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Research Paper

FIXED POINT RESULTS IN INCOMPLETE METRIC SPACES FOR ORTHOGONALITY-PRESERVING GENERALIZED MEIR-KEELER α-CONTRACTIVE MAPPINGS

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Abstract. In this paper, using the orthogonality concept in an orthogonal metric space we establish the existence and uniqueness of common fixed points for a pair of self-mappings f and g under suitable orthogonality-preserving and orthogonality-reflecting assumptions. We show that if f is a continuous orthogonality reflecting self-map of an O-B-complete metric space $(X, \perp, d)$ (not necessarily a complete metric space), then every orthogonally generalized α-f-Meir–Keeler contractive map admits a unique common fixed point with f. Finally, as a corollary, we show that the classical generalized α-f-Meir–Keeler contractive condition can be recovered as a particular case of an orthogonal framework.

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1. Introduction

The celebrated Banach contraction principle [12] constitutes one of the cornerstones of nonlinear analysis and fixed point theory. Since its appearance, numerous generalizations have been obtained by weakening the contractive condition in different directions. Among these extensions, the Meir–Keeler contraction introduced in [9] plays a fundamental role. This type of contraction replaces the linear contraction constant by a more flexible ε–δ condition and allows a wider class of nonlinear operators.

In recent years, the concept of α-admissibility and α-contractive type mappings, initiated in [14], has provided a powerful framework for extending classical fixed point results.

For example, fixed and common fixed point theorems for Meir–Keeler α-contractive mappings were established in [1, 2].

On the other hand, orthogonal metric spaces, introduced in [3] and further investigated in [8, 4], offer an alternative structure in which convergence and contractive behavior are studied only along orthogonally related elements. This approach allows one to replace global metric assumptions by directional or relational ones, leading to new fixed point principles in situations where classical completeness is not available in the usual sense.

Motivated by these developments, first we introduce the notions of orthogonally Meir–Keeler α-f-contractive mappings and orthogonally generalized Meir–Keeler α-f-contractive

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mappings. Then we focus on combining the ideas of Meir–Keeler type contractions, α-admissibility and orthogonal structures in order to obtain new common fixed point results for pairs of mappings.

In 1969, Meir and Keeler [9] presented a generalization of the Banach contraction mapping in a complete metric space (X, d) for mappings $T : X \to X$ satisfying the following condition, known as the Meir–Keeler contractive condition: for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for $x, y \in X$,

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \Rightarrow d(Tx, Ty) < \epsilon,$$

and they presented a fixed point theorem in complete metric spaces as follows.

Theorem 1.1. [9] *Let (X, d) be a complete metric space and let $T : X \to X$ be an operator. Suppose that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $x, y \in X$,*

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \Rightarrow d(Tx, Ty) < \epsilon.$$

Then T admits a unique fixed point $z \in X$, and for any $x \in X$, the sequence $\{T^n(x)\}$ converges to z.

Later, in 2012, Samet et al. [14] introduced Meir–Keeler α-contractive mappings and in 2013, a generalization of fixed point theorems for α-contractive mappings was presented in [1] as follows.

Let (X, d) be a metric space, $f : X \to X$ a self mapping, and $\alpha : X \times X \to [0, \infty)$ a given function. Then f is called *Meir–Keeler α-contractive* if, for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \;\Rightarrow\; \alpha(x, y)\, d(fx, fy) < \epsilon.$$

The mapping f is called *generalized Meir–Keeler α-contractive* if

$$\epsilon \leq M_f(x, y) < \epsilon + \delta(\epsilon) \;\Rightarrow\; \alpha(x, y)\, d(fx, fy) < \epsilon,$$

where

$$M_f(x, y) = \max\{d(x, y),\, d(x, fx),\, d(y, fy),\, \tfrac{1}{2}(d(x, fy) + d(y, fx))\}.$$

For two self mappings $f, g : X \to X$, the pair (f, g) is called *generalized Meir–Keeler α-contractive* if, for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq M_{(f,g)}(x, y) < \epsilon + \delta(\epsilon) \;\Rightarrow\; \alpha(x, y)\, d(fx, gy) < \epsilon,$$

where

$$M_{(f,g)}(x, y) = \max\{d(x, y),\, d(x, fx),\, d(y, gy),\, \tfrac{1}{2}(d(x, gy) + d(y, fx))\}.$$

Let (X, d) be a metric space, f be a continuous self-map of X and $x_0 \in X$ and $C_f = \{g : X \to X; fg = gf\ ,\ gX \subseteq fX\}$. If there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

$$fx_{n+1} = gx_n\ ;\ n = 0, 1, 2 \cdots$$

then the set $\{fx_n\}_{n=1}^{\infty}$ is called the f-iteration of x_0 under g.

Theorem 1.2. [1] *Let f and g be self-mappings of a complete metric space (X, d) and $g \in C_f$. Assume $\alpha(x_n, x_m) \geq 1$ for all $m \geq n$. If g is a generalized Meir-Keeler α-f-contractive map such that α satisfies the condition $(f-H)$: If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_m) \geq 1$ for all $m > n$ and $fx_n \to z$, then $\alpha(fx_n, z) \geq 1$ and $\alpha(fx_n, fz) \geq 1$ for all n. Then f and g have a unique common fixed point.*

In 2017, Eshaghi et al. [8] introduced the notion of orthogonal sets and extended Banach's fixed point theorem in orthogonal metric spaces. The main result of [8] is the following theorem.

Theorem 1.3. *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily a complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp$-continuous, $\perp$-contraction with Lipschitz constant λ, and $\perp$-preserving. Then, f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is,*

$$\lim_{n\to\infty} f^n(x) = x^* \quad \text{for all } x \in X.$$

Now, in this paper, we focus on orthogonal metric spaces and we show that some known fixed and common fixed point results follow immediately from our approach. We establish the existence and uniqueness of common fixed points for a pair of self-mappings f and $g \in C_f$ of an orthogonal B-complete metric space such $(X, \perp, d)$ (not necessarily B-complete or complete) under suitable orthogonality-preserving and orthogonality-reflecting assumptions. In our main result (Theorem 3.9), we prove that for each continuous orthogonality reflecting self-map f of an O-B-complete metric space (X, d) under special assumptions, every continuous orthogonally generalized α-f-Meir–Keeler contractive map admits a unique common fixed point with f.

At the end, we show that the classical generalized α-f-Meir–Keeler contractive condition can be recovered as a particular case of our orthogonal framework. For example, we derive Theorem 3.11, which generalizes Theorem 16 of [1].

2. Preliminaries and Notations

In this section, for the reader's convenience, we provide a summary of the mathematical notations and definitions which will be used in the sequel. First, we recall the definitions which were introduced in [14, 1, 8].

Definition 2.1. Let $f : X \to X$ be a self mapping of a set X and $\alpha : X \times X \to [0, \infty)$ be a mapping. Then the mapping f is called α-admissible if

$$x, y \in X,\ \alpha(x, y) \geq 1 \ \Rightarrow\ \alpha(fx, fy) \geq 1,$$

Definition 2.2. Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition:

$$\exists x_0 \in X \text{ such that } (\forall y \in X,\ y \perp x_0) \text{ or } (\forall y \in X,\ x_0 \perp y),$$

then $\perp$ is called an *orthogonality relation*, the pair $(X, \perp)$ is called an *orthogonal set* (briefly, O-set), and x_0 is called an *orthogonal element*.

Definition 2.3. Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n=1}^{\infty}$ is called an *orthogonal sequence* (briefly, O-sequence) if

$$\forall n,\ x_n \perp x_{n+1} \quad \text{or} \quad \forall n,\ x_{n+1} \perp x_n.$$

If

$$\forall n, k,\ x_n \perp x_{n+k} \quad \text{or} \quad \forall n, k,\ x_{n+k} \perp x_n,$$

then $\{x_n\}$ is called a *strongly orthogonal sequence* (briefly, SO-sequence).

Definition 2.4. Let the triplet $(X, \perp, d)$ be an orthogonal set with metric d, which is called an *orthogonal metric space*. Then X is called *orthogonally complete* (briefly, O-complete) if every Cauchy O-sequence in X is convergent. If every Cauchy SO-sequence in X is convergent, then X is called *strongly orthogonally complete* (briefly, SO-complete).

It is easy to see that every complete metric space is both O-complete and SO-complete. The following example shows that the converse is not true.

Example 2.5. Let $X = (0, \infty)$ with the Euclidean metric, and define the orthogonal relation on X by

$$x \perp y \;\Leftrightarrow\; xy \in \{x, y\}, \quad x, y \in X.$$

Suppose that $\{x_n\}$ is a Cauchy SO-sequence in X. By definition of $\perp$, we have $x_n = 1$ for all $n \geq 2$. Thus, $\{x_n\}$ converges to the constant $1 \in X$, and hence X is an SO-complete metric space. However, it is clear that X is not a complete metric space.

Definition 2.6. Let $(X, \perp, d)$ be an orthogonal metric space and B be a subset of $(X, \perp, d)$. A mapping $f : B \to B$ is called:

i) *Strongly orthogonal continuous* (briefly, SO-continuous) at $a \in B$ if for each SO-sequence $\{a_n\}$ in B with $a_n \to a$, we have $fa_n \to fa$. Moreover, f is *SO-continuous on B* (briefly, SO-B-continuous) if it is SO-continuous at every $a \in B$. Additionally, f is called *orthogonally B-continuous* (briefly, O-B-continuous) if it is orthogonally continuous at each $a \in B$.

ii) *$\perp$-preserving* if $x, y \in B$ and $x \perp y$ imply that $fx \perp fy$.

Example 2.7. Let $X = (0, \infty)$ with the Euclidean metric, and let $\perp$ be the orthogonal relation as in Example 2.5. Define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ \frac{1}{x} & x \in \mathbb{Q}^c. \end{cases}$$

One can easily check that f is $\perp$-preserving on X. Moreover, by definition of $\perp$, it is clear that if $\{x_n\}$ is an SO-convergent sequence in X, then $x_n = 1$ for all $n \geq 2$. Thus, if $x_n \to 1$, we have $fx_n \to f1$. Therefore, f is SO-continuous. However, f is not continuous in the usual sense. For more examples and details, see [3].

3. Main Results

Definition 3.1. Let f be a self-map of a metric space (X, d) and $g \in C_f$ and let $\alpha : X \times X \to [0, \infty)$ be a mapping. Then

i) g is called a Meir-Keeler α-f-contractive map if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$

$$\epsilon \leq d(fx, fy) < \epsilon + \delta \text{ implies } \alpha(x, y)d(gx, gy) < \epsilon,$$

ii) g is called a generalized Meir-Keeler α-f-contractive map if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$

$$\epsilon \leq M_g(f)(x, y) < \epsilon + \delta(\epsilon) \Rightarrow \alpha(x, y)d(gx, gy) < \epsilon,$$

where

$$M_g(f)(x, y) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \tfrac{1}{2}\big(d(fx, gy) + d(fy, gx)\big)\}.$$

Definition 3.2. Let $(X, \perp, d)$ be an orthogonal metric space, and let $f : X \to X$ be a self-map and $g \in C_f$. Let $\alpha : X \times X \to [0, \infty)$ be a mapping. Then

i) g is called a *orthogonally Meir-Keeler α-f-contractive map* if, for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for each $x \neq y$, $x \perp y$,

$$\epsilon \leq d(fx, fy) < \epsilon + \delta(\epsilon) \Rightarrow \alpha(x, y)d(gx, gy) < \epsilon.$$

ii) g is called *orthogonally generalized Meir-Keeler α-f-contractive map* if, given any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for each $x \neq y$, $x \perp y$,

$$\epsilon \leq M_g(f)(x,y) < \epsilon + \delta(\epsilon) \Rightarrow \alpha(x,y)d(gx,gy) < \epsilon.$$

where

$$M_g(f)(x,y) = \max\{d(fx,fy),\ d(fx,gx),\ d(fy,gy),\ \tfrac{1}{2}\big(d(fx,gy) + d(fy,gx)\big)\}.$$

Remark 3.3. It is easy to show that for each orthogonally generalized Meir-Keeler α-f-contractive, we have

$$\alpha(x,y)d(gx,gy) < M_g(f)(x,y)$$

for each $x \neq y$, $x \perp y$, .

Definition 3.4. Let X be a non-empty set, and let f, g be self-mappings on X. A point $y \in X$ is called a *point of coincidence* of f and g if there exists $x \in X$ such that

$$y = fx = gx.$$

Definition 3.5. Let $(X, \perp, d)$ be an orthogonal metric space, $B \subseteq X$, $x_0 \in B$, and let $f, g : B \to B$ be self-mappings. If there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in B such that

$$fx_{n+1} = gx_n\ ;\ n = 0, 1, 2 \cdots$$

then the set $\{fx_n\}_{n=1}^{\infty}$ is called the f-iteration of x_0 under g. Moreover,

i) if every O-Cauchy sequence in B is convergent, then $(X, \perp, d)$ is called *orthogonally B-complete* (briefly, O-B-complete).
ii) if every SO-Cauchy sequence in B is convergent, then $(X, \perp, d)$ is called *strongly orthogonal B-complete* (briefly, SO-B-complete).
iii) if for each $x, y \in B$ with $x \perp y$, we have $fx \perp fy$ then f is called $\perp$-*preserving* on B.
iv) if for each $x, y \in B$ with $fx \perp fy$, we have $x \perp y$ then f is called orthogonality reflecting on B.

It is obvious that every B-continuous map f is both SO-B-continuous and O-B-continuous. Similarly, every B-complete metric space is both O-B-complete and SO-B-complete. However, the converse is not always true. To illustrate these facts, we present the following examples.

Example 3.6. Let $X = (0, 1)$ with the usual metric $d(x, y) = |x - y|$. Define a binary relation $\perp$ on X by

$$x \perp y \quad \Longleftrightarrow \quad x, y \geq \tfrac{1}{2}.$$

Let $B = [1/2, 1) \subset X$. Define $f, g : B \to B$ by

$$fx = \frac{x}{2} + \frac{1}{4}, \qquad gx = \frac{x}{4} + \frac{3}{8}.$$

Fix $x_0 \in B$ and define $\{x_n\}$ by

$$fx_{n+1} = gx_n, \qquad n \geq 0.$$

By computation

$$\frac{x_{n+1}}{2} + \frac{1}{4} = \frac{x_n}{4} + \frac{3}{8},$$

we have

$$x_{n+1} - \frac{1}{2} = \frac{1}{2}(x_n - \frac{1}{2}).$$

Hence

$$x_n = \frac{1}{2} + \frac{x_0 - \frac{1}{2}}{2^n}.$$

Clearly, $x_n \in B$ for all n and moreover, $x_n \to \frac{1}{2}$. So the f-iteration $\{fx_n\}$ is well-defined and infinite.

Also, f is $\perp$-preserving and orthogonality reflecting on B. Because for each $x, y \in B$ with $x \perp y$ we have $x, y \geq 1/2$. Hence $fx, fy \geq 1/2$, so $fx \perp fy$, i.e. f is $\perp$-preserving. Also, if $fx \perp fy$, then $fx, fy \geq 1/2$. Hence $x, y \geq f^{-1}(1/2) = 1/2$, so $x \perp y$. Similarly, g also is $\perp$-preserving and orthogonality reflecting.

In $B = [1/2, 1)$, all elements are mutually orthogonal. Hence O-Cauchy coincides with usual Cauchy. Since B is closed in X and the limit of every Cauchy sequence in B lies in $B \subset X$, then $(X, \perp, d)$ is O-B-complete.
Now, we show that every SO-B-complete metric space is not necessarily O-B-complete.

Example 3.7. Let $X = (0, \infty)$ with the Euclidean metric and the or thogonal relation as.

$$x \perp y \iff xy \in \{x, y\}.$$

Let $\{x_n\}$ be a Cauchy SO-sequence in X, the definition $\perp$ follows that $x_n = 1$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ converges to constant $1 \in X$. Now, consider the sequence

$$x_n = \begin{cases} 1 & n = 2k, \text{ for some } k \in \mathbb{Z} \\ k & n = 2k+1, \text{ for some } k \in \mathbb{Z} \end{cases}$$

is an O-sequence that is not convergent to any element of X

Theorem 3.8. *Let $(X, \perp, d)$ be an orthogonal metric space, and let B be an orthogonal closed (briefly O-closed) nonempty subset of X. Assume $x_0 \in B$ satisfies $x_0 \perp x$ for every $x \in B$. Suppose the mapping g and f be self-mappings on B, such that g is $\perp$-preserving and f is orthogonality reflecting on B. Define inductively a sequence $\{fx_n\}_{n=1}^{\infty} \subseteq B$ by*

$$fx_{n+1} = gx_n \qquad n = 0, 1, 2, \cdots .$$

Then

i) *the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{fx_n\}$ are SO-sequences, i.e.*

$$x_n \perp x_{n+k} \text{ and } fx_n \perp fx_{n+k} \qquad \text{for all } \ n, k \geq 1.$$

ii) *If $z \in B$ is a point of coincidence of f, g, then*

$$fx_n \perp fz \quad \text{for all } \ n \geq 1.$$

and if $z \in B$ is a common fixed point of f, g (that is $fz = gz = z$)), then

$$fx_n \perp z \quad \text{ and } fz \perp z \text{ for all } \ n \geq 1.$$

Proof. Base case: by hypothesis, for arbitrary $k \geq 1$ we have $x_k \in B$ and $x_0 \perp x_k$, hence $\perp$-preservation of g gives $gx_0 \perp gx_k$. Applying the assumption and orthogonality reflecting of f we obtain

$$fx_1 \perp fx_{1+k} \text{ and } x_1 \perp x_{1+k}.$$

By a similar way we can write

$$fx_2 \perp fx_{2+k} \text{ and } x_2 \perp x_{2+k}.$$

Inductive step: assume for some $n \geq 3$ that $fx_{n-1} \perp fx_{n-1+k}$. Since f is orthogonality reflecting, then $x_{n-1} \perp x_{n-1+k}$. Using the $\perp$-preservation of g we obtain $gx_{n-1} \perp gx_{n-1+k}$. So, we have

$$fx_n \perp fx_{n+k} \text{ and } x_k \perp x_{n+k}.$$

This completes the induction.

(ii) Let $z \in B$ satisfy $fz = gz$. Since by hypothesis $x_0 \perp z$ and g is $\perp$-preserving, we get

$$gx_0 \perp gz,$$

i.e. $fx_1 \perp fz$. Applying the orthogonality reflecting of f gives $x_1 \perp z$, whence $gx_1 \perp gz$ and so $fx_2 \perp fz$. Using the same inductive reasoning as in (i) (together with orthogonality reflecting of f at each step) we obtain

$$fx_n \perp fz \quad \text{for all} \quad n \geq 1.$$

It is clear that if $fz = gz = z$ then

$$fx_n \perp z \quad \text{for all} \quad n \geq 1.$$

□

Theorem 3.9. *Let $\perp$ be a binary relation on X, and let B be an O-closed subset of $(X, \perp, d)$. Suppose $(X, \perp, d)$ is O-B-complete (not necessarily B-complete), and let $x_0 \in B$ be an orthogonal element with $x_0 \perp x$ for every $x \in B$. Let f and g be self-mappingsg on B, and the mapping $g \in C_f$ such that*

(i) *g and f are O-continuous, g is $\perp$-preserving and f is orthogonality reflecting;*
(ii) *g is orthogonally generalized f-Meir-Keeler contractive map.*

Assume there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in B satisfying

$$fx_{n+1} = gx_n \ ; \ n = 0, 1, 2 \cdots$$

Then:

(1) *$d(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$;*
(2) *f, g have a unique common fixed point.*

Proof. (1) Set $d_n := d(fx_n, fx_{n+1})$ for $n \geq 1$. By Theorem 3.8, $\{x_n\}_{n=1}^{\infty}$ and $\{fx_n\}_{n=1}^{\infty}$ are O-sequences, hence the quantities appearing in the definition of $M_g(f)(x_n, x_{n+1})$ can be written in terms of the d_k's.

First, we show that d_n is nonincreasing and converges to zero. Using the definition of $M_g(f)$ and $fx_{n+1} = gx_n$ and the fact $\{x_n\}_{n=1}^{\infty}$ is O-sequence, one obtains for each n

$$\begin{aligned} d_n &= d(fx_n, fx_{n+1}) = d(gx_{n-1}, gx_n) < M_g(f)(x_{n-1}, x_n) \\ &= \max\Big\{d(fx_n, fx_{n-1}), d(fx_n, gx_n), d(fx_{n-1}, gx_{n-1}), \\ &\frac{1}{2}\big(d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})\big)\Big\} \\ &= \max\Big\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n-1}), d(fx_{n-1}, fx_n), \\ &\frac{1}{2}\big(d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)\big)\Big\} \\ &= \max\Big\{d(fx_n, fx_{n-1}), \frac{1}{2}\big(d(fx_{n-1}, fx_{n+1}) + 0)\big)\Big\} \\ &\leq \max\Big\{d(fx_{n-1}, fx_n), \frac{1}{2}\big(d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}))\big)\Big\}. \end{aligned}$$

Hence $d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n)$. Thus $\{d_n\}$ is monotone decreasing (hence convergent); let $r := \lim_{n\to\infty} d_n \geq 0$.

Assume to the contrary that $r > 0$. By the orthogonally generalized f-Meir-Keeler condition (ii) there exists $\delta > 0$ such that

$$\epsilon \leq M_g(f)(x, y) < \epsilon + \delta \quad \Rightarrow \quad d(gx, gy) < \epsilon \tag{3.1}$$

for each $\epsilon > 0$ and for orthogonal x, y. Take $\epsilon := r$ and pick N with $d_N \in [r, r + \delta)$. Since $M_g(f)(x_N, x_{N+1}) = d_N \in [r, r+\delta)$ and $x_N \perp x_{N+1}$, the orthogonally generalized f-Meir-Keeler property of g gives

$$g(x_N, x_{N+1}) < r,$$

thus

$$d_{N+1} = d(fx_{N+1}, fx_{N+2}) = g(x_N, x_{N+1}) < r,$$

contradicting $\lim_{n\to\infty} d_n = r$. Hence $r = 0$, i.e. $d_n \to 0$.

(2) Second, we show the existence of a common fixed point. If some $d_k = 0$ then $fx_k = fx_{k+1}$, hence $fx_k = gx_k = fx_{k+1}$. So, x_k is a point of coincidence for f and g. Thus assume $d_n > 0$ for all n. We claim $\{fx_n\}$ is an O-Cauchy sequence. Suppose not. Then there exists $\epsilon > 0$ and indices $n_i < n_{i+1}$ arbitrarily large such that subsequence $\{fx_{n_i}\} \subseteq \{fx_n\}$ satisfies

$$d(fx_{n_i}, fx_{n_{i+1}}) > 2\epsilon. \tag{3.2}$$

For this $\epsilon > 0$ and by assumption (ii), we can choose $0 < \delta < \epsilon$ such that

$$\epsilon \leq M_g(f)(x, y) < \epsilon + \delta \quad \Rightarrow \quad d(gx, gy) < \epsilon, \tag{3.3}$$

where $x \perp y$ and $x, y \in B$. Since $d_n \to 0$, there exists $N(\delta)$ such that $d_k = d(fx_k, fx_{k+1}) < \frac{\delta}{6}$ for all $k > N(\delta)$. For this $N(\delta)$, we can choose $n_j > N(\delta)$ such that

$$d(fx_{n_{j+1}-1}, fx_{n_j}) \geq \epsilon + \frac{\delta}{3},$$

otherwise,

$$\begin{aligned} d(fx_{n_j}, fx_{n_{j+1}}) &\leq d(fx_{n_j}, fx_{n_{j+1}-1}) + d(fx_{n_{j+1}-1}, fx_{n_{j+1}}) \\ &\leq \epsilon + \frac{\delta}{3} + d_{n_{j+1}-1} \\ &< \epsilon + \frac{\delta}{3} + \frac{\delta}{6} \\ &< \epsilon + \frac{\delta}{2} < 2\epsilon, \end{aligned}$$

which contradicts (3.2). Let l be the smallest integer satisfying $n_j < l \leq n_{j+1}$ such that

$$d(fx_{n_j}, fx_l) \geq \epsilon + \frac{\delta}{3}.$$

Hence, we have $d(fx_{n_j}, fx_{l-2}) < \epsilon + \frac{\delta}{3}$. Therefore,

$$\begin{aligned} d(fx_{n_j}, fx_l) &\leq d(fx_{n_j}, fx_{l-2}) + d(fx_{l-2}, fx_{l-1}) + d(fx_{l-1}, fx_l) \\ &< \epsilon + \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} = \epsilon + \frac{2\delta}{3}. \end{aligned}$$

Thus

$$\epsilon + \frac{\delta}{3} \leq d(fx_{n_j}, fx_l) < \epsilon + \frac{2\delta}{3} \tag{3.4}$$

By Theorem 3.8, since $\{x_n\}_{n=1}^{\infty}$ is SO-sequence then $x_{n_j} \perp x_l$. Hence, by the orthogonally generalized f-Meir-Keeler condition for map g, and using the fact that $\{d_n\}$ is monotonically decreasing, we can write

$$\begin{aligned} &\epsilon < d(fx_{n_j}, fx_l) < M_g(f)(x_{n_j}, x_l) \\ &\leq \max\Big\{d(fx_{n_j}, fx_l),\ d(fx_{n_j}, gx_{n_j}),\ d(fx_l, gx_l), \\ &\frac{1}{2}\big(d(fx_{n_j}, gx_l) + d(fx_l, gx_{n_j})\big)\Big\} \\ &= \max\Big\{d(fx_{n_j}, fx_l),\ d(fx_{n_j}, fx_{n_j+1}),\ d(fx_l, fx_{l+1}), \\ &\frac{1}{2}\big(d(fx_{n_j}, fx_{l+1}) + d(fx_l, fx_{n_j+1})\big)\Big\} \\ &\leq \max\Big\{d(fx_{n_j}, fx_l),\ \frac{1}{2}\big(d(fx_{n_j}, fx_{l+1}) + d(fx_l, fx_{n_j+1})\big)\Big\} \\ &\leq \max\Big\{d(fx_{n_j}, fx_l),\ \frac{1}{2}\big(d(fx_{n_j}, fx_l) + d(fx_l, fx_{l+1}) + \\ &d(fx_l, fx_{n_j}) + d(fx_{n_j}, fx_{n_j+1})\big)\Big\} \\ &= \max\Big\{d(fx_{n_j}, fx_l),\ d(fx_{n_j}, fx_l) + \frac{1}{2}(d_l + d_{n_j})\Big\} \\ &= d(fx_{n_j}, fx_l) + \frac{1}{2}(d_l + d_{n_j}) \\ &\leq d(fx_{n_j}, fx_l) + \frac{1}{2}\left(\frac{\delta}{6} + \frac{\delta}{6}\right), \end{aligned}$$

and hence for orthogonal x_{n_j}, x_l, we obtain

$$\epsilon < M_g(f)(x_{n_j}, x_l) < \epsilon + \frac{2\delta}{3} + \frac{\delta}{6} = \epsilon + \frac{5\delta}{6} < \epsilon + \delta.$$

So, by (3.3) and assumption (ii), we have

$$d(fx_{n_j+1}, fx_{l+1}) = d(gx_{n_j}, gx_l) < \epsilon,$$

which shows that

$$\begin{aligned} d(fx_{n_j}, fx_l) &\leq d(fx_{n_j}, fx_{n_j+1}) + d(fx_{n_j+1}, fx_{l+1}) + d(fx_l, fx_{l+1}) \\ &< \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3}, \end{aligned}$$

which contradicts (3.4). Therefore, the sequence $\{fx_n\}$ is a Cauchy SO-sequence in B. Now, by O-B-completeness and O-closedness of B there exists $z \in B$ with $fx_n \to z$.

Since f and g are O-continuous, $fx_{n+1} = gx_n$ and $ffx_{n+1} = fgx_n = gfx_n$, passing to limits gives $fz = gz$. Now we show that fz is a unique common fixed point of f and g. Suppose not. If $fz = gz = u$ and $fu \neq u$, then by part (ii) of Theorem 3.8 we have $fx_n \perp fz$ for all n. Hence by assumption (ii) we can write,

$$\begin{aligned} d(u, fu) = d(gz, fgz) = d(gz, gfz) &< d(gz, gfx_n) + d(gfx_n, gfz) \\ &< d(gz, gfx_n) + M_g(f)(fx_n, fz) \\ &< d(gz, gfx_n) + \max\Big\{d(ffx_n, ffz), d(ffx_n, gfx_n), d(ffz, gfz), \\ &\frac{1}{2}\big(d(ffx_n, gfz) + d(ffz, gfx_n)\big)\Big\} \end{aligned}$$

Using the Pass to the limit as $n \to \infty$ above and use continuity and commutativity of f and g, we obtain $d(u, fu) < d(u, fu)$ hence $fu = u$ or $ffz = fz$. Then

$$gu = gfz = fgz = ffz = fz = u,$$

i.e. $u = fz$ is a common fixed point of f and g.

Finally, we prove the uniqueness of the common fixed point. Let $v \in B$ be another common fixed point. If there exists n_0 such that $u = x_{n_0}$, then by Theorem 3.8, $fx_{n_0} \perp fv$. So, we have $u \perp v$. Hence, by the orthogonally generalized f-Meir–Keeler contractive condition for g, we obtain

$$\begin{aligned} d(u, v) = d(fu, gv) &< M_g(f)(u, v) \\ &= \max\{d(fu, fv), d(fu, gv), d(fv, gv), \frac{1}{2}\big(d(fu, gv) + d(fv, fu)\big)\} \\ &= d(u, v), \end{aligned}$$

which is a contradiction. Otherwise, suppose that $u \neq x_n$ for all n. Then, by part (ii) of Theorem 3.8, we have $x_n \perp u$. Now by the orthogonally generalized f-Meir-Keeler

condition map g, we can write

$$\begin{aligned} d(u,v) &\leq d(fx_n,u) + d(fx_n,v) \\ &= d(fx_n,gu) + d(fx_n,v) \\ &< \max\Big\{d(fx_n,fu),\ d(fx_n,fx_n),\ d(fu,fu), \\ &\frac{1}{2}\big(d(fx_n,fu) + d(fu,fx_n)\big)\Big\} + d(fx_n,v) \\ &< d(fx_n,fu) + d(fx_n,fu) + d(fx_n,v). \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain

$$d(u,v) < d(u,v),$$

which is again a contradiction. Therefore, we conclude that $u = v$ and the common fixed point is unique.

□

Corollary 3.10. *Let $(X, \perp, d)$ be an orthogonal metric space, and let $\alpha : X \times X \to [0,\infty)$ be a function satisfying $\alpha(x,y) \geq 1$ whenever $x \perp y$. If g is an orthogonally generalized α-f-Meir–Keeler contractive mapping, then the conclusions of Theorem 3.9 continue to hold.*

Now, using Theorem 3.9, we present a generalized version of Theorem 16 in [1].

Theorem 3.11. *Let (X,d) be a complete metric space, and let $f : X \to X$ be a continuous orthogonality-reflecting map. Let $g \in C_f$ be a continuous self-map of X, and a point $x_0 \in X$ has an f-iteration $\{fx_n\}_{n=0}^{\infty}$ under g. Let $\alpha : X \times X \to [0,\infty)$ be a function such that $\alpha(x_0,x) \geq 1$ for all $x \in X$. Suppose that*

(i) *f and g are α-admissible;*
(ii) *g is a generalized α-f-Meir–Keeler contractive mapping.*

Then f and g have a unique common fixed point in X.

Proof. Let $B = X$ and define an orthogonality relation $\perp$ on X by

$$x \perp y \quad \Longleftrightarrow \quad \alpha(x,y) \geq 1.$$

Then we observe the following:

(1) Since $\alpha(x_0,x) \geq 1$ for all $x \in X$, it follows that $x_0 \perp x$ for all $x \in B$.
(2) The mapping g is α-admissible; that is, $\alpha(x,y) \geq 1$ implies $\alpha(gx,gy) \geq 1$. Equivalently,
$$x \perp y \quad \Longrightarrow \quad gx \perp gy,$$
so g is $\perp$-preserving. By assumption, f is orthogonality-reflecting.
(3) Since X is complete, it follows that X is B-complete and O-B-complete. Also, every continuous mapping on X is O-B-continuous.
(4) Moreover, the generalized α-f-Meir–Keeler contractive mapping g is also an orthogonally generalized α-f-Meir–Keeler contractive mapping.

Hence, all the hypotheses of Theorem 3.9 are satisfied. Applying Corollary 3.10, we conclude that f and g have a unique common fixed point. □

Note that, by defining the orthogonality relation $\perp$ on X as

$$x \perp y \quad \Longleftrightarrow \quad \alpha(x,y) \geq 1,$$

and using the α-admissibility of f and g in Theorem 3.11, we conclude that f and g are $\perp$-preserving. This establishes parts (i) and (ii) of Theorem 3.8. Consequently, we obtain:

(1) $\alpha(x_n, x_m) \geq 1$ for every $m \geq n$;
(2) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_m) \geq 1$ for every $m \geq n$ and $fx_n \to z$, then $\alpha(fx_n, z) \geq 1$ and $\alpha(fx_n, fz) \geq 1$ for all $n \geq 1$.

These are exactly the assumptions of Theorem 16 in [1]. Therefore, the hypotheses of Theorem 16 in [1] are satisfied under the assumptions of Theorem 3.11, and Theorem 16 in [1] follows immediately as a direct consequence of Theorem 3.11.

Corollary 3.12. *Let (X, d) be a complete metric space, and let $f : X \to X$ be a continuous mapping satisfying the following generalized Meir–Keeler condition: for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x, y \in X$,*

$$\epsilon \leq M_f(x, y) < \epsilon + \delta(\epsilon) \quad \Longrightarrow \quad d(fx, fy) < \epsilon.$$

Then f has a fixed point in X.

Proof. Define a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) := 1 \quad \text{for all } x, y \in X.$$

Then clearly f is α-admissible and satisfies the generalized α-Meir–Keeler contractive condition. Moreover, for any $x_0 \in X$, we have $\alpha(x_0, x) \geq 1$ for all $x \in X$. Since $M_f(x, y) = M_f(I)(x, y)$ and $f \in C_I$, all the hypotheses of Theorem 3.11 are satisfied. Hence, the existence of a fixed point of f follows directly from Theorem 3.11. □

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