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Research Paper

A NEW INEQUALITY FOR THE ANGLES OF A TRIANGLE

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Abstract. In the study, using the Lagrange multiplier method, we investigated the extreme values of $\cos^3 x + \cos^3 y + \cos^3 z$ where x, y, z are the angles of a non-degenerate triangle. The minimum and supremum of $\cos^n x + \cos^n y + \cos^n z$, for integers $n \geq 2$, over the set of acute triangles were also determined.

MSC(2010): 26D05, 26B05.
Keywords: extreme value, geometric inequality, Lagrange multiplier, Cardano formula.

1. Introduction

Geometric inequalities are an important branch of mathematics and have applications in optimization, physics, statistics and other fields, see [2, 4, 5, 6]. There are the large number of inequalities for the triangle (sides, angles, etc.). Bottema et al. (1969), in their book *Geometric Inequalities* collected a significant number of inequalities for the triangle. In 1989 Mitrinović et al. published their book entitled *Recent Advances in Geometric Inequalities*, which is an rich source for Geometric inequalities. Also a top journal for inequalities is *American Mathematical Monthly*. Let us recall some inequalities and theorem here. Suppose that x, y, z be the angles of a non-degenerate triangle, that is, $x+y+z = \pi$ with $x, y, z \in (0, \pi)$. In ([2], pp. 22, 24) the following inequalities proved .

$$1 < \cos x + \cos y + \cos z \leq \frac{3}{2}, \tag{1.1}$$

for all triangles, equality holds if and only if the triangle is equilateral. The right-hand side of the above inequality is attributed to Child, 1939.

$$\frac{3}{4} \leq \cos^2 x + \cos^2 y + \cos^2 z < 3, \tag{1.2}$$

for all triangles.

$$\frac{3}{4} \leq \cos^2 x + \cos^2 y + \cos^2 z < 1, \tag{1.3}$$

for acute triangles.

$$1 < \cos^2 x + \cos^2 y + \cos^2 z < 3, \tag{1.4}$$

for obtuse triangles.
These inequalities are attributed to Kooistra, 1957. In optimization, the method of Lagrange multipliers is a useful tool for finding constrained extremes. In this paper we established

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the maximum and the minimum of $\cos^3 x + \cos^3 y + \cos^3 z$, for each triangle. The Lagrange multiplier method with one constraint is given as follows:

Theorem 1.1. [1] *Let f, g be real value functions of three variable with continuous partial derivatives on the open set containing the smooth surface $g(x, y, z) = 0$, and the function f has a local extremum at the point (x_1, y_1, z_1) subject to the constraint $g(x, y, z) = 0$, and that $\nabla g(x_1, y_1, z_1) \neq 0$. Then, there is a number λ such that*

$$\nabla f(x_1, y_1, z_1) = \lambda \nabla g(x_1, y_1, z_1),$$

λ is the Lagrange multiplier.

The well-known theorem for absolute extrema in Euclidean space is stated as follows

Theorem 1.2. [1] *Let $B \subset \mathbb{R}^n$ be closed and bounded, $f : B \to \mathbb{R}$ be a continuous function on B. Then, there exist points $p, q \in B$ such that f reaches its maximum at p and its minimum at q.*

There have been various methods for finding the roots of a cubic equation, such as decomposition, numerical, geometric, Cardano, etc. The following result is known as Cardano's Formula.

Theorem 1.3. [3, 7] *Let $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$, $a_3 \neq 0$ be the cubic equation with real coefficients a_0, a_1, a_2, a_3, put*

$$\begin{aligned}
Q =& \frac{3a_1a_3 - a_2^2}{9a_3^2},\\
R =& \frac{9a_1a_2a_3 - 27a_0a_3^2 - 2a_2^3}{54a_3^3},\\
S =& \sqrt[3]{R + \sqrt{Q^3 + R^2}},\\
T =& \sqrt[3]{R - \sqrt{Q^3 + R^2}},
\end{aligned}$$

and the discriminant of the equation was $D = Q^3 + R^2$.
If $D > 0$ then one root is real and two are complex conjugates, the solutions are:

$$\begin{aligned}
x_1 =& S + T - \frac{a_2}{3a_3},\\
x_2 =& -\frac{S+T}{2} - \frac{a_2}{3a_3} + i\frac{\sqrt{3}}{2}(S - T),\\
x_3 =& -\frac{S+T}{2} - \frac{a_2}{3a_3} - i\frac{\sqrt{3}}{2}(S - T).
\end{aligned}$$

If $D = 0$ then all roots are real, and at least two are equal. If $D < 0$ then all roots are real and unequal.

The aim of this study is to obtain inequalities similar to inequalities (1.2),(1.3) and (1.4) for the sum of $\cos^3 x + \cos^3 y + \cos^3 z$, with using Lagrange multiplier.

2. Main Results

Let x, y, z be the angles of a non-degenerate triangle, and it is easy to see that

$$\lim_{(x,y,z)\to(0,0,\pi)} (\cos^{2n} x + \cos^{2n} y + \cos^{2n} z) = 3, \tag{2.1}$$

thus

$$\sup_{x,y,z\in(0,\pi)} (\cos^{2n} x + \cos^{2n} y + \cos^{2n} z) = 3,$$

and $\cos^{2n} x + \cos^{2n} y + \cos^{2n} z < 3$, for each natural number n. It is natural to say what is the maximum of $\cos^n x + \cos^n y + \cos^n z$, where $n \geq 3$ is an odd number. In this study, we established the maximum value of $\cos^3 x + \cos^3 y + \cos^3 z$ for the angles of a triangle. However, for odd numbers $n \geq 5$ the problem remains open.

Theorem 2.1. *Let* $E = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = \pi, \quad x, y, z \in [0, \pi]\}$, *then*

$$\frac{3}{8} \leq \cos^3 x + \cos^3 y + \cos^3 z \leq 1.225, \tag{2.2}$$

for every $(x, y, z) \in E$, *where equality holds on the left-hand side if and only if* $x = y = z = \frac{\pi}{3}$, *and on the right-hand side if and only if* $x = y \simeq 23.71°, z \simeq 132.58°$.

Proof. Consider the function $f : E \to \mathbb{R}$ defined by $f(x, y, z) = \cos^3 x + \cos^3 y + \cos^3 z$. Because $E \subset \mathbb{R}$ is closed and bounded by Theorem (1.2), the function f has a maximum and minimum on E. Using Theorem (1.1), we find the extreme points of $\cos^3 x + \cos^3 y + \cos^3 z$ subject to the constraint $x + y + z = \pi$ with $x, y, z \in [0, \pi]$. Suppose that the function F is defined as

$$F(x, y, z, \lambda) = \cos^3 x + \cos^3 y + \cos^3 z + \lambda(x + y + z - \pi).$$

Calculating of partial derivatives shows that

$$\begin{cases} \frac{\partial F}{\partial x} = -3\cos^2 x \sin x + \lambda = 0, \\ \frac{\partial F}{\partial y} = -3\cos^2 y \sin y + \lambda = 0, \\ \frac{\partial F}{\partial z} = -3\cos^2 z \sin z + \lambda = 0, \\ \frac{\partial F}{\partial \lambda} = x + y + z - \pi = 0. \end{cases} \tag{2.3}$$

By multiplying the first equation (2.3) by -1 and summing it to the second equation, we obtain $\cos^2 x \sin x = \cos^2 y \sin y$ which implies that $(1 - \sin^2 x) \sin x = (1 - \sin^2 y) \sin y$, we obtain

$$(\sin y - \sin x)(\sin^2 y + \sin^2 x + \sin y \sin x - 1) = 0. \tag{2.4}$$

Similar to the second and third equations of (2.3), $\cos^2 y \sin y = \cos^2 z \sin z$, hence

$$(\sin y - \sin z)(\sin^2 y + \sin^2 z + \sin y \sin z - 1) = 0. \tag{2.5}$$

From equations (2.4) and (2.5), we obtain the following cases:
Case 1: If

$$\begin{cases} \sin y - \sin x = 0, \\ \sin y - \sin z = 0. \end{cases} \tag{2.6}$$

Because, $x, y, z \in [0, \pi]$ Equation (2.6) implies that $x = y$ or $y = \pi - x$, and the second equation implies that $y = z$ or $y = \pi - z$.
Subcase 1: If

$$\begin{cases} x = y, \\ y = z. \end{cases}$$

Then, $x = y = z$, and the fourth equation (2.3) gives $x = y = z = \frac{\pi}{3}$ and $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ is a critical point with $\lambda \neq 0$.
Subcase 2: If

$$\begin{cases} y = x, \\ y = \pi - z. \end{cases} \tag{2.7}$$

From equations (2.7) and (2.3), we obtain $x = 0$ so $y = 0$, $z = \pi$, thus $(0, 0, \pi)$ is a critical point with $\lambda = 0$.
Subcase 3: If

$$\begin{cases} y = z, \\ y = \pi - x. \end{cases}$$

The solution is $(\pi, 0, 0)$ with $\lambda = 0$.
Subcase 4: If

$$\begin{cases} y = \pi - z, \\ y = \pi - x. \end{cases}$$

Then, $x = 0$, $z = 0$, $y = \pi$. The solution is $(0, \pi, 0)$ with $\lambda = 0$.
Case 2:

$$\begin{cases} \sin y - \sin x = 0, \\ \sin^2 y + \sin^2 z + \sin y \sin z - 1 = 0. \end{cases} \tag{2.8}$$

Equation (2.8) implies $y = \pi - x$ or $y = x$. If $y = \pi - x$, then $z = 0$. Substituting this into the second equation (2.8) implies that $\sin^2 y = 1$, so $y = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$. Thus $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ is a critical point with $\lambda = 0$. If $y = x$ then $z = \pi - 2x$. Taking $z = \pi - 2x$ in equation (2.8), we see that

$$\sin^2 x + \sin^2 2x + \sin x \sin 2x - 1 = 0.$$

Using the identity $\sin 2x = 2 \sin x \cos x$ we get

$$\sin^2 x + 4 \sin^2 x \cos^2 x + 2 \sin^2 x \cos x - 1 = 0,$$

from $\sin^2 x = 1 - \cos^2 x$ we obtain that

$$\cos x(4 \cos^3 x + 2 \cos^2 x - 3 \cos x - 2) = 0.$$

If $\cos x = 0$ then $x = \frac{\pi}{2}$, $y = \frac{\pi}{2}$, $z = 0$, the critical point is $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$. If

$$4 \cos^3 x + 2 \cos^2 x - 3 \cos x - 2 = 0. \tag{2.9}$$

By Theorem (1.3), the real root of cubic equation (2.9) is

$$\cos x = S + T - \frac{a_2}{3a_3} \simeq 0.91559,$$

where

$$S+T=\sqrt[3]{R+\sqrt{Q^3+R^2}}+\sqrt[3]{R-\sqrt{Q^3+R^2}}$$
$$\simeq 1.082255,$$

and

$$R=\frac{9a_1a_2a_3-27a_0a_3^2-2a_2^3}{54a_3^3}=\frac{79}{432},$$
$$Q=\frac{3a_1a_3-a_2^2}{9a_3^2}=-\frac{5}{18}.$$

Here, $a_3=4$, $a_2=2$, $a_1=-3$, and $a_0=-2$. Thus,

$$x=\arccos(0.91559)\simeq\frac{23.71\pi}{180},$$

that gives $y=\frac{23.71\pi}{180}$, $z=\frac{132.58\pi}{180}$. The critical point is $(23.71^\circ, 23.71^\circ, 132.58^\circ)$ with $\lambda\neq 0$.
Case 3:

$$\begin{cases}\sin y-\sin z=0,\\ \sin^2 y+\sin^2 x+\sin y\sin x-1=0.\end{cases}$$

Similar to Case 2 the critical points are $(0,\frac{\pi}{2},\frac{\pi}{2})$ and $(132.58^\circ, 23.71^\circ, 23.71^\circ)$.
Case 4:

$$\begin{cases}\sin^2 y+\sin^2 x+\sin y\sin x-1=0,\\ \sin^2 y+\sin^2 z+\sin y\sin z-1=0.\end{cases}\tag{2.10}$$

Multiplying the second equation (2.10) by -1 and adding it to the first equation, we obtain

$$\sin^2 x-\sin^2 z=\sin y(\sin z-\sin x).$$

Hence

$$(\sin x-\sin z)(\sin x+\sin y+\sin z)=0.$$

If $\sin x-\sin z=0$, then we obtain $x=z$ or $x=\pi-z$. If $x=z$ then $y=\pi-2x$. Taking these in equation (2.10) gives $\sin^2 2x+\sin^2 x+\sin x\sin 2x-1=0$. Similar to Case 2, $(\frac{\pi}{2},0,\frac{\pi}{2})$ and $(23.71^\circ, 132.58^\circ, 23.71^\circ)$ are the critical points. If $x=\pi-z$, then $y=0$. Taking this in equation (2.10), implies that $\sin^2 x=1$, $\sin^2 z=1$, that is, $x=z=\frac{\pi}{2}$, and $(\frac{\pi}{2},0,\frac{\pi}{2})$ is the critical point.
If $\sin x+\sin y+\sin z=0$, from the equation (2.3), the critical points are $(0,\pi,0)$, $(\pi,0,0)$ and $(0,0,\pi)$. By calculating the values of f at its critical points, we see that

$$f(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})=\frac{3}{8},$$
$$f(\pi,0,0)=f(0,\pi,0)=f(0,0,\pi)=1,$$
$$f(\frac{\pi}{2},\frac{\pi}{2},0)=f(\frac{\pi}{2},0,\frac{\pi}{2})=f(0,\frac{\pi}{2},\frac{\pi}{2})=1.$$

Let $x^*=23.71^\circ$, then

$$f(23.71^\circ, 23.71^\circ, 132.58^\circ))=2\cos^3 x^*-\cos^3(2x^*)$$
$$=2\cos^3 x^*-(2\cos^2 x^*-1)^3\simeq 1.225,$$

$$f(23.71^\circ, 132.58^\circ, 23.71^\circ)=f(132.58^\circ, 23.71^\circ, 23.71^\circ)\simeq 1.225.$$

We find the values of f on the boundary of set E. On segment $x+y=\pi$ we have $z=0$, thus

$$\cos^3 x+\cos^3 y+\cos^3 z=\cos^3 x+\cos^3(\pi-x)+\cos^3 z=1,$$

for the sides $y+z=\pi$ and $x+z=\pi$, the values of f were equal to 1. Therefore, the maximum and minimum values of the function f in the region triangular E are 1.225 and $\frac{3}{8}$. The required result for inequality (2.2) was obtained. □

Corollary 2.2. *Let x,y,z be the angles of a non-degenerate triangle. Then*

$$\frac{3}{8}\leq \cos^3 x+\cos^3 y+\cos^3 z\leq 1.225,$$

where equality holds on the left-hand side if and only if the triangle is equilateral, and on the right-hand side if and only if $x=y=23.71^\circ, z=132.58^\circ$.

Proof. Let f be the function defined in Theorem (2.1): Consider the restriction of f on E° (interior of E). As we saw in Theorem (2.1), the extreme values of the function $f \mid E^\circ$ occur at the interior points $(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})$ and $(23.71^\circ, 23.71^\circ, 132.58^\circ)$ of the set E°. This completes this proof. □

Corollary 2.3. *Let E be the set defined in Theorem (2.1); and $E_i\subset E, i=1,2,3,4$ are defined by*

$$E_1=\{(x,y,z): x+y+z=\pi,\quad x,y\in[\frac{\pi}{4},\frac{\pi}{2}], z\in[0,\frac{\pi}{2}]\},$$

$$E_2=\{(x,y,z): x+y+z=\pi,\quad x,z\in[\frac{\pi}{4},\frac{\pi}{2}], y\in[0,\frac{\pi}{4}]\},$$

$$E_3=\{(x,y,z): x+y+z=\pi,\quad x\in[\frac{\pi}{4},\frac{\pi}{2}], y\in[0,\frac{\pi}{4}], z\in[\frac{\pi}{2},\frac{3\pi}{4}]\},$$

$$E_4=\{(x,y,z): x+y+z=\pi,\quad x,y\in[0,\frac{\pi}{4}], z\in[\frac{\pi}{2},\pi]\}.$$

Then we have

$$\frac{3}{8}\leq \cos^3 x+\cos^3 y+\cos^3 z\leq 1, \tag{2.11}$$

where $(x,y,z)\in E_1\cup E_2$.

$$\frac{\sqrt{2}}{2}\leq \cos^3 x+\cos^3 y+\cos^3 z\leq 1.225, \tag{2.12}$$

where $(x,y,z)\in E_3\cup E_4$.

Proof. The boundary points of $E_1\cup E_2$ are $(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2})$, $(\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{4})$, $(\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{4})$, $(\frac{\pi}{2},\frac{\pi}{2},0)$ and $(\frac{\pi}{2},0,\frac{\pi}{2})$. Let f be the function defined in Theorem (2.1). The values of the function $f \mid E_1\cup E_2$ at the boundary points of $E_1\cup E_2$ are as follows:

$$f(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2})=f(\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{4})=f(\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{4})=\frac{\sqrt{2}}{2},$$

$$f(\frac{\pi}{2},\frac{\pi}{2},0)=f(\frac{\pi}{2},0,\frac{\pi}{2})=1.$$

By Theorem (2.1), at the critical points $(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})\in E_1$, the value of f is equal to $\frac{3}{8}$. This proves the inequality (2.11). The boundary points of $E_3\cup E_4$ are $(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2})$, $(\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{4})$, $(0,\frac{\pi}{2},\frac{\pi}{2})$,

$(\frac{\pi}{2}, 0, \frac{\pi}{2})$, $(\frac{\pi}{4}, 0, \frac{3\pi}{4})$, $(0, \frac{\pi}{4}, \frac{3\pi}{4})$ and $(0, 0, \pi)$. The values of function $f \mid E_3 \cup E_4$ at these points are as follows:

$$\begin{aligned} f(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}) = f(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}) &= \frac{\sqrt{2}}{2}, \\ f(\frac{\pi}{2}, 0, \frac{\pi}{2}) = f(0, \frac{\pi}{2}, \frac{\pi}{2}) &= 1, \\ f(\frac{\pi}{4}, 0, \frac{3\pi}{4}) = f(0, \frac{\pi}{4}, \frac{3\pi}{4}) &= 1, \\ f(0, 0, \pi) &= 1. \end{aligned}$$

The value of f at the critical points $(23.71°, 23.71°, 132.58°) \in E_4$ is equal to 1.225. Thus, the result in inequality (2.12) satisfies. □

Corollary 2.4. *Let x, y, z be the angles of a non-degenerate triangle. Then*

$$\frac{3}{8} \leq \cos^3 x + \cos^3 y + \cos^3 z < 1, \tag{2.13}$$

for acute triangles.

$$\frac{\sqrt{2}}{2} < \cos^3 x + \cos^3 y + \cos^3 z \leq 1.225, \tag{2.14}$$

for obtuse triangles. The best upper bound of inequality (2.13) is 1, and the best lower bound of inequality (2.14) is $\frac{\sqrt{2}}{2}$.

Proof. Case 1: Two angles greater than or equal to $\frac{\pi}{4}$ and one less than to $\frac{\pi}{2}$, that is, $x, y \in [\frac{\pi}{4}, \frac{\pi}{2})$ and $z \in (0, \frac{\pi}{2})$, thus $(x, y, z) \in E_1$.
Case 2: Two angles greater than or equal to $\frac{\pi}{4}$ and one less than or equal to $\frac{\pi}{4}$, that is, $x, z \in [\frac{\pi}{4}, \frac{\pi}{2})$ and $y \in (0, \frac{\pi}{4}]$, therefore $(x, y, z) \in E_2$. Moreover

$$\lim_{(x,y,z)\to(\frac{\pi}{2},\frac{\pi}{2},0)} (\cos^3 x + \cos^3 y + \cos^3 z) = 1.$$

Thus, inequality (2.13) follows from inequality (2.11): Similar to Cases 1 and 2, the required results in (2.14) are derived from (2.12). □

Corollary 2.5. *Let x, y, z be the angles of an acute triangle. Then*

$$\cos^n x + \cos^n y + \cos^n z < 1,$$

for every $n \geq 2$.

Proof. For $n = 2$ and $n = 3$, the desired inequality follows from (1.3) and (2.13). If $t \in (0, \frac{\pi}{2})$, then $0 < \cos t < 1$. It follows that $\cos^n t < \cos^3 t$, for $n \geq 4$, this gives us

$$\cos^n x + \cos^n y + \cos^n z < \cos^3 x + \cos^3 y + \cos^3 z,$$

for every natural number $n \geq 4$ and every acute triangle with angles x, y, z. Also

$$\lim_{(x,y,z)\to(\frac{\pi}{2},\frac{\pi}{2},0)} (\cos^n x + \cos^n y + \cos^n z) = 1.$$

The result follows from the right hand-side of inequality (2.13). □

The next theorem gives the minimum value of the $\cos^n x + \cos^n y + \cos^n z$.

Theorem 2.6. *Let x, y, z be the angles of an acute triangle . Then for each $n \geq 2$, one has*

$$\cos^n x + \cos^n y + \cos^n z \geq \frac{3}{2^n},$$

the equality holds if and only if the triangle is equilateral.

Proof. For $n = 2$ and $n = 3$, the desired inequality follows from (1.3), and (2.13).

We prove the result for all integers $n \geq 2$ by strong induction, starting with the base cases $n = 2$ and $n = 3$.

Since $x, y, z \in (0, \frac{\pi}{2})$, we have, for every integer $n \geq 2$,

$$\left(\cos^n x - \frac{1}{2^n}\right)^2 + \left(\cos^n y - \frac{1}{2^n}\right)^2 + \left(\cos^n z - \frac{1}{2^n}\right)^2 \geq 0, \tag{2.15}$$

and

$$\begin{aligned} \cos x \left(\cos^n x - \frac{1}{2^n}\right)^2 + \cos y \left(\cos^n y - \frac{1}{2^n}\right)^2 \\ + \cos z \left(\cos^n z - \frac{1}{2^n}\right)^2 \geq 0. \end{aligned} \tag{2.16}$$

Equality holds in either inequality if and only if

$$x = y = z = \frac{\pi}{3}.$$

Inductive step.

Suppose that the desired inequality holds for all integers m satisfying $2 \leq m < n$, where $n \geq 4$. We consider two cases.

Case 1: n is even.

Let $n = 2k$, where $k \geq 2$. By the induction hypothesis,

$$\cos^k x + \cos^k y + \cos^k z \geq \frac{3}{2^k}.$$

Taking $n = k$ in (2.15) and expanding, we obtain

$$\begin{aligned} \cos^{2k} x + \cos^{2k} y + \cos^{2k} z \geq \frac{2}{2^k}(\cos^k x + \cos^k y + \cos^k z) \\ - \frac{3}{2^{2k}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \cos^{2k} x + \cos^{2k} y + \cos^{2k} z &\geq \frac{2}{2^k} \cdot \frac{3}{2^k} - \frac{3}{2^{2k}} \\ &= \frac{3}{2^{2k}}. \end{aligned}$$

Thus, the result holds for every even integer $n \geq 4$.

Case 2: n is odd.

Let $n = 2k + 1$, where $k \geq 2$. By the induction hypothesis,

$$\cos^{k+1} x + \cos^{k+1} y + \cos^{k+1} z \geq \frac{3}{2^{k+1}}.$$

Taking $n = k$ in (2.16) and expanding, we obtain

$$\begin{aligned}&\cos^{2k+1} x + \cos^{2k+1} y + \cos^{2k+1} z\\ &\geq \frac{2}{2^k}(\cos^{k+1} x + \cos^{k+1} y + \cos^{k+1} z)\\ &\quad - \frac{1}{2^{2k}}(\cos x + \cos y + \cos z).\end{aligned}$$

On the other hand, by (1.1), we have

$$\cos x + \cos y + \cos z \leq \frac{3}{2}.$$

Therefore,

$$\begin{aligned}&\cos^{2k+1} x + \cos^{2k+1} y + \cos^{2k+1} z\\ &\geq \frac{2}{2^k} \cdot \frac{3}{2^{k+1}} - \frac{1}{2^{2k}} \cdot \frac{3}{2}\\ &= \frac{3}{2^{2k+1}}.\end{aligned}$$

Thus, the result also holds for every odd integer $n \geq 5$.

Combining the two cases, we conclude by strong induction that

$$\cos^n x + \cos^n y + \cos^n z \geq \frac{3}{2^n}$$

for every integer $n \geq 2$.

Finally, since equality in (2.15) or (2.16) holds if and only if

$$\cos x = \cos y = \cos z = \frac{1}{2},$$

and $x, y, z \in (0, \frac{\pi}{2})$, equality holds if and only if

$$x = y = z = \frac{\pi}{3}.$$

This completes the proof. □

References

[1] R. G. Bartel, *The Elements of Real Analysis*, John Wiley and Sons, 1976 (Second Edition).

[2] O. Bottema, R. Z. Djordjević, R. P. Janić, D. S. Mitrinović, and P. M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

[3] A. Clark, *Elements of Abstract Algebra*, Dover, New York, 1971.

[4] Z. Cvetkovski, *Inequalities; Theorem, Techniques and Selected Problems*, Springer, New York-Dordrecht-Heidelberg-London, 2012.

[5] D. S. Mitrinović, J. Pečarić and V.Volence, *Recent Advances in Geometric Inequality*, Kluwer, Dordrecht, 1989.

[6] H. Sedrakyan and N. Sedrakyan, *Geometric Inequalities*, Springer, 2017.

[7] M. R. Spiegel, S. Lipschutz and J. Liu, *Mathematical Handbook of Formulas and Tables* McGraw-Hill, New York, 2009 (third edition).

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