dw \\
&\quad + \int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t} h_3(t_0+t,v,w)dvdw - 6f(t_0,t_0,t_0)\Big] \\
&= \frac{1}{24}\Big[6\frac{\partial f}{\partial t}(t,t,t)\,|_{t_0} + 2\Big(g_1(t_0,t_0,t_0) + g_2(t_0,t_0,t_0) + g_3(t_0,t_0,t_0) \\
&\quad + f_1(t_0,t_0,t_0) + f_2(t_0,t_0,t_0) + f_3(t_0,t_0,t_0)\Big)\Big].
\end{aligned} \tag{2.2}
$$

By changing the role of x by y in (2.2), we obtain required results in (2.1). □

The proof of the following lemma is similar to once in lemma 2.1 hence we omit it.

Lemma 2.2. *Let $\Omega := [a_1, b_1] \times [a_1, b_1] \times [a_1, b_1]$ be a cube in $\mathbb{R}^3$ with $a_1 < b_1$, and the function $f : \Omega \to \mathbb{R}$ is continuous, and has continuous four order partial derivatives on Ω°. Choose*

$a, b \in (a_1, b_1)$, with $a < b$, and let $D := [a, b] \times [a, b]$. Suppose that the function $G : D \to \mathbb{R}$ is defined by

$$G(x,y) := \begin{cases} \frac{1}{(y-x)^3}\int_x^y \int_x^y \int_x^y f(r,s,t)drdsdt \\ -f(\frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}), & x \neq y, \;\; x, y \in [a,b], \\ 0, & x = y, \;\; x, y \in [a,b]. \end{cases}$$

Then,

$$\begin{aligned} \frac{\partial G}{\partial x}\Big|_{(t_0,t_0)} =& \frac{\partial G}{\partial y}\Big|_{(t_0,t_0)} \\ =& \frac{1}{24}\Big[-3\frac{\partial f}{\partial t}(t,t,t)\mid_{t_0} +2\Big(g_1(t_0,t_0,t_0) + g_2(t_0,t_0,t_0) + g_3(t_0,t_0,t_0) \\ & + f_1(t_0,t_0,t_0) + f_2(t_0,t_0,t_0) + f_3(t_0,t_0,t_0)\Big)\Big], \end{aligned}$$

for all $t_0 \in [a, b]$, where

$$\begin{aligned} f_1(u, v, t_0 + t) &= \frac{\partial f}{\partial t}(u, v, t_0 + t), \\ f_2(u, t_0 + t, w) &= \frac{\partial f}{\partial t}(u, t_0 + t, w), \\ f_3(t_0 + t, v, w) &= \frac{\partial f}{\partial t}(t_0 + t, v, w), \end{aligned}$$

and

$$\begin{aligned} g_1(u, t_0 + t, t_0 + t) &= \frac{\partial f}{\partial t}(u, t_0 + t, t_0 + t), \\ g_2(t_0 + t, v, t_0 + t) &= \frac{\partial f}{\partial t}(t_0 + t, v, t_0 + t), \\ g_3(t_0 + t, t_0 + t, w) &= \frac{\partial f}{\partial t}(t_0 + t, t_0 + t, w). \end{aligned}$$

We now derive the next results for co-ordinates convex functions.

Theorem 2.3. *Let $D := [a_1, b_1] \times [a_1, b_1] \times [a_1, b_1]$ be a cube in $\mathbb{R}^3$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on D°. Choose $a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a, b] \times [a, b] \times [a, b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $F : [a, b] \times [a, b] \to \mathbb{R}$ defined by*

$$F(x,y) := \begin{cases} \frac{1}{(y-x)^3}\int_x^y \int_x^y \int_x^y f(r,s,t)drdsdt, & x \neq y, \quad x, y \in [a,b], \\ f(x,x,x), & x = y, \quad x, y \in [a,b]. \end{cases} \tag{2.3}$$

is Schur-convex on $[a, b] \times [a, b]$.

Proof. Case 1: If $x, y \in [a, b]$, with $x = y$. Then Lemma 2.1 implies that

$$(y - x)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) = 0.$$

Case 2: If $x, y \in [a, b]$, with $x \neq y$. Then by Lemma 1.11 we have

$$\begin{aligned}\frac{\partial F}{\partial y} =& \frac{-3}{(y-x)^4}\int_x^y\int_x^y\int_x^y f(r,s,t)drdsdt \\ &+ \frac{1}{(y-x)^3}\Big[\int_x^y\int_x^y f(r,s,y)drds \\ &+ \int_x^y\int_x^y f(r,y,t)drdt + \int_x^y\int_x^y f(y,s,t)dsdt\Big],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F}{\partial x} =& \frac{3}{(y-x)^4}\int_x^y\int_x^y\int_x^y f(r,s,t)drdsdt \\ &- \frac{1}{(y-x)^3}\Big[\int_x^y\int_x^y f(r,s,x)drds \\ &+ \int_x^y\int_x^y f(r,x,t)drdt + \int_x^y\int_x^y f(x,s,t)dsdt\Big].\end{aligned}$$

Thus,

$$\begin{aligned}\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) =& \frac{-6}{(y-x)^4}\int_x^y\int_x^y\int_x^y f(r,s,t)drdsdt \\ &+ \frac{1}{(y-x)^3}\Big[\int_x^y\int_x^y \big(f(r,s,x)+f(r,s,y)\big)drds \\ &+ \int_x^y\int_x^y \big(f(r,x,t)+f(r,y,t)\big)drdt \\ &+ \int_x^y\int_x^y \big(f(x,s,t)+f(y,s,t)\big)dsdt\Big].\end{aligned}$$

Then $(y-x)\big(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\big)$ is nonnegative if

$$\begin{aligned}&\frac{1}{y-x}\int_x^y\int_x^y\int_x^y f(r,s,t)drdsdt \\ &\leq \frac{1}{6}\Big[\int_x^y\int_x^y \big(f(r,s,x)+f(r,s,y)\big)drds \\ &\quad+ \int_x^y\int_x^y \big(f(r,x,t)+f(r,y,t)\big)drdt \\ &\quad+ \int_x^y\int_x^y \big(f(x,s,t)+f(y,s,t)\big)dsdt\Big].\end{aligned}$$

Since f is convex on the co-ordinates the last inequality holds by Theorem 1.7. Therefore by Theorem 1.4 the function F is Schur-convex. □

The following theorem also holds:

Theorem 2.4. *Let $D := [a_1, b_1] \times [a_1, b_1] \times [a_1, b_1]$ be a cube in $\mathbb{R}^3$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous four order partial derivatives on D°. Choose $a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a, b] \times [a, b] \times [a, b]$. Suppose that f is convex* *on the co-ordinates on Δ, then the function $G : [a,b] \times [a,b] \to \mathbb{R}$ defined by*

$$(2.4)\qquad G(x,y) := \begin{cases} \frac{1}{(y-x)^3} \int_x^y \int_x^y \int_x^y f(r,s,t)drdsdt \\ -f(\frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}), & x \neq y, \quad x,y \in [a,b], \\ 0, & x = y, \quad x,y \in [a,b]. \end{cases}$$

is Schur-convex on $[a,b] \times [a,b]$.

Proof. Case 1: If $x,y \in [a,b]$, with $x = y$. Then Lemma 2.2 implies that

$$(y-x)\left(\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\right) = 0$$

Case 2: If $x,y \in [a,b]$, with $x \neq y$. Then by Lemma 1.11 we have

$$(y-x)\left(\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\right) \geq 0,$$

if

$$\begin{aligned} &\frac{1}{y-x} \int_x^y \int_x^y \int_x^y f(r,s,t)drdsdt \\ &\leq \frac{1}{6}\Big[\int_x^y \int_x^y \big(f(r,s,x) + f(r,s,y)\big)drds \\ &+ \int_x^y \int_x^y \big(f(r,x,t) + f(r,y,t)\big)drdt \\ &+ \int_x^y \int_x^y \big(f(x,s,t) + f(y,s,t)\big)dsdt \Big]. \end{aligned}$$

The result follows from Theorem 1.7 and Theorem 1.4. □

In the following examples we show that the converses of theorems 2.3 and 2.4 are not true in general.

Example 2.5. Consider the non co-ordinates convex function,

$$f(r,s,t) := r^2 - \frac{1}{2}s^2 + t^2, \quad r,t,s \in [1,2].$$

It is easy to see that for the function F was defined in (2.3) we have $F(x,x) = \frac{3}{2}x^2$, for every $x \in [1,2]$, and

$$F(x,y) = \frac{1}{(y-x)^3} \int_x^y \int_x^y \int_x^y (r^2 - \frac{1}{2}s^2 + t^2)drdsdt = \frac{1}{2}(x^2 + y^2 + xy),$$

for every $x,y \in [1,2]$, with $x \neq y$. Thus,

$$F(x,y) = \frac{1}{2}(x^2 + y^2 + xy),$$

for every $x,y \in [1,2]$. Clearly F is symmetric, continuous and differentiable on $[1,2] \times [1,2]$. If $x,y \in [1,2]$, we have

$$(y-x)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) = \frac{1}{2}(y-x)^2 \geq 0.$$

Therefore by Theorem 1.4 F is Schur-convex.

Remark 2.6. It is easy to see that for the function f was defined in example 2.5 each of the inequalities in theorem 1.7 is valid while f is not convex on the co-ordintes. This means that the converse of theorem 1.7 is not valid in general.

Example 2.7. Consider the non co-ordinated convex function:

$$f(r,s,t) := 2r^2 - s^2 + t^2, \quad r,t,s \in [1,2].$$

It is easy to see that for the function G was defined in (2.4) we have $G(x,x) = 0$, for every $x \in [1,2]$, and

$$\begin{aligned} G(x,y) =& \frac{1}{(y-x)^3} \int_x^y \int_x^y \int_x^y \left(2r^2 - s^2 + t^2\right) drdsdt - \frac{(x+y)^2}{2} \\ =& \frac{2}{3}(x^2+y^2+xy) - \frac{(x+y)^2}{2}, \end{aligned}$$

for every $x, y \in [1,2]$, with $x \neq y$. Thus,

$$G(x,y) = \frac{2}{3}(x^2+y^2+xy) - \frac{(x+y)^2}{2}.$$

Clearly G is symmetric, continuous and differentiable on $[1,2]\times[1,2]$. If $x, y \in [1,2]$, we have

$$(y-x)\left(\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\right) = \frac{2}{3}(y-x)^2 \geq 0.$$

Therefore by Theorem 1.4 the function G is Schur-convex.

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