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Research article

ON ADDITIVE MAPS PRESERVING THE LOCAL SPECTRAL SUBSPACE OF TRIPLE PRODUCT OF OPERATOR

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Abstract. Let $B(H)$ be the algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space H. For $T \in B(H)$ and $\lambda \in \mathbb{C}$, let $H_T(\{\lambda\})$ denotes the local spectral subspace of T associated with $\{\lambda\}$. We prove that if φ_1 and φ_2 be additive maps from $B(H)$ into $B(H)$ such that their ranges contain all operators of rank at most four and satisfies

$$H_{\varphi_1(T)\varphi_2(S)^*\varphi_1(T)}(\{\lambda\}) = H_{TS^*T}(\{\lambda\})$$

for all $T, S \in B(H)$ and $\lambda \in \mathbb{C}$. Then there exist two invertible operators $P : H \longrightarrow H$ and $Q : H \longrightarrow H$ such that $\varphi_1(T) = PTP$ and $\varphi_2(T) = QTQ$ for all $T \in B(H)$.

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1. Introduction

A preserver problem generally deals with characterizing those maps on some specific algebraic structures which preserve a particular subset, property or relation. This subject has a long history and its origins goes back well over a century to the so-called first linear preserver problem, due to Frobenius [14], that determines linear maps preserving the determinant of matrices. As we mentioned earlier, the main of this subject goal is to describe the general form of linear maps between two Banach algebras which preserve a certain property, or a certain class of elements, or a certain relation. One of the most famous related problems is Kaplansky's problem [15] asking whether every surjective unital invertibility preserving linear map between two semisimple Banach algebras is a Jordan homomorphism. Several results on linear or additive preserver problems have been extended to the setting of nonlinear preservers, and, in many cases, their extensions demonstrated to be nontrivial. In particular, the problem of characterizing maps preserving certain functions, subsets, relations and properties of product of matrices or operators has attracted the attention of several authors; see for instance [3, 10, 11, 12, 17, 18, 21] and the references therein. Motivated by problems concerning local automorphisms, L. Molnár characterized, in [18], maps preserving different spectra of operator or matrix products, and his results have been extended by several authors. In [12], J. Hou and Q.H. Di described, in particular, maps preserving the numerical range of products of Hilbert space operators. In [17], maps preserving the nilpotency of operator products are characterized. Throughout this paper, H and K are infinite-dimensional complex Hilbert

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spaces. As usual $B(H,K)$ denotes the space of all bounded linear operators from H into K. When $H=K$ we simply write $B(H)$ instead of $B(H,H)$, and its unit will be denoted by I. The inner product of H or K will be denoted by $\langle , \rangle$ if there is no confusion. For an operator $T \in B(H,K)$, let T^* denote as usual its adjoint.

The local resolvent set, $\rho_T(x)$, of an operator $T \in B(H)$ at a point $x \in H$ is the union of all open subsets U of the complex plane $\mathbb{C}$ for which there is an analytic function $f : U \longrightarrow H$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. We recall that an operator $T \in B(H)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) if, for every open subset U of $\mathbb{C}$, there exists no nonzero analytic solution, $f : U \longrightarrow H$, of the equation

$$(\mu I - T)f(\mu) = 0, \quad \forall\ \mu \in U.$$

Every operator $T \in B(H)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property.

For every subset $F \subseteq \mathbb{C}$ the local spectral subspace $H_T(F)$ is defined by

$$H_T(F) = \{x \in H : \sigma_T(x) \subseteq F\}.$$

Clearly, if $F_1 \subseteq F_2$ then $H_T(F_1) \subseteq H_T(F_2)$. For more information about these notions one may see the books [1, 16].

The study of linear and nonlinear local spectra preserver problems attracted the attention of a number of authors. Bourhim and Ransford were the first ones to consider this type of preserver problem, characterizing in [9] additive maps on $B(X)$, the algebra of all linear bounded operators on infinite-dimensional complex Banach space X, that preserves the local spectrum of operators at each vector of X. Their results motivated several authors to describe maps on matrices or operators that preserve local spectrum, local spectral radius, and local inner spectral radius; see, for instance, the survey articles [6, 20] and the references therein. In the context of local spectral subspace, M. Elhodaibi and A. Jaatit in [13], were the first ones to consider the type of preserver problems. They proved that if the additive map φ on $B(X)$ satisfies $X_{\varphi(T)}(\{\lambda\}) = X_T(\{\lambda\})$ for all $T \in B(X)$ and $\lambda \in \mathbb{C}$, then φ is the identity map on $B(X)$. In [5], Benbouziane et al. characterized the forms of surjective weakly continuous maps φ from $B(X)$ into $B(X)$ which satisfy

$$X_{\varphi(T)-\varphi(S)}(\{\lambda\}) = X_{T-S}(\{\lambda\}), \quad (T, S \in B(X), \lambda \in \mathbb{C}).$$

Afterwards, in [4], the authors studied surjective maps that preserve the local spectral subspace of the sum of two operators associated with non-fixed singletons. In other words, they characterized surjective maps φ on $B(X)$ which satisfy

$$X_{\varphi(T)+\varphi(S)}(\{\lambda\}) = X_{T+S}(\{\lambda\}), \quad (T, S \in B(X), \lambda \in \mathbb{C}).$$

They also gave a characterization of maps on $B(X)$ that preserve the local spectral subspace of the difference of operators associated with non-fixed singletons. Furthermore, they investigated the product case as well as the triple product case. Namely, they described surjective maps φ on $B(X)$ satisfying

$$X_{\varphi(T)\varphi(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}), \quad (T, S \in B(X), \lambda \in \mathbb{C}),$$

and also surjective maps φ on $B(X)$ satisfying

$$X_{\varphi(T)\varphi(S)\varphi(T)}(\{\lambda\}) = X_{TST}(\{\lambda\}) \quad (T, S \in B(X), \lambda \in \mathbb{C}).$$

Additionally, in [2] the authors determined the form of maps preserving the skew product of operators associated with $\{\lambda\}$. In this paper, we investigate the form of all maps φ_1 and φ_2 on $B(H)$ such that, for every T and S in $B(H)$, the local spectral subspaces of TS^*T and $\varphi_1(T)\varphi_2(T)^*\varphi_1(T)$, associated with the singleton $\{\lambda\}$, coincide.

2. Preliminaries

The first lemma summarizes some known basic and properties of the local spectrum.

Lemma 2.1. *(See [1, 16].) Let $T \in B(H)$. For every $x, y \in H$ and a scalar $\alpha \in \mathbb{C}$ the following statements hold.*

(i) *If T has SVEP, then $\sigma_T(x) \neq \emptyset$ provided that $x \neq 0$.*
(ii) *$\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.*
(iii) *If $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$. In particular, if $x \neq 0$ and T has SVEP, then $\sigma_T(x) = \{\lambda\}$.*

In the next lemma we collect some of the basic properties of the subspaces $H_T(F)$.

Lemma 2.2. *(See [1, 16].) Let $T \in B(H)$. For $F \subseteq \mathbb{C}$ the following statements hold.*

(i) *$H_T(F)$ is a T-hyperinvariant subspace of H.*
(ii) *$(T - \lambda I)H_T(F) = H_T(F)$ for every $\lambda \in \mathbb{C}\backslash F$.*
(iii) *If $x \in H$ satisfies $(T - \lambda I)x \in H_T(F)$, then $x \in H_T(F)$.*
(v) *$ker(T - \lambda I) \subseteq H_T(F)$.*
(iv) *$H_{\alpha T}(\lambda) = H_T(\frac{\lambda}{\alpha})$ for every $\lambda \in \mathbb{C}$ and non-zero scalar α.*

For a nonzero $h \in H$ and $T \in B(H)$, we use a useful notation defined by Bourhim and Mashreghi in [7]:

$$\sigma_T^*(h) := \begin{cases} \{0\} & \text{if} \quad \sigma_T(h) = \{0\}, \\ \sigma_T(h) \setminus \{0\} & \text{if} \quad \sigma_T(h) \neq \{0\}. \end{cases}$$

For two nonzero vectors u and v in H, let $u \otimes v$ stands for the operator of rank at most one defined by

$$(u \otimes v)h = \langle h, v\rangle\, u, \qquad \forall\ h \in H.$$

Note that every rank one operator in $B(H)$ can be written in this form, and that every finite rank operator $T \in B(H)$ can be written as a finite sum of rank one operators; i.e., $T = \sum_{i=1}^{n} u_i \otimes v_i$ for some $u_i, v_i \in H$ and $i = 1, 2, ..., n$. By $F(H)$ and $F_n(H)$, we mean the set of all finite rank operators in $B(H)$. and the set of all operators of rank at most n, n is a positive integer, respectively.

The following lemma is an elementary observation which discribes the nonzero local spectrum of any rank one operator.

Lemma 2.3. *(See [7, Lemma 2.2].) Let h_0 be a nonzero vector in H. For every vectors $u, v \in H$, we have*

$$\sigma_{u\otimes v}^*(h_0) := \begin{cases} \{0\} & \text{if} \quad \langle h_0, v\rangle = 0, \\ \langle u, v\rangle & \text{if} \quad \langle h_0, v\rangle \neq 0. \end{cases}$$

The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators in term of local spectrum.

Theorem 2.4. *(See [8].) For a nonzero vector $h \in H$ and a nonzero operator $R \in B(H)$, the following statements are equivalent.*
(a) R has rank one.
(b) $\sigma^*_{TRT}(h)$ contains at most one element for all $T \in B(H)$.
(c) $\sigma^*_{TRT}(h)$ contains at most one element for all $T \in F_4(H)$.

The following Lemma is a key tool for the proofs in the sequel.

Lemma 2.5. *(See [4, Lemma 1.6].) Let h be a nonzero vector in H and $T, S \in B(H)$. If $H_T(\{\lambda\}) = H_S(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Then, $\sigma_T(h) = \{\mu\}$ if and only if $\sigma_S(h) = \{\mu\}$ for all $\mu \in \mathbb{C}$.*

Moreover, this theorem will be useful in the proofs of our main result.

Theorem 2.6. *(See [4, Theorem 2.1].) Let $T, S \in B(H)$. The following statements are equivalent.*
(1) $T = S$.
(2) $H_{RTR}(\{\lambda\}) = H_{RSR}(\{\lambda\})$ *for all* $\lambda \in \mathbb{C}$ *and* $R \in F_1(H)$.

The next theorem describes additive maps on $B(H)$ that preserve the local spectral subspace of operators associated with any singleton.

Theorem 2.7. *(See [13, Theorem 2.1].) Let $\varphi : B(H) \to B(H)$ be an additive map such that $H_{\varphi(T)}(\{\lambda\}) = H_T(\{\lambda\})$ for all $T \in B(H)$ and $\lambda \in \mathbb{C}$. Then $\varphi(T) = T$ for all $T \in B(H)$.*

The following theorem will be useful in the sequel. We recall that if $h : \mathbb{C} \to \mathbb{C}$ is a ring homomorphism, then an additive map $A : H \to H$ satisfying $A(\alpha x) = h(\alpha)x, (x \in H, \alpha \in \mathbb{C})$ is called an h-quasilinear operator.

Theorem 2.8. *(See [19, Theorem 3.3].) Let $\varphi : F(H) \to F(H)$ be a bijective additive map preserving rank one operators in both directions. Then there exist a ring automorphism $h : \mathbb{C} \to \mathbb{C}$, and either there are h-quasilinear bijective maps $A : H \to H$ and $B : H \to H$ such that*

$$\varphi(u \otimes v) = Au \otimes Bv, \ u, v \in H,$$

or there are h-quasilinear bijective maps $C : H \to H$ and $D : H \to H$ such that

$$\varphi(u \otimes v) = Cv \otimes Du, \ u, v \in H.$$

Note that, if in Theorem 2.8 the map φ is linear, then h is the identity map on $\mathbb{C}$ and so the maps A, B, C and D are linear.

3. Main Results

In the following theorem we give a concrete form of maps that preserve the local spectral subspace of triple product of operators.

Theorem 3.1. *Let φ_1 and φ_2 be additive maps from $B(H)$ into $B(H)$ which satisfy*

$$H_{\varphi_1(T)\varphi_2(S)^*\varphi_1(T)}(\{\lambda\}) = H_{TS^*T}(\{\lambda\}), \ (T, S \in B(H), \lambda \in \mathbb{C}). \tag{3.1}$$

If the range of φ_1 and φ_2 contain $F_4(H)$, then, there exist two invertible operators $P : H \longrightarrow H$ and $Q : H \longrightarrow H$ such that $\varphi_1(T) = PTP$ and $\varphi_2(T) = QTQ$ for all $T \in B(H)$.

Proof. The proof is rather long and we break it into several claims.

Claim 1. φ_2 is injective and $\varphi_2(0) = 0$.

If $\varphi_2(T) = \varphi_2(S)$ for some $T, S \in B(H)$, we get that

$$H_{RT^*R}(\{\lambda\}) = H_{\varphi_1(R)\varphi_2(T)^*\varphi_1(R)}(\{\lambda\}) = H_{\varphi_1(R)\varphi_2(S)^*\varphi_1(R)}(\{\lambda\}) = H_{RS^*R}(\{\lambda\})$$

for all $R \in F_1(H)$ and $\lambda \in \mathbb{C}$. By Theorem 2.6, we see that $T = S$ and hence φ_2 is injective. In a similar way, we show that $\varphi_2(0) = 0$. Indeed,

$$H_{\varphi_1(T)\varphi_2(0)^*\varphi_1(T)}(\{\lambda\}) = H_{T0^*T}(\{\lambda\}) = H_0(\{\lambda\}) = H_{\varphi_1(T)0\varphi_1(T)}(\{\lambda\})$$

for all $T \in B(H)$. Since the range of φ_1 contains all rank one operators, it follows by Theorem 2.6 that $\varphi_2(0) = 0$.

Claim 2. φ_2 preserves rank one operators in both directions.

Assume that $R \in B(H)$ be a rank one operator, so, there exist $u, v \in H$ such that $R = u \otimes v$. Since φ_2 is injective and $\varphi_2(0) = 0$, we have $\varphi_2(R) \neq 0$. Let $T \in B(H)$ be an arbitrary operator. Since $TR^*T(Tv) = \langle Tv, T^*u\rangle\, Tv$ and TR^*T has the SVEP, then $\sigma_{TR^*T}(T^*u) = \{\langle Tv, T^*u\rangle\}$. We have

$$Tv \in H_{TR^*T}(\{\langle Tv, T^*u\rangle\}) = H_{\varphi_1(T)\varphi_2(R)^*\varphi_1(T)}(\{\langle Tv, T^*u\rangle\}).$$

As the range of φ_1 contains $F_1(H)$, using Lemma 2.5, $\sigma^*_{S\varphi_2(R)^*S}(T^*u)$ contains at most one element for all operators $S \in F_4(H)$. By Theorem 2.4, we see that $\varphi_2(R)$ has rank one. The converse holds in a similar way and thus φ_2 preserves the rank one operators in both directions.

Claim 3. φ_2 is linear.

We show that φ_2 is homogeneous. Let R be an arbitrary rank-one operator. By the previous claim, there exists a rank one operator S in $B(H)$ such that $\varphi_1(S) = R$. For every $\alpha, \lambda \in \mathbb{C}$ with $\alpha \neq 0$ and $T \in B(H)$, we have

$$\begin{aligned} H_{R\alpha\varphi_2(T)^*R}(\{\lambda\}) &= H_{\varphi_1(S)\alpha\varphi_2(T)^*\varphi_1(S)}(\{\lambda\}) = H_{\varphi_1(S)\varphi_2(T)^*\varphi_1(S)}(\{\frac{\lambda}{\alpha}\}) \\ &= H_{ST^*S}(\{\frac{\lambda}{\alpha}\}) = H_{S(\alpha T)^*S}(\{\lambda\}) \\ &= H_{\varphi_1(S)\varphi_2(\alpha T)^*\varphi_1(S)}(\{\lambda\}) = H_{R\varphi_2(\alpha T)^*R}(\{\lambda\}). \end{aligned}$$

By Theorem 2.6, we see that $\varphi_2(\alpha T) = \alpha\varphi_2(T)$. Since φ_2 is assumed to be additive, the map φ_2 is, in fact, linear.

Claim 3. There are bijective linear mappings $A : H \to H$ and $B : H \to H$ such that $\varphi_2(u \otimes v) = Au \otimes Bv$ for all $u, v \in H$.

By the previous claims, $\varphi_2 : F(H) \to F(H)$ is a bijective linear map which preserves rank one operators in both directions. Thus by Theorem 2.8, φ_2 has one of the following forms.
(1) There exist bijective linear maps $A : H \to H$ and $B : H \to H$ such that

$$\varphi_2(u \otimes v) = Au \otimes Bv, \;\; u, v \in H. \tag{3.2}$$

(2) There exist bijective linear maps $C : H \to H$ and $D : H \to H$ such that

$$\varphi_2(u \otimes v) = Cv \otimes Du, \ u, v \in H. \tag{3.3}$$

Assume that φ_2 takes the form (3.3). Let y be a nonzero vector in H, choose a nonzero vector $v \in H$ such that $\langle \varphi_1(I)y, v\rangle = 0$. Set $u = C^{-1}v$, since u and y are nonzero vectors in H, there exists a vector $x \in H$ such that $\langle y, x\rangle \neq 0$ and $\langle u, x\rangle \neq 0$. Since $\sigma^*_{(Dx\otimes v)\varphi_1(I)}(y) = \{0\}$, we have

$$y \in H_{\varphi_1(I)(Dx\otimes v)\varphi_1(I)}(\{0\}) = H_{\varphi_1(I)(Dx\otimes Cu)\varphi_1(I)}(\{0\}) = H_{\varphi_1(I)(Cu\otimes Dx)^*\varphi_1(I)}(\{0\}) = H_{(x\otimes u)^*}(\{0\}).$$

Using Lemma 2.5, $\sigma_{u\otimes x}(y) = \{0\}$. But lemma 2.3 implies that

$$\sigma^*_{u\otimes x}(y) = \{\langle u, x\rangle\} \neq \{0\}.$$

This contradiction shows that φ_2 only takes the form (3.2).

Claim 4. For every $u, v \in H$, $\langle v, u\rangle = \{\left\langle Bv, (\varphi_1(I)^2)^*(Au)\right\rangle\}$.

Assume that u and v are arbitrary vectors in H. We have, $\sigma_{v\otimes u}(v) = \{\langle v, u\rangle\}$, so the previous claim and (3.1) imply that

$$v \in H_{v\otimes u}(\{\langle v, u\rangle\}) = H_{(u\otimes v)^*}(\{\langle v, u\rangle\}) = H_{\varphi_1(I)\varphi_2(u\otimes v)^*\varphi_1(I)}(\{\langle v, u\rangle\}) = H_{\varphi_1(I)(Au\otimes Bv)^*\varphi_1(I)}(\{\langle v, u\rangle\}).$$

Assume first that $\langle v, u\rangle \neq 0$, using lemma 2.5,

$$\{0\} \neq \{\langle v, u\rangle\} = \sigma_{v\otimes u}(v) = \sigma_{\varphi_1(I)(By\otimes Au)\varphi_1(I)}(v),$$

which means that $\left\langle v, (\varphi_1(I)^2)^*)(Au)\right\rangle \neq 0$. Then Lemma 2.3 implies that

$$\{\langle v, u\rangle\} = \sigma_{v\otimes u}(v) = \sigma_{(Bv\otimes Au)\varphi_1(I)^2)}(v) = \{\left\langle Bv, (\varphi_1(I)^2)^*(Au)\right\rangle\}.$$

Now, if $\langle v, u\rangle = 0$, then using the Hahn–Banach theorem, there exist $x_1, x_2 \in H$ such that $\langle v, u\rangle = \langle v, x_1\rangle + \langle v, x_1\rangle$, where $\langle y, x_1\rangle \neq 0$ and $\langle v, x_2\rangle \neq 0$. Thus $\langle v, u\rangle = \left\langle Bv, (\varphi_1(I)^2)^*(Au)\right\rangle$, for every $u, v \in H$.

Claim 5. $\varphi_1(I)^*$ is invertible.

It is clear that $\varphi_1(I)^*$ is injective, if not, there is a nonzero vector $y \in H$ such that $\varphi_1(I)^*y = 0$. Take $x = A^{-1}y$, and let $u \in H$ be a vector such that $\langle u, x\rangle = 1$. By the previous claim, we have $1 = \langle u, x\rangle = \left\langle Bu, (\varphi_1(I)^2)^*(Ax)\right\rangle = \left\langle Bu, (\varphi_1(I)^2)^*y\right\rangle = 0$. This contradiction tells us that $\varphi_2(I)^*$ is injective. Now, let us show that B is continuous. Let $(x_n)_n$ be a sequence in H such that $\lim_{n\longrightarrow\infty} x_n = x \in X$ and $\lim_{n\longrightarrow\infty} Bx_n = y \in H$. By claim 4, we have

$$\begin{aligned}\left\langle y, (\varphi_1(I)^2)^*)(Az)\right\rangle &= \lim_{n\longrightarrow\infty} \left\langle Bx_n, (\varphi_1(I)^2)^*)(Az)\right\rangle \\ &= \lim_{n\longrightarrow\infty} \langle x_n, z\rangle = \langle x, z\rangle = \left\langle Bx, (\varphi_1(I)^2)^*)(Az)\right\rangle\end{aligned}$$

For every $z \in H$. Since A is bijective and z is an arbitrary vector in H, the closed graph theorem shows that B is continuous. Moreover, for every $x, z \in H$, we have

$$\langle y, x\rangle = \left\langle By, (\varphi_1(I)^2)^*)(Ax)\right\rangle = \left\langle y, B^*(\varphi_1(I)^2)^*)(Ax)\right\rangle,$$

and thus $I = B^*(\varphi_1(I)^2)^* A$. It follows that $(\varphi_1(I)^2)^*$ is invertible and hence $\varphi_1(I)^*$ is invertible.

Claim 6. There exist two invertible operators $P : H \longrightarrow H$ and $Q : H \longrightarrow H$ such that $\varphi_1(T) = PTP$ and $\varphi_2(T) = QTQ$ for all $T \in B(H)$.

We define the map $\psi_1 : B(H) \to B(H)$ by $\psi_1(T) = \varphi_1(I)\varphi_2(T^*)^*\varphi_1(I)$ for all $T \in B(H)$. We have,

$$H_{\psi_1(T)}(\{\lambda\}) = H_{\varphi_1(I)\varphi_2(T^*)^*\varphi_1(I)}(\{\lambda\}) = H_T(\{\lambda\})$$

for all $T \in B(H)$ and $\lambda \in \mathbb{C}$. So by Theorem 2.7, $\psi_1(T) = T$ for all $T \in B(H)$, and so $\varphi_2(T^*)^* = \varphi_1(I)^{-1}T\varphi_1(I)^{-1}$ for all $T \in B(H)$. Therefore $\varphi_2(T) = PTP$ for all $T \in B(H)$, where $P = \varphi_1^{-1}(I)^*$. It follows that $\varphi_2(I) = (\varphi_1^{-1}(I)^*)^2$, and so $\varphi_2(I)^*$ is invertible. Once again, we consider the map $\psi_2 : B(H) \to B(H)$ defined by $\psi_2(T) = \varphi_2(I)\varphi_1(T^*)^*\varphi_2(I)$ for all $T \in B(H)$. We see that for all $T \in B(H)$ and $\lambda \in \mathbb{C}$,

$$H_{\psi_2(T)}(\{\lambda\}) = H_T(\{\lambda\}),$$

by Theorem 2.7, $\psi_2(T) = T$ for all $T \in B(H)$. Hence

$$\varphi_1(T^*)^* = \varphi_2(I)^{-1}T\varphi_2(I)^{-1},$$

and so $\varphi_1(T) = QTQ$ for all $T \in B(H)$, where $Q = \varphi_2^{-1}(I)^*$. □

Theorem 3.1 leads directly to the following corollary.

Corollary 3.2. *Suppose $U \in B(H, K)$ be a unitary operator. Let φ_1 and φ_2 be two additive map from $B(H)$ into $B(K)$ which satisfy*

$$K_{\varphi_1(T)\varphi_2(S)^*\varphi_1(T)}(\{\lambda\}) = UH_{TS^*T}(\{\lambda\}),\ (T, S \in B(H), \lambda \in \mathbb{C}).$$

If the range of φ_1 and φ_2 contain $F_4(K)$, then $\varphi_1(T) = \varphi_2^{-1}(I)^*UTU^*\varphi_2^{-1}(I)^*$ and $\varphi_2(T) = \varphi_1^{-1}(I)^*UTU^*\varphi_1^{-1}(I)^*$ for all $T \in B(H)$.

Proof. We consider the maps $\psi_1 : B(H) \to B(H)$ defined by $\psi_1(T) = U^*\varphi_1(T)U$ and $\psi_2 : B(H) \to B(H)$ defined by $\psi_2(T) = U^*\varphi_2(T)U$ for all $T \in B(H)$. We have,

$$\begin{aligned} H_{\psi_1(T)\psi_2(T)^*\psi_1(S)}(\{\lambda\}) &= H_{U^*\varphi_1(T)UU^*\varphi_2(S)^*UU^*\varphi_1(T)U}(\{\lambda\}) \\ &= H_{U^*\varphi_1(T)\varphi_2(S)^*\varphi_1(T)U}(\{\lambda\}) \\ &= U^{-1}K_{\varphi_1(T)\varphi_2(S)^*\varphi_1(T)}(\{\lambda\}) = H_{TS^*T}(\{\lambda\}) \end{aligned}$$

for every $T, S \in B(H)$ and $\lambda \in \mathbb{C}$. So by Theorem 3.1, $\psi_1(T) = \psi_2^{-1}(I)^*T\psi_2^{-1}(I)^*$ and $\psi_2(T) = \psi_1^{-1}(I)^*T\psi_1^{-1}(I)^*$ for all $T \in B(H)$. Therefore $\varphi_1(T) = \varphi_2^{-1}(I)^*UTU^*\varphi_2^{-1}(I)^*$ and $\varphi_2(T) = \varphi_1^{-1}(I)^*UTU^*\varphi_1^{-1}(I)^*$ for all $T \in B(H)$. □

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