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Research Paper

ON LOCALLY DUALLY FLAT GENERALIZED BERWALD (α, β)-MANIFOLDS

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Abstract. Locally dually flat Finsler metrics form an interesting class of Finsler metrics in Finsler information geometry. Every locally Minkowskian metric is locally dually flat. But there are many locally dually flat metrics that are not locally Minkowskian. In this paper, we study the class of locally dually flat generalized Berwald (α, β)-metrics and find the conditions under which these metrics reduce to locally Minkowskian metrics.

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1. Introduction

The base of information geometry come back to the interesting and seminal work of Rao in 1992, where he considered the Fisher matrix as a Riemannian metric [9]. But the modern theory is due to the Amari's lecture notes in statistics, whose work has been major influential on the development of the field [1]. In information geometry, a parametrized statistical model considered as a Riemannian manifold. For such models, there is a natural choice of Riemannian metric, known as the Fisher information metric. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory.

The class of dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which plays a very important role in studying flat Finsler information structure. A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point $x \in M$ there is a standard coordinate system (x^i, y^i) in TM satisfies following

$$[F^2]_{x^k y^l}\, y^k = 2\,[F^2]_{x^l}\,. \tag{1.1}$$

In this case, at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

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where $H = H(x, y)$ is a C^∞ scalar function on slit tangent bundle $TM_0 = TM \setminus \{0\}$ satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system [8][24].

There is an important non-Riemannian quantity that is very close to the Berwald metrics and generalized Berwald metrics, namely, S-curvature. It is constructed by Z. Shen to study of comparison theorems for Finsler manifolds [11]. A natural problem is to study and characterize Finsler metrics of vanishing S-curvature. It is known that some of Randers metrics are of vanishing S-curvature [10][17]. This is one of our motivations to consider Finsler metrics with vanishing S-curvature. Shen proved that every Berwald metric satisfies $\mathbf{S} = 0$ [11]. In [2], Bao-Shen find a class of non-Berwaldian Randers metrics with vanishing S-curvature.

Locally Minkowskian metrics are trivially Berwald metrics. Also, it is easy to see that every locally Minkowskian metric satisfies (1.1), hence is locally dually flat [18][19][20]. A natural question arises:

Question. *Under which conditions locally dually flat metrics are locally Minkowskian?*

To find some solutions for the above question, one can consider the class of generalized Berwald metrics. A Finsler metric F on a manifold M is called a generalized Berwald metric if there exists a covariant derivative ∇ on M such that the parallel translations induced by ∇ preserve the Finsler function F [14][22]. In this case, (M, F) is called a generalized Berwald manifold. If ∇ is also torsion-free, then F reduces to a Berwald metric.

To find some solutions for the above question, one can consider the class of (α, β)-metrics. An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M (see [13]). The simplest (α, β)-metrics are the Randers metrics $F = \alpha + \beta$ which were discovered by G. Randers when he studied 4-dimensional general relativity. For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, let us define

$$k_1 := \Pi(0), \qquad k_2 := \frac{\Pi'(0)}{Q(0)}, \qquad k_3 := \frac{1}{6Q(0)^2}\Big[3Q''(0)\Pi'(0) - 6\Pi'(0)^2 - Q(0)\Pi'''(0)\Big],$$

where

$$Q(s) := \frac{\phi'(s)}{\phi(s) - s\phi'(s)}, \qquad \Pi(s) := \frac{\phi'(s)\phi'(s) + \phi(s)\phi''(s)}{\phi(s)\big(\phi(s) - s\phi'(s)\big)}.$$

Here, we find some conditions under which a locally dually flat non-Randers type (α, β)-metric reduces to a locally Minkowskian metric. More precisely, we prove the following.

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type generalized Berwald (α, β)-metric on a manifold M of dimension $n \geq 3$ such that*

$$\phi'(0) \neq 0, \quad b^2(k_2 - k_3 b^2) + 1 \neq 0. \tag{1.2}$$

Then, F is a locally dually flat with vanishing S-curvature if and only if it is a locally Minkowskian metric.

In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively [6][7][16].

2. Preliminary

A Finsler metric on a manifold M is a nonnegative function F on TM having the following properties

(a) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;
(c) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\Big[F^2(y+su+tv)\Big]\Big|_{s,t=0}, \qquad u,v \in T_xM.$$

Then the pair (M,F) is called a Finsler manifold.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on TM_0 satisfying

$$G^i(x,\lambda y) = \lambda^2 G^i(x,y) \quad \lambda > 0.$$

$\mathbf{G}$ is called the associated spray to (M,F). The projection of an integral curve of G is called a geodesic in M.

A Finsler metric $F = F(x,y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j},$$

where $H = H(x,y)$ is a C^∞ scalar function on TM_0 satisfying $H(x,\lambda y) = \lambda^3 H(x,y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system [5]. In [10], Shen proved that the Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies

$$(F^2)_{x^k y^l}y^k = 2(F^2)_{x^l}.$$

In this case, $H = -\frac{1}{6}[F^2]_{x^m}y^m$.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1\cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i)\in\mathbb{R}^n \;\Big|\; F\Big(y^i|_x\Big) < 1\Big\}}.$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary functions, even F is locally expressed by elementary functions.

Let $G^i(x,y)$ denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{G^i}{y^i}(x,y) - y^i\frac{}{x^i}\Big[\ln\sigma_F(x)\Big].$$

where $\mathbf{y} = y^i|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric [10]. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$ [2].

Given a Riemannian metric α, a 1-form β on a manifold M, and a C^∞ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o := \sup_{x\in M}\|\beta\|_x$, one can define a function on TM by

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}.$$

If ϕ and b_o satisfy (**??**) and (**??**) below, then F is a Finsler metric on M. Finsler metrics in this form are called (α, β)-metrics. Randers metrics are special (α, β)-metrics.

Now we consider (α, β)-metrics. Let $\alpha = \sqrt{a_{ij}y^iy^j}$ be a Riemannian metric and $\beta = b_iy^i$ a 1-form on a manifod M. Let

$$\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

For a C^∞ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o = \sup_{x\in M}\|\beta\|_x$, define

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}.$$

By a direct computation, we obtain

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j - \rho_1(b_i\alpha_j + b_j\alpha_i) + s\rho_1\alpha_i\alpha_j,$$

where $\alpha_i := a_{ij}y^j/\alpha$, and

$$\begin{aligned}\rho &:= \phi(\phi - s\phi'),\\ \rho_0 &:= \phi\phi'' + \phi'\phi',\\ \rho_1 &:= s(\phi\phi'' + \phi'\phi') - \phi\phi'.\end{aligned}$$

By further computation, one obtains

$$\det(g_{ij}) = \phi^{n+1}\left(\phi - s\phi'\right)^{n-2}\Big[(\phi - s\phi') + (\|\beta\|_x^2 - s^2)\phi''\Big]\det(a_{ij}).$$

Using the continuity, one can easily show that

Lemma 2.1. *Let $b_o > 0$. $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric on M for any pair $\{\alpha, \beta\}$ with $\sup_{x\in M}\|\beta\|_x \le b_o$ if and only if $\phi = \phi(s)$ satisfies the following conditions:*

$$\begin{aligned}&\phi(s) > 0, \qquad (|s| \le b_o),\\ &\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \le b \le b_o).\end{aligned}$$

Let

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \quad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big).$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad r_{i0} := r_{ij}y^j, \quad r_{00} := r_{i0}y^i, \quad s_{i0} := s_{ij}y^j, \quad r_0 := r_jy^j, \quad s_0 := s_jy^j.$$

Suppose that $G^i = G^i(x, y)$ and $\bar{G}^i = \bar{G}^i(x, y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we obtain the following identity

$$G^i = \bar{G}^i + Py^i + Q^i, \tag{2.1}$$

where

$$\begin{aligned}
P &= \alpha^{-1}\Theta\Big[r_{00}-2Q\alpha s_0\Big]\\
Q^i &= \alpha Q s^i{}_0+\Psi\Big[r_{00}-2Q\alpha s_0\Big]b^i,\\
Q &= \frac{\phi'}{\phi-s\phi'},\\
\Theta &= \frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi\Big((\phi-s\phi')+(b^2-s^2)\phi''\Big)},\\
\Psi &= \frac{1}{2}\frac{\phi''}{(\phi-s\phi')+(b^2-s^2)\phi''}.
\end{aligned}$$

Clearly, if β is parallel with respect to α ($r_{ij}=0$ and $s_{ij}=0$), then $P=0$ and $Q^i=0$. In this case, $G^i=\bar{G}^i$ are quadratic in y, and F is a Berwald metric.

3. Proof of Theorem 1.1

In this section, we will prove a generalized version of Theorem 1.1. Indeed, we prove the following.

Theorem 3.1. *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a non-Riemannian regular generalized Berwald (α,β)-metric on a manifold M of dimension $n\geq 3$ such that*

$$\phi'(0)\neq 0,\qquad (k_2-k_3b^2)b^2+1\neq 0. \tag{3.1}$$

Suppose that F has isotropic S-curvature

$$\mathbf{S}=(n+1)cF, \tag{3.2}$$

where $c=c(x)$ is a scalar function on M. Then F is locally dually flat if and only it is a locally Minkowskian metric.

To prove Theorem 3.1, we need the following key lemma.

Lemma 3.2. ([21]) An (α,β)-metric satisfying $\phi'(0)\neq 0$ is a generalized Berwald manifold if and only if β has constant length with respect to α.

A Finsler metric F on an n-dimensional manifold M is called of isotropic S-curvature, if

$$\mathbf{S}=(n+1)cF,$$

where $c=c(x)$ is a scalar function on M. In [4], Cheng-Shen characterized (α,β)-metrics with isotropic S-curvature on a manifold M of dimension $n\geq 3$. Soon, they found that their result holds for the class of (α,β)-metrics with constant length one-forms, only. In [15], we give a new characterization of the class of generalized Berwald metrics with vanishing S-curvature and prove the following.

Lemma 3.3. ([15]) Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a generalized Berwald (α,β)-metric on an n-dimensional manifold M. Suppose that $\phi'(0)\neq 0$. Then $\mathbf{S}=0$ if and only if β is a Killing form with constant length, namely

$$r_{ij}=0,\qquad s_j=0. \tag{3.3}$$

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α,β)-metric on a manifold M. Let us define

$$\Pi := \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')},$$
$$k_1 := \Pi(0),$$
$$k_2 := \frac{\Pi'(0)}{Q(0)},$$
$$k_3 := \frac{1}{6Q(0)^2}\Big[3Q''(0)\Pi'(0) - 6\Pi'(0)^2 - Q(0)\Pi'''(0)\Big].$$

For the class of non-Riemannian (α,β)-metrics, we have the following.

Lemma 3.4. ([23]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$, where $\phi'(0) \neq 0$ and $\beta \neq 0$. Then F is locally dually flat if and only if α, β and ϕ satisfy

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l), \tag{3.4}$$

$$r_{00} = \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\tau\beta^2, \tag{3.5}$$

$$G^l_\alpha = \frac{1}{3}\Big[2\theta + (3k_1 - 2)\tau\beta\Big]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l, \tag{3.6}$$

$$\tau\Big[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')\Big] = 0, \tag{3.7}$$

where $\tau := \tau(x)$ is a scalar function, $\theta := \theta_i(x)y^i$ is a 1-form on M and $\theta^l = a^{lm}\theta_m$.

By using the same argument applied in Lemma 3.4, Xia proved the following.

Lemma 3.5. ([23]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M of dimension $n \geq 3$ with the same assumption as Lemma 3.4. Let ϕ satisfies

$$s(k_2 - k_3 s^2)(\phi\phi' - s\phi' - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0.$$

Then F is locally dually flat on M if and only if

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l), \tag{3.8}$$

$$r_{00} = \frac{2}{3}\left[\theta\beta - (\theta_l b^l)\alpha^2\right], \tag{3.9}$$

$$G^l_\alpha = \frac{1}{3}\left[2\theta y^l + \theta^l\alpha^2\right]. \tag{3.10}$$

where k_i, $(1 \leq i \leq 3)$ are the same as those of Lemma 3.4.

Proof of Theorem 3.1: Let $F := \alpha\phi(\beta/\alpha)$ be a regular (α,β)-metric of non-Randers type on an n-dimensional manifold M. In [3], Cheng proved that F is of isotropic S-curvature,

$$\mathbf{S} = (n+1)\sigma F,$$

if and only if β satisfies

$$r_{ij} = 0, \quad s_j = 0,$$

where $\sigma=\sigma(x)$ is a scalar function on M. In this case, $\mathbf{S}=0$, regardless of the choice of a particular $\phi=\phi(s)$. Now, we can prove Theorem 3.1. For this aim, we consider two main cases as follows:

Case(1): Let $\phi'(0)\neq 0$ such that the following holds

$$s(k_2-k_3s^2)(\phi\phi'-s\phi'-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')\neq 0.$$

By Lemma 3.5, we have

$$s_{l0}=\frac{1}{3}\Big[\beta\theta_l-\theta b_l\Big], \tag{3.11}$$

$$r_{00}=\frac{2}{3}\Big[\theta\beta-(\theta_l b^l)\alpha^2\Big], \tag{3.12}$$

$$G^l_\alpha=\frac{1}{3}\Big[2\theta y^l+\theta^l\alpha^2\Big], \tag{3.13}$$

Contracting (3.11) with b^l yields

$$s_0=\frac{1}{3}\Big[\theta_l b^l\beta-b^2\theta\Big]. \tag{3.14}$$

According to assumption, $s_i=0$. By putting it in (3.14), we get

$$\theta=\frac{1}{b^2}\theta_m b^m\beta. \tag{3.15}$$

Putting (3.15) into (3.12) implies that

$$r_{00}=\frac{1}{b^2}\Big[\beta^2-b^2\alpha^2\Big]\theta_m b^m. \tag{3.16}$$

Putting $r_{ij}=0$ in (3.16) gives us

$$\theta_m b^m=0. \tag{3.17}$$

Comparing (3.17) and (3.15) we get

$$\theta=0. \tag{3.18}$$

Putting (3.18) in (3.11) and (3.13) imply

$$s_{ij}=0,\quad \text{and}\quad G^l_\alpha=0, \tag{3.19}$$

respectively. Putting (3.19) in (2.1) yields $G^l=0$. In this case, F reduces a locally Minkowskian metric.

Case(2): Let $\phi'(0)\neq 0$ such that the following holds

$$s(k_2-k_3s^2)(\phi\phi'-s\phi'-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')=0.$$

According to Lemma 3.4, we have

$$s_{l0}=\frac{1}{3}\Big(\beta\theta_l-\theta b_l\Big), \tag{3.20}$$

$$r_{00}=\frac{2}{3}\theta\beta+\Big[\tau+\frac{2}{3}(b^2\tau-\theta_l b^l)\Big]\alpha^2+\frac{1}{3}\left(3k_2-2-3k_3b^2\right)\tau\beta^2, \tag{3.21}$$

$$G^l_\alpha=\frac{1}{3}\Big[2\theta+(3k_1-2)\tau\beta\Big]y^l+\frac{1}{3}(\theta^l-\tau b^l)\alpha^2+\frac{1}{2}k_3\tau\beta^2b^l. \tag{3.22}$$

Contracting (3.20) with b^l yields

$$s_0 = \frac{1}{3}\Big[\theta_l b^l \beta - b^2\theta\Big]. \tag{3.23}$$

Putting $s_i = 0$ in (3.23) implies that

$$\theta = \frac{1}{b^2} b^l \theta_l \beta. \tag{3.24}$$

Also, by putting (3.24) in (3.21) it follows that

$$r_{00} = \frac{2}{3b^2}\theta_l b^l \beta^2 + \Big[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\Big]\alpha^2 + \frac{1}{3}\Big(3k_2 - 2 - 3k_3 b^2\Big)\tau\beta^2. \tag{3.25}$$

Since $r_{ij} = 0$, then (3.25) reduces to following

$$\frac{2}{3b^2}\theta_l b^l \beta^2 + \Big[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\Big]\alpha^2 + \frac{1}{3}\Big(3k_2 - 2 - 3k_3 b^2\Big)\tau\beta^2 = 0. \tag{3.26}$$

Differentiating (3.26) with respect to y^m yields

$$\frac{4}{3b^2}\theta_l b^l \beta b_m + 2\Big[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\Big]y_m + \frac{2}{3}\Big(3k_2 - 2 - 3k_3 b^2\Big)\tau\beta b_m = 0. \tag{3.27}$$

Multiplying (3.27) with b^m gives us

$$2\Big[(k_2 - k_3 b^2)b^2 + 1\Big]\tau\beta = 0. \tag{3.28}$$

By assumption, we have $(k_2 - k_3 b^2)b^2 + 1 \neq 0$. Then (3.28) reduces to following

$$\tau = 0. \tag{3.29}$$

Putting (3.29) in (3.20), (3.21) and (3.22), yield

$$s_{l0} = \frac{1}{3}\Big(\beta\theta_l - \theta b_l\Big), \tag{3.30}$$

$$r_{00} = \frac{2}{3}\Big(\theta\beta - \theta_l b^l \alpha^2\Big), \tag{3.31}$$

$$G^l_\alpha = \frac{2}{3}\theta y^l + \frac{1}{3}\theta^l \alpha^2, \tag{3.32}$$

which are (3.8), (3.9) and (3.10), exactly. Then, by the same argument used in Case (1), we get the proof. □

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