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Research Paper

BASIS PROPERTY AND MINIMALITY OF THE EIGENFUNCTIONS OF THE MODIFIED FRANKL PROBLEM WITH A NONLOCAL ODDNESS CONDITION IN THE SPACE $\overline{W}_p^{2l}(0,\pi)$

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ABSTRACT. The classical Frankl problem was considered in [3]. The problem was further developed in [2, pp.339-345], [8, pp.235-252]. The modified Frankl problem with a nonlocal boundary condition of the first kind was studied in [1, 6]. The basis property of an eigenfunctins of the Frankl problem with a nonlocal parity conditions in the space sobolev was studied in[7]. In the present paper, we prove the completeness, the basis property, and the minimality of the eigenfunctions in the space $\overline{W}_p^{2l}(0,\pi)$. This analysis may be of interest in itself.

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1. STATEMENT OF THE MODIFIED FRANKL PROBLEM

Definition 1.1. In the domain $D = (D_+ \cup D_{-1} \cup D_{-2})$, we seek a solution of the modified generalized Frankl problem

$$u_{xx} + \operatorname{sgn}(y)u_{yy} + \mu^2 \operatorname{sgn}(x+y)u = 0 \qquad \text{in} \quad (D_+ \cup D_{-1} \cup D_{-2}), \tag{1.1}$$

with the boundary conditions

$$u(1,\theta) = 0, \theta \in [0, \frac{\pi}{2}], \tag{1.2}$$

$$\frac{\partial u}{\partial x}(0,y) = 0, y \in (-1,0) \cup (0,1) \tag{1.3}$$

$$ku(0,y) = u(0,-y), y \in [0,1].\ ku(0,+0) = u(x,-0). \tag{1.4}$$

where $u(x,y)$ is a regular solution in the class

$$u \in C^0(\overline{D_+ \cup D_{-1} \cup D_{-2}}) \cap C^2(D_{-1}) \cap C^2(D_{-2}),$$

and where

$$D_+ = \{(r,\theta) : 0 < r < 1, 0 < \theta < \frac{\pi}{2}\},$$

$$D_{-1} = \{(x,y) : -y < x < y+1, \frac{-1}{2} < y < 0\},$$

$$D_{-2} = \{(x,y) : x-1 < y < -x, 0 < x < \frac{1}{2}\},$$

$$\kappa \frac{\partial u}{\partial y}(x,+0) = \frac{\partial u}{\partial y}(x,-0), -\infty < \kappa < \infty, 0 < x < 1. \tag{1.5}$$

Definition 1.2. .System $\{x_n\}_{n\in N} \subset X$ is called complete in X if $\overline{L[\{x_n\}_{n\in N}]} = X$.

Definition 1.3. .System $\{x_n\}_{n\in N} \subset X$ is called minimal in X if $x_k \notin \overline{L[\{x_n\}_{n\in N}]}, \forall k \in N$.

Remark 1.4. If the system $\{x_n\}_{n\in N} \subset X$ is called minimal in $L_p(I)$, then it is also minimal in $L_p(J)$, for $J \supset I$, and if it is complete in $L_p(I)$, then it is also complete in $L_p(J)$, for $J \subset I$.

Theorem 1.5 ([5]). *The eigenvalues and eigenfunctins of problem (1-5) can be written out in two serise. In the first series, the eigenvalues $\lambda = \mu_{nk}^2$ are found from the equation*

$$J_{4n}(\mu_{nk}) = 0, \tag{1.6}$$

where $\mu_{nk}, n = 0,1,2,..., k = 1,2,...,$ are roots of the Bessel equation (6), $J_\alpha(z)$ is the Bessel function [4], and the eigenfunctins are given by the formula

$$u_{nk} = \begin{cases} A_{nk}J_{4n}(\mu_{nk}r)\cos(4n)(\frac{\pi}{2}-\theta), & in \quad D^+; \\ kA_{nk}J_{4n}(\mu_{nk}\rho)\cosh(4n)\psi, & in \quad D_{-1}; \\ kA_{nk}J_{4n}(\mu_{nk}R)\cosh(4n)\varphi, & in \quad D_{-2}, \end{cases} \tag{1.7}$$

where $x = r\cos\theta, y = r\sin\theta$ for $0 \le \theta \le \frac{\pi}{2}, r^2 = x^2+y^2$ in D_+, $x = \rho\cosh\psi, y = \rho\sinh\psi$, for, $0 < \rho < 1, -\infty < \psi < 0, \rho^2 = x^2-y^2$, in D_{-1} and $x = R\sinh\varphi, y = -R\cosh\varphi$, for, $0 < \varphi < +\infty, R^2 = y^2-x^2$, in D_{-2}.

In the second series, the eigenvalues $\tilde{\lambda} = \tilde{\mu}_{nk}^2$ are found from the equation.

$$J_{4(n+\triangle)}(\tilde{\mu}_{nk}) = 0. \tag{1.8}$$

Where $n = 1,2,...,$ and $k = 1,2,...,$ and the $(\tilde{\mu}_{nk})$ are the roots of the Bessel equation (8).

$$\tilde{u}_{nk} = \begin{cases} \tilde{A}_{nk}J_{4(n+\triangle)}(\tilde{\mu}_{nk}r)\cos 4(n+\triangle)(\frac{\pi}{2}-\theta), & in \quad D^+; \\ \tilde{A}_{nk}J_{4(n+\triangle)}(\tilde{\mu}_{nk}\rho)[\cosh 4(n+\triangle)\varphi\cos 4(n+\triangle)\frac{\pi}{2} & \\ \qquad +\kappa\sinh 4(n+\triangle)\psi\cos 4(n+\triangle)], & in \quad D_{-1}; \\ k\tilde{A}_{nk}J_{4(n+\triangle)}(\tilde{\mu}_{nk}R)\cosh 4(n+\triangle)\varphi[\cos 4(n+\triangle)\frac{\pi}{2} & \\ \qquad -\sin 4(n+\triangle)\frac{\pi}{2}], & in \quad D_{-2}, \end{cases} \tag{1.9}$$

where, $\Delta = \frac{1}{\pi}\arcsin\frac{\kappa}{\sqrt{1+\kappa^2}}, \Delta \in (0, \frac{1}{2})$, and

$$A_{nk}^2 \int_0^1 J_{4n}^2(\mu_{nk}r)rdr = 1,$$

$$\tilde{A}_{nk}^2 \int_0^1 J_{4n+\triangle}^2(\tilde{\mu}_{nk}r)rdr = 1,$$

$A_{nk} > 0$ and $\tilde{A}_{nk} > 0$.

Definition 1.6. Let $\beta < 2 - \frac{1}{p}$. Let $(\widetilde{W}_p^{2l}(0,\pi))$ be the subspace of the space $(W_p^{2l}(0,\pi))$ consisting of functions $f \in (W_p^{2l}(0,\pi))$ satisfying the following boundary conditions:

$$f^{2k}(0) = 0, \qquad (k = 0, 1, ..., l-1) \tag{1.10}$$

And, for $\beta < 1$, let them satisfy condition:

$$\int_0^\pi f^{(2k-1)(\theta)} \widetilde{H}_0^\beta d\theta = 0, \qquad (k = 1, 2, 3, ..., l)$$

where

$$\widetilde{H}_0^\alpha = \frac{\Gamma^2(1-\frac{\alpha}{2})}{\Gamma(1-\alpha)\pi(2\cos\frac{\theta}{2})^\alpha}, \qquad (\alpha = \beta - 2).$$

This restriction on β is connected with applied problems and is natural in this sense.

Definition 1.7. Let $\beta < 2 - \frac{1}{p}$, and let $\overline{W}_p^{2l}(0,\pi)$ be the set of functions $f \in (W_p^{2l}(0,\pi))$ satisfying the following conditions:

$$f^{2k-1}(0) = 0, \qquad (k = 1, ..., l)$$

And also the following conditions depending on the parameter β : For $\beta < 1$,

$$\int_0^\pi f^{(2k)}(\theta)\widetilde{H}_0^\beta d\theta = 0, \qquad (k = 1, 2, 3, ..., l-1) \tag{1.11}$$

and for $\beta \geq 1$,

$$\int_0^\pi (f^{(2k)} - \frac{f^{2l}(-1)^{l-k}}{(1-\frac{\beta}{2})^{2l-2k}})H_0^{\beta-2} d\theta = 0, \qquad (k = 1, 2, 3, ..., l-1) \tag{1.12}$$

$$H_n^\alpha = \frac{2}{\pi(2\cos\frac{\theta}{2})^\alpha}\{\sum_{k=0}^n C_\alpha^k \cos(n-k)\theta - \frac{C_\alpha^n}{2}\} \qquad (n \geq 0)$$

and

$$h_n^\beta = \frac{2}{\pi(2\cos\frac{\theta}{2})^\beta}\sum_{k=0}^{n-1} C_\beta^k \sin(n-k)\theta.$$

Remark 1.8. For $\beta = 1$, condition (12) transforms to the condition $f^{2k-2}(\pi) = 0, k = 2, 3, ..., l$ and for $l = 1$ conditions (11) and (12) had not exist.

2. THE BASIS PROPERTY AND MINIMALITY OF THE EIGENFUNCTIONS

Theorem 2.1. *The function system*

$$\{\cos(4n)(\frac{\pi}{2} - \theta)\}_{n=0}^\infty, \{\cos 4(n+\triangle)(\frac{\pi}{2} - \theta)\}_{n=1}^\infty, \tag{2.1}$$

is complete and a Riesz basis in $L_2(0, \frac{\pi}{2})$, provided that $\triangle \in (\frac{-1}{4}, \frac{1}{2})$.

For $\triangle < \frac{-1}{4}$ the system is not complete but is minimal,for $\triangle > \frac{3}{4}$ is complete but is not minimal,and if $\triangle = \frac{-1}{4}$ is complete and minimal.

Proof. The proof of this theorem we use the convergence function

$$f(\theta)=\sum_{n=0}^{\infty}A_n\cos 4n(\frac{\pi}{2}-\theta)+\sum_{n=1}^{\infty}B_n\cos 4(n+\triangle)(\frac{\pi}{2}-\theta), \tag{2.2}$$

in $L_2(0,\frac{\pi}{2})$, Riesz basis the system $(\sin 4(n+\triangle)(\frac{\pi}{2}-\theta))$ for $\triangle\in(\frac{-1}{4},\frac{3}{4})$,(see [1, 6]). □

Theorem 2.2. *The system of eigenfunctions*

$$u_{nk}(r,\theta)=A_{nk}J_{4n}(\mu_{nk}r)\cos(4n)(\frac{\pi}{2}-\theta),$$

$$\tilde{u}_{nk}(r,\theta)=\tilde{A}_{nk}J_{4(n+\triangle)}(\tilde{\mu}_{nk}r)[\cosh 4(n+\triangle)\varphi\cos 4(n+\triangle)\frac{\pi}{2},$$

is complete and basis in the space $L^2(0,\frac{\pi}{2})$, *therefor*

$$\int_0^{\frac{\pi}{2}}f(r,\theta)u_{nk}(r,\theta)rdrd\theta=0,$$

$$\int_0^{\frac{\pi}{2}}f(r,\theta)\tilde{u}_{nk}(r,\theta)rdrd\theta=0,$$

and $f\in L(0,\frac{\pi}{2})$ *then f=0 , in* $(0,\frac{\pi}{2})$.

Proof. Using fobini theorem and Lebesgue‘s integral for any n,k=1,2,... we have

$$\begin{aligned}0=&\int_0^{\frac{\pi}{2}}f(r,\theta)u_{nk}(r,\theta)rd\theta dr\\ =&\int_0^1(rJ_{4n}(\mu_{nk}r)\int_0^{\frac{\pi}{2}}f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta)dr,\end{aligned}$$

again since $f\in\ L^2(0,\frac{\pi}{2})$ so;

$$\int_0^1\int_0^{\frac{\pi}{2}}|f(r,\theta)|^2d\theta dr<\infty.$$

In so much system $\{\sqrt{r}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$ in $L^2(0,1)$ is orthogonal and complete, it is enough to prove;

$$\sqrt{r}\int_0^{\frac{\pi}{2}}f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta\ \in\ L^2(0,1).$$

Using the Holder inequality

$$\begin{aligned}|\sqrt{r}\int_0^{\frac{\pi}{2}}f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta|^2&<\frac{1}{2}r\int_0^{\frac{\pi}{2}}|f^2(r,\theta)|d\theta\int_0^{\frac{\pi}{2}}d\theta\\ &=\frac{\pi}{4}r\int_0^{\frac{\pi}{2}}|f(r,\theta)|^2d\theta=\frac{\pi}{4}r\int_0^{\frac{\pi}{2}}|f(r,\theta)|^2d\theta,\end{aligned}$$

with the integration interval (0,1).

$$\int_0^1|\sqrt{r}\int_0^{\frac{\pi}{2}}f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta|^2dr\ <\frac{\pi}{4}\int_0^1\int_0^{\frac{\pi}{2}}r|f(r,\theta)|^2drd\theta<\infty.$$

This inequality is equivalent to

$$\{\int_0^1 \sqrt{r}|\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta|^2 dr\}^{\frac{1}{2}} < \infty.$$

Also system $\{\sqrt{r}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$ is orthogonal and complete in $L^2(0,\frac{\pi}{2})$ of relation

$$\int_0^1 (\sqrt{r}J_{4n}(\mu_{nk}r)\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta)dr = 0,$$

imply that

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta = 0.$$

According to theorem 2,we conclude that $f(r,\theta)=0$ in $L^2(0,1)$. Similarly,if we consider the above calculations for sequence $\{\cos 4(n+\triangle)(\frac{\pi}{2}-\theta)\}_{n=1}^{\infty}$, we have;

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos 4(n+\triangle)(\frac{\pi}{2}-\theta)d\theta = 0.$$

Because completeness $\{\cos 4(n+\triangle)(\frac{\pi}{2}-\theta)\}_{n=0}^{\infty}$, $f(r,\theta)=0$ in $L^2(0,1)$.

The proof of the theorem is complete.

□

Theorem 2.3. *The system of eigenfunctions $u_{nk}(r,\theta)$ and $\tilde{u}_{nk}(r,\theta)$ of the problem (1)-(5)is a Riesz basis in the space $L(0,\frac{\pi}{2})$, where,*

$$A_{nk}^2 = (\int_0^1 J_{4n}^2(\mu_{nk}r)rdr)^{-1},\ \widetilde{A_{nk}^2} = (\int_0^1 J_{4(n+\triangle)}^2(\widetilde{\mu_{nk}}r)rdr)^{-1},$$

Proof. Theorem 3.1 results from Theorem 3.2 and the completeness and orthogonality of the system$\{A_{nk}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$ for $n>0$ and$\{\widetilde{A_{nk}}J_{4(n+\triangle)}(\widetilde{\mu_{nk}})r\}_{k=1}^{\infty}$ for $n>1$ in $L^2(0,1)$.

□

Theorem 2.4. *The system of function $\{\cos(n-\frac{\beta}{2})\theta\}_{n=0}^{\infty}$ is a Riesz basis in $W_p^1(0,\pi)$ if and only if $\beta\in(-\frac{1}{p},2-\frac{1}{p}),\beta\neq 1$.*

Proof. (see [6, 7]).

□

Theorem 2.5. *The cosine system $\{\cos(n-\frac{\beta}{2})\theta\}_{n=0}^{\infty}$ forms a basis in the space $\overline{W}_p^{2l}(0,\pi)$,if and only if $\beta\in(-\frac{1}{p},2-\frac{1}{p}),\beta\neq 1$.The expansion into cosines has the form*

$$f(\theta)=\sum_{n=0}^{\infty} D_n\cos(n-\frac{\beta}{2})\theta. \tag{15}$$

Where the coefficients D_n are calculated according to the formulas

$D(0)=\int_0^{\pi} f(\theta)H_0^{\beta}(\theta)d(\theta)$ for $\beta<1$,

$$D(0)=\frac{8(1-\beta)}{\pi\beta(2-\beta)}\int_0^{\pi}\frac{\sin(\theta)\sin(\frac{\beta\theta}{2})}{(2\cos\frac{\theta}{2})^{\beta}}d(\theta)=\int_0^{\pi} f(\theta)H_0^{\beta}(\theta)d(\theta)+ \tag{16}$$

$\int_0^{\pi}\frac{f'(\theta)h_1^{\beta}}{1-\frac{\beta}{2}}d(\theta)$ for $\beta>1$.

$$D_n = -\int_0^\pi (f' + D_0(\frac{\beta}{2})\sin(\frac{\beta\theta}{2}))h_n^\beta d(\theta)(n-\frac{\beta}{2})^{-1}. \tag{17}$$

Where H_n^β and $h_n^\beta(\theta)$ were studied in [7].

Proof. Analogously to the proof of relation (16),we obtain the relation

$$f(\theta) - f(0) = \sum_{n=0}^{\infty} D_n \cos(n-\frac{\beta}{2})\theta - \sum_{n=1}^{\infty} D_n. \tag{18}$$

The convergence of numerical series$\sum_{n=0}^{\infty} D_n$ is proved analogously to the proof of convergence of series$\sum_{n=1}^{\infty} B_n$. This implies the uniform convergence of series(6).

First let $\beta < 1$, then multiplying series(6) byH_0^β.Integrating over the closed interval $[0,\pi]$, and taking into account relations(6)[6, 9]) and(29), we have the relation

$$f(0) = \sum_{n=0}^{\infty} D_n.$$

Therefore,instead of relation(18),we can write

$$f(\theta) = \sum_{n=0}^{\infty} D_n \cos(n-\frac{\beta}{2})\theta. \tag{19}$$

For $\beta > 0$, we multiply series(31)by $H_0^{\beta-2}(\theta)$ and integrate the obtained relation over the closed interval$[0,\pi]$. Using relation $(9,[11])$, we obtain

$$\int_0^\pi f(\theta)H_0^{\beta-2}(\theta)d(\theta) = D_0\int_0^\pi \cos\frac{\beta\theta}{2}H_0^{\beta-2}(\theta)d(\theta) + D_1 +$$
$$(f(0) - \sum_{n=0}^{\infty} D_n)\int_0^\pi H_0^{\beta-2}(\theta)d(\theta).$$

Substituting the expression for $D_1$from (17)in the latter relation, we obtain

$$\int_0^\pi f(\theta)H_0^{\beta-2}(\theta)d(\theta) - D_0\int_0^\pi \cos\frac{\beta\theta}{2}H_0^{\beta-2}(\theta)d(\theta) +$$
$$\int_0^\pi f'(\theta)h_1^\beta(\theta)d(\theta)\frac{1}{1-\frac{\beta}{2}} + D_0\int_0^\pi \sin\frac{\beta\theta}{2}h_1^\beta(\theta)d(\theta)\frac{\beta}{2(1-\frac{\beta}{2})} =$$
$$(f(0) - \sum_{n=0}^{\infty} D_n)\int_0^\pi H_0^{\beta-2}(\theta)d(\theta). \tag{20}$$

Now let us show that the left-hand side of relation (20)vanishes,this will imply

$$f(0) = \sum_{n=0}^{\infty} D_n.$$

Indeed integrating relation $(9,[8])$ by parts, we obtain the relation

$$\frac{\beta}{2(1-\frac{\beta}{2})}\int_0^\pi H_0^{\beta-2}(\theta)\cos\frac{\beta\theta}{2}d(\theta) = (1-\frac{\beta}{2})\frac{2}{\beta}\int_0^\pi \sin\frac{\beta\theta}{2}h_1^\beta(\theta)d(\theta).$$

Furthermore, substituting this formula in (20),we immediately see that

$$\int_0^{\pi}(f(\theta)H_0^{\beta-2}(\theta)+\frac{f'(\theta)h_1^{\beta}(\theta)}{1-\frac{\beta}{2}})d(\theta)+$$
$$D_0\int_0^{\pi}\sin\frac{\beta\theta}{2}h_1^{\beta}(\theta)d(\theta)(\frac{2}{2-\beta}-\frac{2-\beta}{\beta})=$$
$$\int_0^{\pi}(f(\theta)H_0^{\beta-2}(\theta)+\frac{f'(\theta)h_1^{\beta}(\theta)}{1-\frac{\beta}{2}})d(\theta)+$$
$$(\frac{4D_0(\beta-1)}{\beta(2-\beta)})\int_0^{\pi}\sin\frac{\beta\theta}{2}h_1^{\beta}(\theta)d(\theta).$$

By using relation(16)and(9, [8]), we annihilate the latter relation,i.e.,we obtain relation (19)for$\beta>1$. The remaining part of theorem(3.5)is proved analogously to theorem(3.4). □

Remark 2.6. Case $\kappa>0$. The system of function(10)a Riesz basis in $\overline{W}_p^{2l}(0,\pi)$, if $\triangle\in(\frac{-1}{4},0)\cup(0,\frac{3}{4})$.
If $\triangle\geq\frac{3}{4},\triangle\neq 1,2,3,...$, then system(10)is complete, but is not minimal in $\overline{W}_p^{2l}(0,\pi)$.
If $\triangle=\frac{-1}{4}$, then system(10)is complete and minimal but is not basis in $\overline{W}_p^{2l}(0,\pi)$.
If $\triangle<\frac{-1}{4},\triangle\neq 1,2,3,...$, then system(10)is not complete, but is minimal in $\overline{W}_p^{2l}(0,\pi)$.

Proof. The proof of remark (3.6) reproduces that of theorem(3.4)and theorem(3.5). □

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