Mathematical Analysis & Convex Optimization

Vol. xxxxxx
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.6.1.3

Research Paper

ON QUASI-EINSTEIN-REVERSIBLE RANDERS METRICS

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ABSTRACT. *Quasi*-Einstein metrics play an important role in Finsler geometry, acting as a natural extension of their counterparts in Riemannian geometry [**?**].In this paper, we introduce the concept of *quasi*-Einstein reversibility for Finsler metrics, which extends the standard *quasi*-Einstein condition in this context. We further investigate *quasi*-Ricci-flat and *quasi*-Einstein Randers metrics in detail. Finally, we demonstrate that every *quasi*-Einstein Randers metric possesses isotropic S-curvature, and we establish an equivalence between *quasi*-Einstein reversibility and the *quasi*-Einstein property for Randers metrics.

MSC(2010): 53C60.
Keywords: *quasi*-Einstein, isotropic S-curvature, *quasi*-Einstein-reversible.

1. Introduction

The applications of Finsler geometry in physics primarily include areas such as mechanics, gravitation, and dynamics. Notably, the more generalized gravitational equations were derived by Asanov[**?**]. using the osculating curvature scalar. To this day, the most widely accepted and comprehensive framework for understanding gravity remains Einstein's theory of general relativity [**?**]. The dynamics of Finsler spaces can be formulated using the Einstein–Hilbert functional, which, from a Finslerian standpoint, appears as a special case within a broader class of non-metric Finsler structures [**?**]. The equations that describe the interplay between geometry and matter are none other than Einstein's field equations, which govern the metric of a Riemannian manifold. Within Riemannian geometry, Einstein metrics themselves are solutions to the Einstein field equations in general relativity. Viewed in this way, Finsler metrics can be regarded as a generalization of Riemannian metrics without the constraint of quadratic restriction. In 1997, H. Yasuda and H. Shimada explored Randers spaces with scalar curvature [**?**]. In practical applications, these spaces have also been used in fields like electron optics, particularly in the presence of magnetic fields [**?**].One of the key ideas in Finsler geometry is the Riemann curvature, which naturally leads to the definition of the Ricci curvature, commonly written as $Ric := R^i_i$. A Finsler metric is said to have constant flag curvature if its Ricci curvature satisfies the condition $Ric_{\mathbb{F}} = (n-1)k\mathbb{F}^2$, where $k = k(x)$ is a scalar function. Alongside the Riemann curvature, the S-curvature is another important feature that sets Finsler geometry apart from its Riemannian counterpart. A Finsler metric is described as having isotropic S-curvature on a manifold $\mathbb{M}$ if the S-curvature can be written as $S(x, y) = (n + 1)c\mathbb{F}(x, y)$, where $c = c(x)$ is a smooth scalar function. When this condition holds, the metric is also said to have scalar flag curvature. In general, a Finsler

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metric is considered to have isotropic curvature if and only if its flag curvature is constant. Interestingly, there are special cases such as the well-known Funk metric that show it is possible for a metric to have isotropic flag curvature without being of constant flag curvature. The Funk metric, which is a classic example of a Randers metric, illustrates this exception clearly [?]. In Riemannian geometry, a typical smooth measure space is described by the triple $(\mathbb{M}^n, g, e^{-f}dV_g)$. Here, g on an n-dimensional manifold, dV_g is the standard volume element, and f a smooth real-valued function defined on $\mathbb{M}$. To generalize the Ricci tensor in this setting, one uses the m-Bakry–Émery Ricci tensor, which is defined as

$$Ric_f^m = Ric + Hessf - \frac{1}{m} df \otimes df. \tag{1.1}$$

If the function f is constant, this expression reduces to the ordinary Ricci tensor [?, ?]. Such a space
$(\mathbb{M}^n, g, e^{-f}dV_g)$, is called m- *quasi*-Einstein if it satisfies

$$Ric_f^m = \lambda g,$$

for some $\lambda \in R$. Notably, this condition is exactly the same as the gradient Ricci soliton equation in the special case where $m = \infty$ [?].

Recently, Ohta proposed a definition of N-Ricci curvature, which brings together the concepts of S-curvature and the Ricci curvature in Finsler geometry [?, ?]. To illustrate this, consider a Finsler measure space $\big(\mathbb{M}, \mathbb{F}, e^{-f}dV_{BH}\big)$, where f is a smooth function on the manifold $\mathbb{M}$, and $dV_{BH} = \sigma_{BH}dx$ denotes the Busemann–Hausdorff volume measure Such a space is said to be N–weakly *quasi*-Einstein if it satisfies an equation of the form

$$Ric_{\mathbb{F}} + \dot{S} - \frac{S^2}{N-n} = \big[n-1\big]\bigg[c + \frac{3\theta}{\mathbb{F}}\bigg]\mathbb{F}^2, \qquad n < N \leq +\infty$$

where θ is a 1-form on $\mathbb{M}$ and $\dot{S}$ is the covariant derivative of S along geodesic of Finsler metric $\mathbb{F}$; when $\theta = 0$ and $N = \infty$, Finsler metric $\mathbb{F}$ is said *quasi*-Einstein. Also, *quasi*-Einstein Finsler metric $\mathbb{F}$ is called *quasi*-Ricci flat if $c = 0$. Within Finsler geometry, reversible metrics are of particular interest when studying Einstein metrics. For example, in 2005, Crampin demonstrated that a Randers metric possesses reversible geodesics if and only if the associated 1-form β is parallel [?]. A few years later,in 2008, Mo-Yu determined Randers metrics on compact manifold n-dimensional of constant S-curvature. If $Ric < -(n-1)c^2$ then $\mathbb{F} = \alpha$ is a Riemannian metric[?]. In 2009, Cheng and Shen classified Randers metrics with scalar flag curvature [?] and, Li-Shen later investigated Ricci-quadratic Randers metrics [?]. In 2010, B. Njafi and A. Tayebi introduced a family of Einstein Randers metrics [?]. Two years later, Shen and Yang proved that a Randers metric is Ricci-reversible if and only if it is Ricci-quadratic, and they also studied weak Einstein Randers metrics [?]. In 2015, it was shown that if $\mathbb{F} = \alpha + \beta$ is a Randers metric on an n-dimensional manifold M and β is closed, then F is a generalized Einstein metric precisely when it is a Berwald metric [?]. More recently, in 2022, Zhu analyzed *quasi*-Einstein and *quasi*-Ricci flat square metrics [?]. In this work, we establish necessary and sufficient conditions for a Randers metric to be *quasi*-Einstein and *quasi*-Ricci flat. Furthermore, we examine the structure of *quasi*-Einstein reversibility in Randers metrics and demonstrate that, for this class, *quasi*-Einstein reversibility coincides with the *quasi*-Einstein property. In fact we prove:

Theorem 1.1. *Any quasi-Einstein Randers metrics is of isotropic S-curvature.*

In 2018, the reversibility of Einstein scalar metrics in Finsler geometry was investigated by Yang [?]. In this work, we examine the notion of reversibility for *quasi*-Einstein metrics in the context of Finsler geometry. A Finsler metric is called to be *quasi*-Einstein-reversible if the *quasi*-Einstein scalar is reversible, i.e., $c(x,y)=c(x,-y)$. It is clear, if the *quasi*-Einstein scalar $c(x,y)$ be a scalar on $\mathbb{M}$, namely, $c(x,y)=c(x)$, then F is called an *quasi*-Einstein metric. The main theorem as follows:

Theorem 1.2. *Let $\mathbb{F}=\alpha+\beta$ be a Randers metric on an n-dimensional manifold M with volume form $dV=e^{-f}dV_\alpha$. Then $\mathbb{F}$ is quasi -Einstein -reversible if and only if $\mathbb{F}$ be quasi-Einstein.*

2. **Preliminaries**

Consider the Finsler space $\big(\mathbb{M},\mathbb{F}\big)$, then a smooth vector field G on $T\mathbb{M}_0$ expressed in a standard local coordinate system (x^i,y^i) in $T\mathbb{M}_0$ is given by

$$G(x,y)=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i},$$

where geodesic coefficients defined by

$$G^i(x,y):=\frac{1}{4}g^{im}\Big[[\mathbb{F}^2]_{x^ly^m}y^l-[\mathbb{F}^2]_{x^m}\Big],$$

and $G^i(x,y)$ are local functions on TM satisfying

$$G^i(x,\lambda y)=\lambda^2G^i(x,y)\qquad \lambda>0.$$

The Riemann curvature $R_y=R^i_k(y)\frac{\partial}{\partial x^i}\otimes dx^k$ of F is defined by

$$\boldsymbol{R}^i_k(x,y):=2\frac{\partial G^i}{\partial x^k}-y^j\frac{\partial^2G^i}{\partial x^j\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Ricci curvature is the trace of the Riemann curvature, which is defined by

$$\boldsymbol{R}ic:=\boldsymbol{R}^m_m.$$

For Finsler space $(\mathbb{M},\mathbb{F})$, the Busemann-Hausdorff volume form $dV_{BH}:=\sigma_{BH}(x)dx^1\wedge.......\wedge dx^n$, is defined by

$$\sigma_{BH}:=\frac{w_n}{\big\{Vol(y^i)\in R^n|F\big(x,y^i\frac{\partial}{\partial x^i}|_x\big)\big\}},$$

where $Vol\{.\}$ indicates the Euclidean volum and $w_n:=Vol\big(B^n(1)\big)$ indicates the Euclidean volume of the unit ball on R^n. The Busemann-Hausdorff volume form dV_{BH} is related to the scalar function $\tau=\tau(x,y)$ on $T\mathbb{M}_0$ is defined by

$$\tau(x,y):=ln\left[\frac{\sqrt{det\big(g_{ij}(x,y)\big)}}{\sigma_{BH}}\right],$$

is called the distortion. The S- curvature is defined by

$$S(x,y):=\frac{d}{dt}\big[\tau(c(t),c^{\cdot}(t))\big]|_0,$$

where $c(t)$ is the geodesic with $c(0) = x$ and $c^{\cdot}(x) = y$.
The S-curvature of Finsler metric $\mathbb{F}$ is given by

$$S(x,y) := \frac{\partial G^k}{\partial y^k} - y^k \frac{\partial}{\partial x^k}\big(ln[\sigma_{BH}]\big). \tag{2.1}$$

For a Finsler measure space $(\mathbb{M}, \mathbb{F}, e^{-f}dV_{BH})$, where f is a smooth function on $\mathbb{F}$. The *quasi*-Ricci curvature can be described as

$$\boldsymbol{QRic} := \boldsymbol{R}ic + \dot{S}, \tag{2.2}$$

Finsler metric F is *quasi*-Einstein if it satisfies

$$\boldsymbol{QRic} = (n-1)c\mathbb{F}^2, \tag{2.3}$$

where $c = c(x)$ is a scalar function. In particular, if $\mathbb{F}$ is satisfying (**??**), F is said to be *quasi*-Ricci constant, where $c = const$ and $\mathbb{F}$ is called *quasi*-Ricci flat, where $c = 0$.

Definition 2.1. Let $\mathbb{F}$ be Finsler metirc on manifold $\mathbb{M}$ with volume form $dV_{\mathbb{F}} = e^{-f}dV_{BH}$ which f is scalar function on $\mathbb{M}$ then its *quasi*-Einstein-reversible, if

$$\boldsymbol{QRic} = c(x,y)\mathbb{F}^2, \tag{2.4}$$

with

$$c(x,y) = c(x,-y). \tag{2.5}$$

An (α,β)-metric can be expressed by the form

$$\mathbb{F} := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha},$$

where α is Riemann metric and β is one form on $\mathbb{M}$.
Finsler metric $\mathbb{F}$ is known to be positive and strongly convex on $T\mathbb{M}_0$ if and only if

$$\phi\big(s\big) - s\phi'(s) + \big[B - s^2\big]\phi''(s) > 0,$$

where $B := a^{ij}b_ib_j = \|\beta\|_\alpha^2$.
For (α,β)-metrics, the spray coefficients are expressed by [**?**]

$$G^i = G^i_\alpha + Q^i, \tag{2.6}$$

where

$$Q^i := \alpha Q s^i_0 + \theta\Bigg[r_{00} - 2\alpha Q s_0\Bigg]\frac{y^i}{\alpha} + \psi\Bigg[r_{00} - 2\alpha Q s_0\Bigg]b^i,$$

and

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \theta := \frac{\big[\phi - s\phi'\big]\phi' - s\phi'\phi''}{2\phi\Bigg[\phi - s\phi' + \big[B - s^2\big]\phi''\Bigg]}$$

$$\psi := \frac{\phi''}{2\Bigg[\phi - s\phi' + \big[B - s^2\big]\phi''\Bigg]},$$

where

$$G^i_\alpha = \frac{1}{4}a^{ij}\Big[[\alpha^2]_{x^l y^j}y^k - [\alpha^2]_{x^j}\Big].$$

For (α, β)-metric, the Ricci curvature is given by

$$\boldsymbol{R}ic := \boldsymbol{R}ic_\alpha + T^i_i.$$

For more details, see [**?**]. We use some notations for (α, β)-metrics as follows,

$$r_{ij} = \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \quad s_{ij} = \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big), \quad r_{00} = r_{ij}y^i y^j, \quad s^i_0 = a^{ij}s_{jk}y^k,$$
$$r_i = b^i r_{ji}, \quad s_i = b^j r_{ji}, \quad s_0 = s_i y^i, \quad r^i = a^{ij}r_j, \quad s^i = a^{ij}s_j, \quad r = b^i r_i,$$

where $(a^{ij}) := (a_{ij})^{-1}$, $b^i := a^{ij}b_j$ and "$|$" denotes the covariant derivative with respect to Levi-Civita connection of α, .
For (α, β)-metric with a volume form $dV_{\mathbb{F}} = e^{-f}dV_\alpha$, where $dV_\alpha = \sigma_\alpha dx^1 \wedge ... \wedge dx^n$, $\sigma_\alpha = \sqrt{det(a_{ij}(x))}$ and f is a smooth function, the S-curvature is defined by

$$\begin{aligned} S(x,y): &= \frac{\partial G^k}{\partial y^k} - y^k\frac{\partial}{\partial x^k}\big(ln[e^{-f}\sigma_\alpha]\big) \\ &= 2\psi\big(r_0 + s_0\big) + \big[(n+1)\theta + \psi_s\big(B - s^2\big)\big]\big(r_{00} - 2\alpha Q s_0\big)\frac{1}{\alpha} + f_0, \end{aligned} \tag{2.7}$$

where $f_0 := f_{x^i}y^i$. The covariant derivative of S along a geodesic of (α, β)-metric is calculated. In this paper, we prove the following:

Lemma 2.2. *Suppose*

$$\Big(\frac{r_{00}}{2\alpha} - s_0\Big)^2 \cong 0 \quad mod \quad (1+s), \qquad s = \frac{\beta}{\alpha}. \tag{2.8}$$

Then

$$\big(\frac{1}{2}r_{00} + s_0\beta\big) = \sigma\big(\alpha^2 - \beta^2\big). \tag{2.9}$$

Proof. Equation (**??**) implies that there is a scalar function $\sigma = \sigma(x)$ and 1-form η satisfying

$$\big(\frac{r_{00}}{2\alpha} - s_0\big) = \big(\eta + \sigma\alpha\big)(1+s).$$

Since

$$0 = \frac{1}{2}r_{00} - \sigma\alpha^2 - \alpha\big(s_0 + \eta + \sigma\beta\big) - \eta\beta.$$

Hence, we have

$$0 = s_0 + \eta + \sigma\beta, \tag{2.10}$$
$$0 = -\sigma\alpha^2 + \frac{1}{2}r_{00} - \eta\beta. \tag{2.11}$$

We obtain η from (**??**), and then plugging it into (**??**) yields (**??**). □

3. Quasi-Einstein-Randers metrics

We consider a special class of (α,β)-metric. The Ricci curvature of Randers metric is followed

$$\begin{aligned}\boldsymbol{Ric}_{\mathbb{F}}: &= \boldsymbol{Ric}_{\alpha}+\frac{(n-1)}{A}\Bigg\{\Bigg(\frac{3r_{00}^2}{4\alpha^2}-\frac{3r_{00}s_0}{\alpha}+3s_0^2\Bigg)\frac{1}{A}\\ &\quad -\frac{r_{00|0}}{2\alpha}+2r_{0i}s_0^i+s_{0|0}-2\alpha s_i s_0^i\Bigg\}-2s_{0i}s_0^i,\end{aligned} \tag{3.1}$$

and $A:=1+s$. The S-curvature of Randers metric is given by

$$S=\frac{n+1}{1+s}\big(r_{00}-2\alpha s_0\big)\frac{1}{\alpha}+f_0. \tag{3.2}$$

For Randers metrics, the covariant derivative of S, we can obtain as follows

$$\begin{aligned}S^{\cdot}: &= \frac{(n+1)}{A}\Bigg\{\big(-\frac{r_{00}^2}{\alpha^2}+4\frac{1}{\alpha}-4s_0^2\big)\frac{1}{A}+\frac{r_{00|0}}{2\alpha}-2r_{0i}s_0^i\\ &\quad -s_{0|0}+2\alpha s_i s_0^i-\frac{f_0}{\alpha}r_{00}+2f_0s_0-A\big(2\alpha f_{x^i}s_0^i-f_{0|0}\big)\Bigg\},\end{aligned} \tag{3.3}$$

where $A=1+s$.
We will prove necessary and sufficient condition under which a Randers metric be *quasi*-Einstein.

Lemma 3.1. *Let $\mathbb{F}=\alpha+\beta$ be a Randers metric on an $n-$dimensional manifold $\mathbb{M}$ with volume form $dV=e^{-f}dV_{\alpha}$. Then quasi-Ricci curvature associated with $\mathbb{F}$ can be written as*

$$\begin{aligned}\boldsymbol{QRic} &= -(n+7)\frac{\big(\frac{1}{2}r_{00}-\alpha s_0\big)^2}{F^2}+\frac{2\alpha\big(-2r_{0i}s_0^i-s_{0|0}+2\alpha s_i s_0^i+f_0s_0\big)-f_0r_{00}+r_{00|0}}{F}\\ &\quad +Ric_{\alpha}+f_{0|0}-2s_{0i}s_0^i+\alpha\big(2s_{0|i}^i-\alpha s_j^i s_i^j-2f_{x^i}s_0^i\big),\end{aligned} \tag{3.4}$$

where $f=f(x)$ is a smooth function, and $f_0=f_{x^i}y^i$ and $f_{0|0}=f_{0|x^i}y^i$.

Proof. By (**??**) and (**??**), we obtain

$$\begin{aligned}\boldsymbol{QRic} &= -\frac{(n+7)}{A^2}\Bigg(-\frac{r_{00}^2}{4\alpha^2}+\frac{r_{00}s_0}{\alpha}-s_0^2\Bigg)+\frac{1}{A}\Bigg(\frac{r_{00|0}}{\alpha}-4r_{0i}s_0^i-2s_{0|0}\\ &\quad +4s_is_0^i\alpha-\frac{r_{00}f_0}{\alpha}+2f_0s_0\Bigg)-2f_{x^i}s_0^i\alpha+f_{0|0}+2s_{0|i}^i\alpha\\ &\quad -s_j^is_i^j\alpha^2+Ric_{\alpha},\end{aligned}$$

where $A:=(1+s)$. By replace $A=\frac{\mathbb{F}}{\alpha}$ into equation above , we can get (**??**). ☐

Corollary 3.2. *Consider $\mathbb{F}=\alpha+\beta$ be a Randers metric on an n-dimensional manifold $\mathbb{M}$ with volume form $dV=e^{-f}dV_{\alpha}$. The corresponding isotropic S-curvature with $\mathbb{F}$ is expressed* *as*

$$
\begin{aligned}
\boldsymbol{QRic} &= Ric_\alpha + 2\alpha s^m_{0|m} - 2s_{0i}s^i_0 - \alpha^2 s^i_j s^j_i + (n-1)\Big[\frac{3\big(r_{00}-2\alpha s_0\big)^2}{4\mathbb{F}^2} \\
&\quad + \frac{4\alpha\big(r_{0i}s^i_0 - \alpha s_i s^i_0\big) - r_{00|0} + 2\alpha s_{0|0}}{2\mathbb{F}}\Big] + (n+1)\big[c_{|0}\mathbb{F} + c r_{00}\big],
\end{aligned} \tag{3.5}
$$

where Ric_α denotes the Ricci curvature of α.

Now we want to prove theorem **??**:

Proof. By Lemma **??** and equation (**??**), we have

$$
\begin{aligned}
0 &= -\frac{(n+7)}{A^2}\left[\frac{r_{00}}{2\alpha} - -s_0\right]^2 \\
&\quad + \frac{2}{A}\left[\frac{r_{00|0}}{2\alpha} - 2r_{0i}s^i_0 - s_{0|0} + 2\alpha s_i s^i_0 - \frac{r_{00}s_0}{2\alpha} + f_0 s_0\right] - 2s_{0i}s^i_0 \\
&\quad - 2\alpha f_{x^i} s^i_0 + f_{0|0} + 2\alpha s^i_{0|i} - \alpha^2 s^i_j s^j_i + Ric_\alpha - (n-1)cF^2,
\end{aligned} \tag{3.6}
$$

where $A := (1+s)$. By lemma **??** we can obtain

$$
r_{00} = 2\big[c(\alpha^2-\beta^2) - s_0\beta\big]. \tag{3.7}
$$

□

Theorem 3.3. *Suppose $\mathbb{F} = \alpha + \beta$ be a quasi-Einstein Randers metric on an n-dimensional manifold $\mathbb{M}$ with the volume form $dV_{\mathbb{F}} = e^{-f}dV_\alpha$. Then $\mathbb{F}$ is quasi-Ricci flat, if*

$$
\begin{aligned}
\boldsymbol{R}ic_\alpha &= \eta\alpha^2 - \beta\big[(n+7)c^2\beta + 4s_i s^i_0 + 8s_0 c + 2s^i_{0|i} - f_{x^i}s^i_0\big] + s_0\big[(n+3)s_0 - 2f_0\big] \\
&\quad + 2s_{0|0} + 2s_{0i}s^i_0 - f_{0|0},
\end{aligned} \tag{3.8}
$$

where $\eta := \eta(x)$ is a scalar function and f is a smooth function.

Proof. Suppose $\mathbb{F}$ be a *quasi*-Einstein metric, Then by (**??**) we can obtain

$$
r_{00|0} = 4\beta s_0^2 - 2\beta s_{0|0} + 8c\beta^2 s_0 + 2\big(c_0 - 2cs_0 - 4\beta c^2\big)(\alpha^2-\beta^2), \tag{3.9}
$$

$$
r_{0i}s^i_0 = -s_0^2 - s_i s^i_0\beta - 2cs_0\beta. \tag{3.10}
$$

Now, by putting (**??**), (**??**) and (**??**) into (**??**) we have

$$
0 = \boldsymbol{R}att + \alpha \boldsymbol{I}rrat, \tag{3.11}
$$

where

$$\begin{aligned}
(3.12)\qquad \boldsymbol{R}att &= \boldsymbol{B}_2\alpha^2+\boldsymbol{B}_4\beta,\\
&\quad \boldsymbol{B}_2 := (n+7)c^2\beta+2cs_0(n+7)+4s_is^i_0-6cs_0-2\big(c_0-4c^2\beta\big)\\
&\quad -2f_{x^i}s^i_0+2s^i_{0|i}-s^i_js^j_i\beta,\\
&\quad \boldsymbol{B}_4 := (n+7)\big(c\beta-s_0\big)^2+4s_0^2-2s_{0|0}+14cs_0\beta-2(c_0-4c^2\beta)\beta\\
&\quad +2s_0^2-2s_{0i}s^i_0+f_{0|0}+Ric_\alpha,\\
(3.13)\qquad \boldsymbol{I}ratt &= \alpha^2\boldsymbol{B}_1+\boldsymbol{B}_3,\\
&\quad \boldsymbol{B}_1 := -(n+7)c^2-s^i_js^j_i,\\
&\quad \boldsymbol{B}_3 := +(n+7)c^2\beta^2-(n+7)s_0^2+4s_0^2-2s_{0|0}+8cs_0\beta+4s_is^i_0\beta\\
&\quad +2f_0s_0-2s_{0i}s^i_0-f_{x^i}s^i_0\beta+f_{0|0}+2s^i_{0|i}\beta+Ric_\alpha.
\end{aligned}$$

By equation (**??**) there is a scalar function $\eta=\eta(x)$ such that

$$\boldsymbol{B}_3=\eta\alpha^2.$$

We can get (**??**). This completes the proof. □

In [**?**], Tayebi, Majidi and Haji-Badali studied Einstein -reversible on m -root Finsler metrics. They proved a 3-th root (α,β)-metric $\mathbb{F}=\sqrt[3]{c_1\beta\alpha^2+c_2\beta^3}$ is an Einstein -reversible metric if and only if $s_i=0$,$r_{ij}=0$ and $Ric_\alpha=\frac{3c_2}{c_1^3}\Big[\lambda\beta^2-(c_1^2\alpha^2+3c_1\beta^2)t^m_m-8c_1^2t_{00}\Big]$ where c_i are real contants and $\lambda=\lambda(x)$ is scalar function on manifold $\mathbb{M}$.

4. Proof of Theorem ??

Suppose $\mathbb{F}=\alpha+\beta$ be *quasi*- Einstein - reversible, by equation (**??**), we have

$$c(x,y):=\frac{\boldsymbol{QRic}}{\mathbb{F}^2}.$$

By definition of *quasi*-Einstein-reversible, we have

$$c(x,y)=c(x,-y). \tag{4.1}$$

By equation (**??**) and Lemma **??** we can get

$$\begin{aligned}
0 &= (\alpha^2-\beta^2)^3\Big[2(s^i_{0|i}+s^i_js^j_i-f_{x^i}s^i_0+2s_is^i_0)\Big]\\
&\quad +(\alpha^2-\beta^2)^2\Big[2\beta\big([2s^i_{0|i}+s^i_js^j_i\beta-\\
&\quad 3f_{x^i}s^i_0+10s_is^i_0]+6r_{0i}s^i_0+3s_{0|0}-3f_0s_0-Ric_\alpha-f_{0|0}\\
&\quad +2s_{0i}s^i_0+2s_0^2(n+7)-f_0r_{00}+(n+7)r_{00}s_0\big)\Big]\\
&\quad +(\alpha^2-\beta^2)\Big[4\beta^3(4r_{0i}s^i_0+2s_{0|0}+4s_is^i_0\beta-2f_0s_0+6(n+7)s_0^2)\\
&\quad -4\beta^2(f_0r_{00}-r_{00|0}-2(n+7)r_00s_0)+r_{00}^2(n+7)\beta\Big]. \qquad (4.2)
\end{aligned}$$

By the equation above, there is a scalar function $c = c(x)$ such that

$$r_{00} = 2c(\alpha^2 - \beta^2) - 2s_0\beta. \tag{4.3}$$

By (**??**) we get

$$r_{00|0} = 4\beta s_0^2 - 2\beta s_{0|0} + 8c\beta^2 s_0 + 2\big(c_0 - 2cs_0 - 4\beta c^2\big)(\alpha^2 - \beta^2), \tag{4.4}$$

$$r_{0i}s_0^i = -s_0^2 - s_i s_0^i \beta - 2cs_0\beta. \tag{4.5}$$

Now by plugging (**??**), (**??**) and (**??**) in (**??**). We obtain

$$\begin{aligned} 0 \;=\; & \alpha^2\Big[2s^i_{0|i} - 2f_{x^i}s_0^i + 4s_i s_0^i - 2cf_0 + 2(c_0 - 2cs_0) + 2(n+7)cs_0\Big] \\ & +\beta\Big[\big[2s^i_j s^j_i - 8c^2 + 4c^2(n+7)\big]\alpha^2 - 4f_0 s_0 - 8s_0^2 + 4s_{0|0} \\ & +2(n+7)s_0^2 - 2Ric_\alpha - 2f_{0|0} + 4s_{0i}s_0^i\Big] - \beta^2\Big[2s^i_{0|i} - 2f_{x^i}s_0^i \\ & +4s_i s_0^i - 2cf_0 + 2c_0 - 4cs_0 + 2(n+7)cs_0 + 32cs_0 - 8(n+7)cs_0 \\ & +8cf_0 - 8c_0 - 8s_i s_0^i - 4s^i_{0|i} + 4f_{x^i}s_0^i - \Big(2s^i_j s^j_i \\ & +8c^2 + 4(n+7)c^2 + 32c^2 - 8(n+7)c^2 - 2s^i_j s^j_i\Big)\beta\Big]. \end{aligned} \tag{4.6}$$

It follows from (**??**) that

$$s^i_{0|i} = \frac{1}{2}\Big(2f_{x^i}s_0^i + 4s_i s_0^i - 2cf_0 + 2c_0 - 2s_0 c + 2(n+7)cs_0 + 2\sigma\beta\Big), \tag{4.7}$$

such that $\sigma = \sigma(x)$ is a scalar function on M. Plugging (**??**) in (**??**), we obtain

$$\begin{aligned} Ric_\alpha \;=\; & \alpha^2\Big[\sigma - 4c^2 + 2(n+7)c^2 + s^i_j s^j_i\Big] - 2\beta\Big[cf_0 - c_0 + 6s_0 c - cs_0(n+7) \\ & +\beta c^2\big(n+1\big)\Big] - 2f_0 s_0 - 4s_0^2 + 2s_{0|0} + (n+7)s_0^2 - f_{0|0} + 2s_{0i}s_0^i. \end{aligned} \tag{4.8}$$

Finally, plugging (**??**), (**??**), (**??**) and (**??**) in (**??**) and simplify this equation, it yields

$$\boldsymbol{QRic} = \mathbb{F}^2\Big(\sigma + c^2[n+3]\Big).$$

Corollary 4.1. *Let $\mathbb{F} = \alpha + \beta$ be a S-isotropic Randers metric on an n-dimensional manifold $\mathbb{M}$ with volume form $dV_{\mathbb{F}} = e^{-f}dV_\alpha$. Then $\mathbb{F}$ is quasi-Einstein-reversible if and only if it be quasi-Einstein.*

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