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Research Paper

PAIR (Q, h) ON SOME FIXED POINT THEOREMS FOR α-ADMISSIBLE MAPPINGS IN S-METRIC SPACES

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Abstract. In this paper, we introduce a novel class of contraction mappings defined on S-metric spaces. Our approach relies on the concept of α-admissible mappings and the newly defined notion of upper class pairs of type III. We prove a generalized fixed point theorem that extends and unifies several existing results in the literature, most notably the theorems of Shimpi and Dapke [6]. The main result is formulated in terms of a pair (Q, h), where h belongs to a subclass of type III and Q satisfies appropriate conditions. By choosing specific forms for h and Q, we obtain a series of corollaries that encompass well-known fixed point theorems as particular cases. The findings not only enhance the applicability of fixed point theory in S-metric spaces but also establish a unified framework for future research in this area.

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1. Introduction and Preliminaries

Fixed point theory occupies a central position in nonlinear analysis due to its wide range of applications in areas such as differential equations, optimization, and dynamical systems. The Banach contraction principle stands as a cornerstone of this theory, ensuring the existence and uniqueness of fixed points for certain self-mappings defined on complete metric spaces. Over the decades, this fundamental result has been extended in several directions, including generalizations to more abstract metric-type spaces and the formulation of weaker contractivity conditions.

A notable extension is the concept of α-admissible mappings, first introduced by Samet et al. [19]. This notion provides greater flexibility in formulating contraction conditions and has since been investigated in various metric settings. In particular, the study of α-admissible mappings within the framework of S-metric spaces—a generalization of ordinary metric spaces proposed by Sedghi et al. [21]—has received considerable attention. Recently, Shimpi and Dapke [6] obtained several fixed point results for α-admissible mappings in S-metric spaces under diverse contractive hypotheses.

Motivated by the work of Shimpi and Dapke [6], in this paper we introduce a new family of contraction mappings defined by means of a function pair (Q, h). Here h is taken from a subclass of type III, while Q constitutes an upper class of type III. This formulation enables us to unify and extend a number of existing fixed point theorems. We establish a general fixed

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point result and, by selecting particular instances of h and Q, we deduce several corollaries that cover known theorems as special cases.

To make the presentation self-contained, we first recall some essential definitions and facts concerning S-metric spaces and α-admissible mappings.

Definition 1.1 ([21])**.** Let X be a nonempty set and let $S : X^3 \to \mathbb{R}^+$ be a function satisfying the following conditions:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;
(ii) $S(x, y, z) \le S(x, x, a)+S(y, y, a)+S(z, z, a)$ for all $a, x, y, z \in X$ (rectangle inequality).

Then the pair (X, S) is called an *S-metric space.*

Definition 1.2 ([21])**.** Let (X, S) be an S-metric space.

(i) A sequence $\{x_n\}$ in X is said to *converge* to $x \in X$ if $\lim\limits_{n\to\infty} S(x_n, x_n, x) = 0$. Equivalently, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x) < \epsilon$ for all $n \ge n_0$. In this case we write $\lim_{n\to\infty} x_n = x$ and call x the limit of $\{x_n\}$.
(ii) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \ge n_0$.
(iii) The S-metric space (X, S) is said to be *complete* if every Cauchy sequence in X converges to some point of X.

Lemma 1.3 ([21])**.** *If (X, S) is an S-metric space, then for all $x, y \in X$ we have*

$$S(x, x, y) = S(y, y, x).$$

Lemma 1.4 ([21])**.** *Let (X, S) be an S-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y respectively (i.e., $x_n \to x$ and $y_n \to y$ as $n \to \infty$), then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Lemma 1.5 ([21])**.** *Let (X, S) be an S-metric space. If a sequence $\{x_n\}$ in X converges to some $x \in X$, then $\{x_n\}$ is a Cauchy sequence.*

Definition 1.6 ([11])**.** Let (X, S) be an S-metric space and let $T : X \to X$ be a mapping. We say that T is *α-admissible* if for all $x, y, z \in X$,

$$\alpha(x, y, z) \ge 1 \quad \text{implies} \quad \alpha(Tx, Ty, Tz) \ge 1.$$

Example 1.7. Consider $X = [0,\infty)$. Define a mapping $T : X \to X$ and a function $\alpha : X \times X \times X \to [0,\infty)$ by

$$Tx = x^2 \quad \text{and} \quad \alpha(x, y, z) = \begin{cases} 2, & \text{if } x \ge y \ge z, \\ 0, & \text{otherwise.} \end{cases}$$

Then T is α-admissible.

Proof. Assume $\alpha(x, y, z) \ge 1$. By the definition of α, this implies $x \ge y \ge z$. Since the squaring function is increasing on $[0,\infty)$, we obtain

$$Tx = x^2 \ge y^2 = Ty \ge z^2 = Tz,$$

i.e., $Tx \ge Ty \ge Tz$. Consequently, $\alpha(Tx, Ty, Tz) = 2 \ge 1$. Hence T is α-admissible. $\square$

Lemma 1.8 ([3])**.** *Let $A : X \to X$ be a triangular α-admissible mapping. Suppose there exists a point $x_1 \in X$ such that $\alpha(x_1, x_1, Ax_1) \geq 1$. Define the sequence $\{x_n\}$ recursively by $x_{n+1} = Ax_n$ for $n \geq 1$. Then for all $m, n \in \mathbb{N}$ with $n < m$, we have*

$$\alpha(x_n, x_n, x_m) \geq 1.$$

Next, we recall the definitions of functions belonging to subclasses of types I and II, as well as pairs forming upper classes of types I and II. For a more detailed exposition we refer the reader to [4].

Definition 1.9 ([4, 5])**.** A function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be of *subclass of type I* if for all $x, y \in \mathbb{R}^+$,

$$x \geq 1 \quad \Longrightarrow \quad h(1, y) \leq h(x, y).$$

Below we list several examples of functions that belong to the subclass of type I; see [4].

Example 1.10 ([4, 5])**.** The following functions are of subclass of type I:

(1) $h(x, y) = (y + l)^x$, where $l > 1$;
(2) $h(x, y) = (x + l)^y$, where $l > 0$;
(3) $h(x, y) = x^n y^k$, where $k > 0$ and $n \in \mathbb{N} \cup \{0\}$;
(4) $h(x, y) = \dfrac{x^n + x^{n-1} + \cdots + x + 1}{n + 1}\, y$;
(5) $h(x, y) = \Big(\dfrac{x^n + x^{n-1} + \cdots + x + 1}{n + 1} + l\Big)^y$, where $l > 1$;
(6) $h(x, y) = \dfrac{mx + n}{m + n}\, y$, where $m, n \in \mathbb{N}$.

Remark 1.11. Note that for any function h of subclass of type I we have the inequality $y \leq h(1, y)$ for all $y \in \mathbb{R}^+$.

Definition 1.12 ([4])**.** A function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be of *subclass of type II* if for all $x, y, z \in \mathbb{R}^+$,

$$x, y \geq 1 \quad \Longrightarrow \quad h(1, 1, z) \leq h(x, y, z).$$

Example 1.13 ([4])**.** The following functions are of subclass of type II:

(1) $h(x, y, z) = (z + l)^{xy}$, where $l > 1$;
(2) $h(x, y, z) = (xy + l)^z$, where $l > 0$;
(3) $h(x, y, z) = x^m y^n z^p$, where $m, n \in \mathbb{N} \cup \{0\}$ and $p > 0$;
(4) $h(x, y, z) = \Big(\dfrac{1}{n + 1} \displaystyle\sum_{i=0}^{n} x^{n-i} y^i\Big) z$;
(5) $h(x, y, z) = \Big(\dfrac{1}{n + 1} \displaystyle\sum_{i=0}^{n} x^{n-i} y^i + l\Big)^z$, where $l > 1$;
(6) $h(x, y, z) = \dfrac{x^m + x^n y^p + y^q}{3}\, z^k$, where $m, n, p, q, k \in \mathbb{N}$.

Remark 1.14. For any function h belonging to the subclass of type II, one has the inequality $z \leq h(1, 1, z)$ for all $z \in \mathbb{R}^+$.

Definition 1.15 ([4])**.** Let $h, Q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be two functions. The pair (Q, h) is called an *upper class of type I* if h is a subclass of type I and the following conditions are satisfied:

(1) For all $s, t \in \mathbb{R}^+$ with $0 \leq s \leq 1$, we have $Q(s, t) \leq Q(1, t)$;

(2) For all $x, y, s, t \in \mathbb{R}^+$, the implication $h(1, y) \leq Q(s, t) \implies y \leq st$ holds.

If in condition (2) we take $s = 1$, then the pair (Q, h) is referred to as a *special upper class of type I*.

Example 1.16 ([4])**.** The following pairs (Q, h) form upper classes of type I:

(1) $h(x, y) = (y + l)^x$ with $l > 1$, and $Q(s, t) = st + l$;
(2) $h(x, y) = (x + l)^y$ with $l > 0$, and $Q(s, t) = (1 + l)^{st}$;
(3) $h(x, y) = x^n y^k$ and $Q(s, t) = s^m t^k$, where $k > 0$;
(4) $h(x, y) = \dfrac{mx + n}{m + n}\, y$ with $m, n \in \mathbb{N}$, and $Q(s, t) = st$;
(5) $h(x, y) = \Big(\dfrac{x^n + x^{n-1} + \cdots + x + 1}{n + 1} + l\Big)^y$ with $l > 1$, and $Q(s, t) = (1 + l)^{st}$;
(6) $h(x, y) = (y + l)^x$ with $l > 1$, and $Q(s, t) = st + \dfrac{l}{k}$ where $k \geq 1$.

Definition 1.17 ([4])**.** Let $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $Q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be functions. We say that the pair (Q, h) is an *upper class of type II* provided h is a subclass of type II and the following two conditions hold:

(1) For all $s, t \in \mathbb{R}^+$ with $0 \leq s \leq 1$, we have $Q(s, t) \leq Q(1, t)$;
(2) For all $z, s, t \in \mathbb{R}^+$, the implication $h(1, 1, z) \leq Q(s, t) \implies z \leq st$ holds.

If in condition (2) we set $s = 1$, then the pair (Q, h) is called a *special upper class of type II*.

Example 1.18 ([4])**.** The following pairs (Q, h) are examples of upper classes of type II:

(1) $h(x, y, z) = (z + l)^{xy}$ with $l > 1$, and $Q(s, t) = st + l$;
(2) $h(x, y, z) = (xy + l)^z$ with $l > 0$, and $Q(s, t) = (1 + l)^{st}$;
(3) $h(x, y, z) = x^m y^n z^p$ and $Q(s, t) = s^p t^p$, where $m, n \in \mathbb{N} \cup \{0\}$ and $p > 0$;
(4) $h(x, y, z) = \dfrac{x^m + x^n y^p + y^q}{3}\, z^k$ with $m, n, p, q, k \in \mathbb{N}$, and $Q(s, t) = (st)^k$;
(5) $h(x, y, z) = \Big(\dfrac{1}{n + 1} \displaystyle\sum_{i=0}^{n} x^{n-i} y^i + l\Big)^z$ with $l > 1$, and $Q(s, t) = (1 + l)^{st}$;
(6) $h(x, y, z) = \Big(\dfrac{1}{n + 1} \displaystyle\sum_{i=0}^{n} x^{n-i} y^i\Big) z$, and $Q(s, t) = st$;
(7) $h(x, y, z) = (z + l)^{xy}$ with $l > 1$, and $Q(s, t) = st + \dfrac{l}{k}$ where $k \geq 1$.

Theorem 1.19 ([6])**.** *Let (X, S) be a complete S-metric space and let $T : X \to X$ be an α-admissible mapping. Assume there exists a function $\beta : [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and for all $x, y, z \in X$ with $l \geq 1$,

$$(S(Tx, Ty, Tz) + l)^{\alpha(x,x,Tx)\,\alpha(y,y,Ty)\,\alpha(z,z,Tz)} \leq \beta\big(S(x, y, z)\big)\, S(x, y, z) + l. \tag{1.1}$$

Suppose moreover that one of the following conditions holds:

(i) *T is continuous; or*
(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, x, Tx) \geq 1$.*

If there exists $x_0 \in X$ with $\alpha(x_0, x_0, Tx_0) \geq 1$, then T possesses a fixed point.

Theorem 1.20 ([6])**.** *Let (X,S) be a complete S-metric space and let $T: X \to X$ be an α-admissible mapping. Assume there exists a function $\beta: [0,\infty) \to [0,1]$ such that for any bounded sequence $\{t_n\}$ of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and for all $x, y, z \in X$,

$$\big(\alpha(x,x,Tx)\,\alpha(y,y,Ty)\,\alpha(z,z,Tz)+1\big)^{S(Tx,Ty,Tz)} \le 2^{\beta(S(x,y,z))\,S(x,y,z)}. \tag{1.2}$$

Suppose moreover that one of the following conditions holds:

(i) *T is continuous; or*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\alpha(x_n, x_n, x_{n+1}) \ge 1$ for all n, then $\alpha(x,x,Tx) \ge 1$.*

If there exists $x_0 \in X$ with $\alpha(x_0, x_0, Tx_0) \ge 1$, then T possesses a fixed point.

Theorem 1.21 ([6])**.** *Let (X,S) be a complete S-metric space and let $T: X \to X$ be an α-admissible mapping. Assume there exists a function $\beta: [0,\infty) \to [0,1]$ such that for any bounded sequence $\{t_n\}$ of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and for all $x, y, z \in X$,

$$\alpha(x,x,Tx)\,\alpha(y,y,Ty)\,\alpha(z,z,Tz)\; S(Tx,Ty,Tz) \le \beta\big(S(x,y,z)\big)\, S(x,y,z). \tag{1.3}$$

Suppose moreover that one of the following conditions is satisfied:

(i) *T is continuous; or*

(ii) *if $\{x_n\}$ is a sequence in X converging to x and $\alpha(x_n, x_n, x_{n+1}) \ge 1$ for every n, then $\alpha(x,x,Tx) \ge 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, x_0, Tx_0) \ge 1$, then T possesses a fixed point.

2. Main Results

In this section, we establish several new fixed point theorems for α-admissible mappings in the framework of S-metric spaces. To set the stage for our analysis, we begin by introducing the following definitions.

Definition 2.1. A function $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be of *subclass of type III* if for all $x, y, z, t \in \mathbb{R}^+$,

$$x, y, z \ge 1 \quad \Longrightarrow \quad h(1,1,1,t) \le h(x,y,z,t).$$

Example 2.2. The following are examples of functions belonging to the subclass of type III:

(1) $h(x,y,z,t) = (t+l)^{xyz}$, where $l > 1$;

(2) $h(x,y,z,t) = (xyz+l)^t$, where $l > 0$;

(3) $h(x,y,z,t) = x^m y^n z^q t^p$, where $m, n, q \in \mathbb{N} \cup \{0\}$ and $p > 0$.

Remark 2.3. For any function h of subclass of type III, the inequality $t \le h(1,1,1,t)$ holds for every $t \in \mathbb{R}^+$.

Definition 2.4. Let $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $Q: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be functions. We say that the pair (Q,h) is an *upper class of type III* provided h is a subclass of type III and the following two conditions are satisfied:

(1) For all $s, t \in \mathbb{R}^+$ with $0 \le s \le 1$, we have $Q(s,t) \le Q(1,t)$;

(2) For all $z, s, t \in \mathbb{R}^+$, the implication $h(1,1,1,z) \le Q(s,t) \implies z \le st$ holds.

If in condition (2) we take $s = 1$, then the pair (Q,h) is called a *special upper class of type III*.

Example 2.5. The following pairs (Q,h) are examples of upper classes of type III:

(1) $h(x,y,z,t) = (t+l)^{xyz}$ with $l \ge 1$, and $Q(s,t) = st + l$;
(2) $h(x,y,z,t) = (xyz+l)^t$ with $l > 0$, and $Q(s,t) = (1+l)^{st}$;
(3) $h(x,y,z,t) = x^m y^n z^q t^p$ and $Q(s,t) = s^p t^p$, where $m,n,q \in \mathbb{N} \cup \{0\}$ and $p > 0$.

Theorem 2.6. *Let (X,S) be a complete S-metric space and let $T : X \to X$ be an α-admissible mapping. Assume there exists a function $\beta : [0,\infty) \to [0,1]$ such that for any bounded sequence $\{t_n\}$ of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and suppose that for all $x, y, z \in X$,

$$h\Big(\alpha(x,x,Tx),\, \alpha(y,y,Ty),\, \alpha(z,z,Tz),\, S(Tx,Ty,Tz)\Big) \le Q\Big(\beta\big(S(x,y,z)\big),\, S(x,y,z)\Big), \tag{2.1}$$

where the pair (Q,h) is an upper class of type III. Moreover, assume that one of the following conditions holds:

(i) *T is continuous; or*
(ii) *if $\{x_n\}$ is a sequence in X converging to x and $\alpha(x_n,x_n,x_{n+1}) \ge 1$ for every n, then $\alpha(x,x,Tx) \ge 1$.*

If there exists a point $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \ge 1$, then T possesses a fixed point.

Proof. Define the sequence $\{x_n\}$ recursively by $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$ for $n \ge 1$. By hypothesis we have $\alpha(x_0,x_0,Tx_0) \ge 1$. Since T is α-admissible, we obtain $\alpha(x_1,x_1,x_2) = \alpha(Tx_0,Tx_0,Tx_1) \ge 1$. Proceeding by induction, we conclude that

$$\alpha(x_n,x_n,x_{n+1}) \ge 1 \qquad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

Now, for each $n \in \mathbb{N}$, applying inequality (2.1) with $x = y = x_{n-1}$ and $z = x_n$, we obtain

$$\begin{aligned}
&h\Big(1,1,1,S(x_n,x_n,x_{n+1})\Big)\\
&= h\Big(1,1,1,S(Tx_{n-1},Tx_{n-1},Tx_n)\Big)\\
&\le h\Big(\alpha(x_{n-1},x_{n-1},Tx_{n-1}),\, \alpha(x_{n-1},x_{n-1},Tx_{n-1}),\, \alpha(x_n,x_n,Tx_n),\, S(Tx_{n-1},Tx_{n-1},Tx_n)\Big)\\
&\le Q\Big(\beta\big(S(x_{n-1},x_{n-1},x_n)\big),\, S(x_{n-1},x_{n-1},x_n)\Big).
\end{aligned}$$

Since h is of subclass of type III, the inequality $h(1,1,1,t) \le h(x,y,z,t)$ holds whenever $x,y,z \ge 1$. In view of (2.2) we have $\alpha(x_{n-1},x_{n-1},Tx_{n-1}) = \alpha(x_{n-1},x_{n-1},x_n) \ge 1$ and $\alpha(x_n,x_n,Tx_n) = \alpha(x_n,x_n,x_{n+1}) \ge 1$, therefore the first estimate above is justified. Consequently, from the definition of an upper class of type III we deduce

$$S(x_n,x_n,x_{n+1}) \le \beta\big(S(x_{n-1},x_{n-1},x_n)\big)\, S(x_{n-1},x_{n-1},x_n).$$

It follows that

$$S(x_n,x_n,x_{n+1}) \le S(x_{n-1},x_{n-1},x_n) \qquad \text{for all } n. \tag{2.3}$$

Thus the sequence $\{S(x_n,x_n,x_{n+1})\}$ is non-increasing and bounded below by zero; hence it converges to some limit $\ell \ge 0$.

From (2.3) we have

$$\frac{S(x_n, x_n, x_{n+1})}{S(x_{n-1}, x_{n-1}, x_n)} \leq \beta\big(S(x_{n-1}, x_{n-1}, x_n)\big) \leq 1,$$

so letting $n \to \infty$ gives $\lim_{n\to\infty} \beta\big(S(x_{n-1}, x_{n-1}, x_n)\big) = 1$. By the assumed property of β this forces $\lim_{n\to\infty} S(x_{n-1}, x_{n-1}, x_n) = 0$, and therefore

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{2.4}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Assume, for contradiction, that it is not. Then there exist $\epsilon > 0$ and two increasing sequences of indices $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq \epsilon \quad \text{and} \quad S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) < \epsilon \qquad \text{for all } k.$$

Using the rectangle inequality of the S-metric we obtain

$$\begin{aligned}\epsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = S(x_{m(k)}, x_{m(k)}, x_{n(k)})\\ &\leq S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) + S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) + S(x_{n(k)}, x_{n(k)}, x_{m(k)-1})\\ &< 2S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) + \epsilon.\end{aligned}$$

Letting $k \to \infty$ and employing (2.4) yields

$$\epsilon \leq \liminf_{k\to\infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \leq \limsup_{k\to\infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \leq \epsilon,$$

hence

$$\lim_{k\to\infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \epsilon. \tag{2.5}$$

Again applying the rectangle inequality,

$$\begin{aligned}S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) &\leq 2S(x_{n(k+1)}, x_{n(k+1)}, x_{n(k)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{m(k)})\\ &\quad + S(x_{m(k+1)}, x_{m(k+1)}, x_{m(k)}) + S(x_{n(k)}, x_{n(k)}, x_{m(k)})\\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k+1)}) + 2S(x_{m(k)}, x_{m(k)}, x_{m(k+1)}) + S(x_{n(k)}, x_{n(k)}, x_{m(k)}).\end{aligned}$$

Taking $k \to \infty$ and using (2.4) together with (2.5) gives

$$\lim_{k\to\infty} S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) = \epsilon. \tag{2.6}$$

Now, from Lemma 1.8 we have $\alpha(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq 1$. Applying (2.1) with $x = y = x_{n(k)}$ and $z = x_{m(k)}$ and using the fact that h is of subclass of type III, we obtain

$$\begin{aligned}&h\Big(1, 1, 1, S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)})\Big)\\ &\quad = h\Big(1, 1, 1, S(Tx_{n(k)}, Tx_{n(k)}, Tx_{m(k)})\Big)\\ &\quad \leq h\Big(\alpha_{n(k)}, \alpha_{n(k)}, \alpha_{m(k)}, S(Tx_{n(k)}, Tx_{n(k)}, Tx_{m(k)})\Big)\\ &\quad \leq Q\Big(\beta\big(S(x_{n(k)}, x_{n(k)}, x_{m(k)})\big), S(x_{n(k)}, x_{n(k)}, x_{m(k)})\Big),\end{aligned}$$

Since (Q, h) is an upper class of type III, the last inequality implies

$$S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) \leq \beta\big(S(x_{n(k)}, x_{n(k)}, x_{m(k)})\big)\, S(x_{n(k)}, x_{n(k)}, x_{m(k)}).$$

Consequently,

$$\frac{S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)})}{S(x_{n(k)}, x_{n(k)}, x_{m(k)})} \le \beta\big(S(x_{n(k)}, x_{n(k)}, x_{m(k)})\big) \le 1.$$

Letting $k \to \infty$ and using (2.5) and (2.6) we obtain $\lim_{k\to\infty} \beta\big(S(x_{n(k)}, x_{n(k)}, x_{m(k)})\big) = 1$. By the property of β this forces $\lim_{k\to\infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = 0$, contradicting (2.5). Hence $\{x_n\}$ is a Cauchy sequence.

Because (X, S) is complete, there exists $z \in X$ such that $x_n \to z$. If condition (i) holds, i.e., T is continuous, then

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z,$$

so z is a fixed point of T.

Now suppose that condition (ii) holds. Then, from the convergence $x_n \to z$ and (2.2) we infer $\alpha(z, z, Tz) \ge 1$. Applying (2.1) with $x = y = z$ and z replaced by x_n, we get

$$\begin{aligned} h\Big(1, 1, 1, S(Tz, Tz, x_{n+1})\Big) &= h\Big(1, 1, 1, S(Tz, Tz, Tx_n)\Big) \\ &\le h\Big(\alpha(z, z, Tz),\ \alpha(z, z, Tz),\ \alpha(x_n, x_n, Tx_n),\ S(Tz, Tz, Tx_n)\Big) \\ &\le Q\Big(\beta\big(S(z, z, x_n)\big),\ S(z, z, x_n)\Big). \end{aligned}$$

Hence, using again that (Q, h) is an upper class of type III,

$$S(Tz, Tz, x_{n+1}) \le \beta\big(S(z, z, x_n)\big)\, S(z, z, x_n).$$

Finally, by the rectangle inequality,

$$\begin{aligned} S(Tz, Tz, z) &\le S(Tz, Tz, x_{n+1}) + S(Tz, Tz, x_{n+1}) + S(z, z, x_{n+1}) \\ &= 2S(Tz, Tz, x_{n+1}) + S(z, z, x_{n+1}) \\ &\le 2\beta\big(S(z, z, x_n)\big)\, S(z, z, x_n) + S(z, z, x_{n+1}). \end{aligned}$$

Letting $n \to \infty$ and using the fact that $x_n \to z$ (so $S(z, z, x_n) \to 0$ and $S(z, z, x_{n+1}) \to 0$) together with $\beta(0) \in [0, 1]$, we obtain $S(Tz, Tz, z) \le 0$. Since S is non-negative, this forces $S(Tz, Tz, z) = 0$, i.e., $Tz = z$. Thus z is a fixed point of T in both cases. $\square$

3. Corollaries

Choosing the functions

$$h(x, y, z, t) = x^m y^n z^q t^p \quad \text{and} \quad Q(s, t) = s^p t^p,$$

where $m, n, q \in \mathbb{N} \cup \{0\}$ and $p > 0$, in Theorem 2.6 we obtain the following corollary.

Corollary 3.1. *Let (X, S) be a complete S-metric space and let $T : X \to X$ be an α-admissible mapping. Assume there exists a function $\beta : [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and for all $x, y, z \in X$,

$$\alpha(x, x, Tx)^m\, \alpha(y, y, Ty)^n\, \alpha(z, z, Tz)^q\, S(Tx, Ty, Tz)^p \le \beta\big(S(x, y, z)\big)^p\, S(x, y, z)^p, \tag{3.1}$$

where $m, n, q \in \mathbb{N} \cup \{0\}$ and $p > 0$. Moreover, suppose that one of the following conditions holds:

(i) T *is continuous; or*
(ii) *if* $\{x_n\}$ *is a sequence in* X *converging to* x *and* $\alpha(x_n,x_n,x_{n+1}) \geq 1$ *for every* n*, then* $\alpha(x,x,Tx) \geq 1$.

If there exists a point $x_0 \in X$ *such that* $\alpha(x_0,x_0,Tx_0) \geq 1$*, then* T *possesses a fixed point.*

Example 3.2 (Illustrating Corollary 3.1)**.** Let $X = [0,\infty)$ be equipped with the S-metric

$$S(x,y,z) = \max\big\{|x-y|,\ |y-z|,\ |z-x|\big\}.$$

Define $T : X \to X$ by $Tx = \dfrac{x}{2}$ and $\alpha : X^3 \to [0,\infty)$ by

$$\alpha(x,y,z) = \begin{cases} 2, & \text{if } x,y,z \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Take $\beta(t) = \dfrac{1}{2}$ for all $t \geq 0$, and choose the exponents $m = n = q = 1$, $p = 2$. For $x,y,z \leq 1$ we have $\alpha(x,x,Tx) = \alpha(y,y,Ty) = \alpha(z,z,Tz) = 2$, and

$$\begin{aligned}\alpha(x,x,Tx)^1\alpha(y,y,Ty)^1\alpha(z,z,Tz)^1\, S(Tx,Ty,Tz)^2 &= 2\cdot 2\cdot 2\cdot \Big[\max\big\{|\tfrac{x}{2}-\tfrac{y}{2}|, |\tfrac{y}{2}-\tfrac{z}{2}|, |\tfrac{z}{2}-\tfrac{x}{2}|\big\}\Big]^2 \\ &= 8\cdot\Big[\tfrac{1}{2}\max\{|x-y|,|y-z|,|z-x|\}\Big]^2 \\ &= 2\,S(x,y,z)^2.\end{aligned}$$

On the other hand,

$$\beta\big(S(x,y,z)\big)^2\, S(x,y,z)^2 = \Big(\frac{1}{2}\Big)^2 S(x,y,z)^2 = \frac{1}{4}\,S(x,y,z)^2.$$

Thus inequality (3.1) becomes

$$2\,S(x,y,z)^2 \leq \frac{1}{4}\,S(x,y,z)^2,$$

which is satisfied only when $S(x,y,z) = 0$, i.e., when $x = y = z$. Nevertheless, for any $x_0 \in [0,1]$ the iteration $x_{n+1} = Tx_n = \frac{x_n}{2}$ produces a sequence converging to 0, and 0 is indeed a fixed point of T.

Taking $m = n = q = p = 1$ in Corollary 3.1 we obtain the following special case.

Corollary 3.3. *Let* (X,S) *be a complete* S*-metric space and let* $T : X \to X$ *be an* α*-admissible mapping. Assume there exists a function* $\beta : [0,\infty) \to [0,1]$ *such that for any bounded sequence* $\{t_n\}$ *of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and for all $x,y,z \in X$*,*

$$\alpha(x,x,Tx)\,\alpha(y,y,Ty)\,\alpha(z,z,Tz)\,S(Tx,Ty,Tz) \leq \beta\big(S(x,y,z)\big)\,S(x,y,z). \tag{3.2}$$

Moreover, suppose that one of the following conditions holds:

(i) T *is continuous; or*
(ii) *if* $\{x_n\}$ *is a sequence in* X *converging to* x *and* $\alpha(x_n,x_n,x_{n+1}) \geq 1$ *for every* n*, then* $\alpha(x,x,Tx) \geq 1$.

If there exists a point $x_0 \in X$ *such that* $\alpha(x_0,x_0,Tx_0) \geq 1$*, then* T *possesses a fixed point.*

Now, selecting the functions

$$h(x,y,z,t) = (t+l)^{xyz} \quad \text{and} \quad Q(s,t) = st + l,$$

with $l > 1$, in Theorem 2.6 yields another corollary.

Corollary 3.4. *Let (X,S) be a complete S-metric space and let $T : X \to X$ be an α-admissible mapping. Assume there exists a function $\beta : [0,\infty) \to [0,1]$ such that for any bounded sequence $\{t_n\}$ of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and for all $x,y,z \in X$ with $l \geq 1$,

$$\big(S(Tx,Ty,Tz) + l\big)^{\alpha(x,x,Tx)\,\alpha(y,y,Ty)\,\alpha(z,z,Tz)} \leq \beta\big(S(x,y,z)\big)\, S(x,y,z) + l. \tag{3.3}$$

Moreover, suppose that one of the following conditions holds:

(i) T is continuous; or
(ii) if $\{x_n\}$ is a sequence in X converging to x and $\alpha(x_n,x_n,x_{n+1}) \geq 1$ for every n, then $\alpha(x,x,Tx) \geq 1$.

If there exists a point $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$, then T possesses a fixed point.

Example 3.5 (Illustrating Corollary 3.4)**.** Consider $X = [0,1]$ endowed with the S-metric

$$S(x,y,z) = |x-y| + |y-z| + |z-x|.$$

Define $T : X \to X$ by $Tx = \dfrac{x}{3}$ and $\alpha : X^3 \to [0,\infty)$ by

$$\alpha(x,y,z) = \begin{cases} 3, & \text{if } x,y,z \in \big[0,\tfrac{1}{2}\big], \\ 0, & \text{otherwise.} \end{cases}$$

Take $l = 2$ and $\beta(t) = \dfrac{1}{3}$ for all $t \geq 0$. For $x,y,z \in [0,\frac{1}{2}]$ we have $\alpha(x,x,Tx) = \alpha(y,y,Ty) = \alpha(z,z,Tz) = 3$, and

$$S(Tx,Ty,Tz) = \frac{1}{3}S(x,y,z).$$

Hence inequality (3.3) becomes

$$\Big(\frac{1}{3}S(x,y,z) + 2\Big)^{27} \leq \frac{1}{3}S(x,y,z) + 2.$$

For $S(x,y,z) = 0$ the left-hand side equals $2^{27} \approx 1.34 \times 10^8$, while the right-hand side is 2, so the inequality fails. This shows that the parameters l and the form of β must be chosen carefully for the contraction condition to be satisfied. In fact, with the given choices the hypothesis of Corollary 3.4 is not fulfilled; the example serves to highlight the need for verifying the assumptions of the theorem in concrete situations.

Selecting the functions

$$h(x,y,t) = (x+l)^{y} \quad \text{and} \quad Q(s,t) = (1+l)^{st},$$

with $l > 0$, in Theorem 2.6 we derive the next corollary. Note that here h is regarded as a function of three variables (the first two being α-values and the third being the S-metric), which corresponds to a subclass of type III where one of the first three arguments is constant.

Corollary 3.6. *Let (X, S) be a complete S-metric space and let $T : X \to X$ be an α-admissible mapping. Assume there exists a function $\beta : [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and for all $x, y, z \in X$,

$$\big(\alpha(x, x, Tx)\, \alpha(y, y, Ty)\, \alpha(z, z, Tz) + l\big)^{S(Tx,Ty,Tz)} \le (1 + l)^{\beta(S(x,y,z))\, S(x,y,z)}. \tag{3.4}$$

Moreover, suppose that one of the following conditions holds:

(i) T is continuous; or

(ii) if $\{x_n\}$ is a sequence in X converging to x and $\alpha(x_n, x_n, x_{n+1}) \ge 1$ for every n, then $\alpha(x, x, Tx) \ge 1$.

If there exists a point $x_0 \in X$ such that $\alpha(x_0, x_0, Tx_0) \ge 1$, then T possesses a fixed point.

Putting $l = 1$ in Corollary 3.6 we obtain a further specialization.

Corollary 3.7. *Let (X, S) be a complete S-metric space and let $T : X \to X$ be an α-admissible mapping. Assume there exists a function $\beta : [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals,*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0,$$

and for all $x, y, z \in X$,

$$\big(\alpha(x, x, Tx)\, \alpha(y, y, Ty)\, \alpha(z, z, Tz) + 1\big)^{S(Tx,Ty,Tz)} \le 2^{\beta(S(x,y,z))\, S(x,y,z)}. \tag{3.5}$$

Moreover, suppose that one of the following conditions holds:

(i) T is continuous; or

(ii) if $\{x_n\}$ is a sequence in X converging to x and $\alpha(x_n, x_n, x_{n+1}) \ge 1$ for every n, then $\alpha(x, x, Tx) \ge 1$.

If there exists a point $x_0 \in X$ such that $\alpha(x_0, x_0, Tx_0) \ge 1$, then T possesses a fixed point.

Remark 3.8. The corollaries presented above—specifically Corollary 3.3, Corollary 3.4 and Corollary 3.7—recover the main theorems of Shimpi and Dapke [6]. Thus our Theorem 2.6 provides a unified framework that encompasses those earlier results as particular cases.

4. Conclusion and Future Work

In this paper, we have introduced a novel class of contraction mappings defined by a pair (Q, h) of type III and established a generalized fixed point theorem for α-admissible mappings in complete S-metric spaces. Our main result extends and unifies several known fixed point theorems, most notably those of Shimpi and Dapke [6]. By selecting particular forms of the functions h and Q, we have derived a series of corollaries that demonstrate the versatility and broad applicability of our theorem.

The framework developed here provides a unified perspective on fixed point theory in S-metric spaces and suggests several directions for future investigation. Possible extensions include:

- studying the presented contractions in partially ordered S-metric spaces;
- investigating coupled, tripled or quadruple fixed point problems within this setting;
- applying the results to solve integral equations, differential equations, or other non-linear functional equations;

- exploring the corresponding fixed point results in other generalized metric spaces, such as b-metric spaces, cone metric spaces, or complex-valued metric spaces.

We believe that the ideas presented in this work can stimulate further research in this active area of nonlinear analysis.

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