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Research Paper

BOUNDEDNESS OF THE DERIVATIVE OF THE FUNCTION ON WAVELET COAPPROXIMATION

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Abstract

In this paper, we define Chebyshev wavelets and Chebyshev wavelet coapproximation. We obtain some generalized results on wavelet coapproximation. We show that if the series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|t_{n,m}|^2$ is convergent, then there exists a wavelet coapproximation for a set. We assume that functions have bounded derivative and we obtain wavelet coapproximation for a set.

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1. Introduction

Let the sequences $\{T_n(x\},\ \{U_n(x)\},\ \{V_n(x)\}\ and\ \{W_n(x)\}$ be Chebyshev functions of first, second, third and fourth kinds, where $x = \cos\theta$ and $\theta \in [0, 2\pi]$. In the continue we consider wavelet coapproximation.(see [1-12]).

$Chebyshev\ Polynomial$	$Weight\ function$
$T_n(x) = cos\ n\ \theta$	$\frac{1}{\sqrt{1-x^2}}$
$U_n(x) = \frac{sin\ (n+1)\ \theta}{sin\ \theta}$	$\sqrt{1-x^2}$
$V_n(x) = \frac{cos(n+\frac{1}{2})\theta}{cos(\frac{\theta}{2})}$	$\sqrt{\frac{1+x}{1-x}}$
$W_n(x) = \frac{sin(n+\frac{1}{2})\theta}{sin(\frac{\theta}{2})}$	$\sqrt{\frac{1-x}{1+x}}$

(see [1-5]).
With the above definition, we obtain the fundamental recurrence relation

$$X_{n+1}(x) = 2xX_n(x) - X_{n-1}(x),\ n \geq 2.$$

Also we know that definitions Chebyshev wavelets on the interval $[0, L] \subset [0, 2\pi]$, are defied $\{T_{n,m}(x\},\ \{U_{n,m}(x)\},\ \{V_{n,m}(x)\}\ and\ \{W_{n,m}(x)\}$ (see [1-9], [11-13]).

Definition 1.1. [10] Suppose $W \subseteq L^2([0, L])$ and $f(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\Psi_{n,m}(t)$ in terms of Chebyshev wavelets. We say that the function $g \in L^2([0, L])$ is a wavelet coapproximation of f concerning W, If

$$F_f(p) := \|g - p\|_2 - \|f - p\|_2 = o(\phi(n)),$$

for every $p \in W$.

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2. Main Results

In this section, we consider wavelet approximation in Chebyshev polynomials of first, second, third and fourth kinds. We estimate wavelet approximation of a function f have bounded derivative and obtain wavelet coapproximation for a set.

Theorem 2.1. *Let $f \in L^2[0, L]$ be a continuous function and there is a $Q > 0$ such that $|f'(t)| \leq Q$.*

(i) If $f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t_{n,m}T_{n,m}(t)$. Then Chebyshev wavelet coapproximation f, for every l with respect to any set W is the partial sums $s_{2^k,l-1}(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{l-1} t_{n,m}T_{n,m}(t)$. and

$$F_f(p) = o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}\frac{1}{m^2}),$$

for any set $p \in W$.

(ii) If $f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t_{n,m}U_{n,m}(t)$. Then Chebyshev wavelet coapproximation f, for every l with respect to any set W is the partial sums $s_{2^k,l-1}(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{l-1} t_{n,m}U_{n,m}(t)$. and

$$F_f(p) = o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(\frac{1}{m}+\frac{1}{m+1})^2,$$

for every $p \in W$.

Proof. Since f is continuous, there is a $P > 0$ such thtat $|f(t| \leq P$.
(i) We have

$$\begin{aligned}
|c_{n,m}| &= |< f(t), T_{n,m} >| \\
&= |\int_0^L f(t) T_{n,m}(t) \frac{1}{\sqrt{1-t^2}} dt| \\
&= |\sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} f(t) T_{n,m}(t) \frac{1}{\sqrt{1-t^2}} dt| \\
&= |\frac{2^{\frac{k+1}{2}}}{\sqrt{n}} \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} f(t) T_m((\frac{2^k}{L}t - 2n + 1)) \frac{1}{\sqrt{1-(\frac{2^k}{L}t - 2n+1)^2}} dt| \\
put \quad cosu &= \frac{2^k}{L}t - 2n + 1 \\
&= \frac{2^{\frac{-k+1}{2}} L}{\sqrt{n}} |\int_{-1}^1 f(\frac{cosuL + 2nL - L}{2^k}) T_m(u) \frac{1}{\sqrt{1-(cosu)^2}} (-sinu) du| \\
&\leq \frac{2^{\frac{-k+1}{2}} L}{\sqrt{n}} |\int_{-1}^1 |f(\frac{cosuL + 2nL - L}{2^k}|| \frac{e^{imu} + e^{-imu}}{2} |du| \\
&\leq \frac{2^{\frac{-k+1}{2}} L}{\sqrt{n}} |[(f(\frac{cosuL + 2nL - L}{2^k}) \frac{1}{im} (-e^{imu} + e^{-imu})_{-1}^1| \\
&+ \frac{2^{\frac{-k+1}{2}} L}{\sqrt{n}} |\int_{-1}^1 if'(\frac{cosuL + 2nL - L}{2^k}) \frac{1}{im} (-e^{imu} + e^{-imu}) du| \\
&\leq \frac{2^{\frac{-k+3}{2}} L}{\sqrt{n}} (\frac{2}{m} + \frac{2^{\frac{-k+3}{2}} L}{\sqrt{n}} \frac{Q}{m} \\
&= \frac{k_1}{m}
\end{aligned}$$

therefore

$$|c_{n,m}|^2 \leq \frac{k_1^2}{m^2}$$

and

$$\begin{aligned}
\|T_{n,m}\|_2^2 &= \int_0^L |T_{n,m}(t)|^2 dt \\
&= 2^{-\frac{k}{2}} \sum_{n=1}^{2^k} \int_{\frac{(n-1)L}{2^k}}^{\frac{nL}{2^k}} |T_m(\frac{2^k}{L}t - 2n - 1)|^2 dt \\
put & \quad cosu = \frac{2^k}{L}t - 2n - 1 \\
&= 2^{\frac{k}{2}} \sum_{n=1}^{2^k} \int_{-1}^{1} |T_m(u)|^2 du \\
&\leq 2^{\frac{k}{2}} \sum_{n=1}^{2^k} \|T_m\|_2^2 \\
&= \pi^2 2^{\frac{k}{2}-1} \sum_{n=1}^{2^k} 1
\end{aligned}$$

$$\begin{aligned}
\| & \quad p - s_{2^k,l-1}\|_2 - \|f-p\|_2 \\
\leq & \quad \|f - s_{2^k,l-1}\|_2 \\
= & \quad \|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} c_{n,m}T_{n,m} - \sum_{n=1}^{2^k}\sum_{m=0}^{l-1} c_{n,m}T_{n,m}\|_2^2 \\
= & \quad \|\sum_{n=1}^{2^k}\sum_{m=l}^{\infty} c_{n,m}T_{n,m}\|_2^2 \\
\leq & \quad \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |c_{n,m}|^2 \|T_{n,m}\|_2^2 \\
\leq & \quad \frac{k_2}{m^2}
\end{aligned}$$

It follows that

$$F_f(p) = o(\sum_{n=1}^{2^k}(\frac{k_2^2}{m^2}))$$

and

$$lim_{l\to\infty} \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} \frac{k_2^2}{m^2} = 0.$$

(ii) We have

$$\begin{aligned}
|c_{n,m}| &= |< f(t), U_{n,m} >| \\
&= 2^{\frac{k}{2}} |\int_0^L f(t) U_{n,m}(t) |\sqrt{1-t^2} dt| \\
&= |\sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} f(t) U_{n,m}(t) \sqrt{1-t^2} dt| \\
&= |\sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} f(t) U_m(\frac{2^k}{L} t - 2n + 1) \sqrt{1 - (\frac{2^k}{L} t - 2n + 1)^2} dt| \\
put \quad & cosu = \frac{2^k}{L} t - 2n + 1 \\
&= 2^{\frac{k}{2}} |\int_0^1 f(\frac{cosuL + 2nL - L}{2^k}) U_m(u) sin^2 u du| \\
&= 2^{\frac{k}{2}} |\int_0^1 f(\frac{cosuL + 2nL - L}{2^k}) U_m(u) (\frac{e^{iu} - e^{-iu}}{2})^2 du \\
| &= 2^{\frac{k}{2}} |\int_0^L f(\frac{uL + 2nL - L}{2^k}) \frac{e^{i(m+1)u} - e^{-i(m+1)u}}{e^{iu} - e^{-iu}} \frac{(e^{iu} - e^{-iu})(e^{iu} - e^{iu})}{4} du| \\
&= 2^{\frac{k}{2}-2} |\int_0^L f(\frac{uL + 2nL - L}{2^k}) (e^{i(m+1)u} - e^{-i(m+1)u})(e^{-iu} - e^{iu}) du| \\
&= 2^{\frac{k}{2}-2} |\int_0^L f(\frac{uL + 2nL - L}{2^k}) (e^{imu} - e^{i(m+2)u} + e^{-imu} - e^{i(m+2)u}) du| \\
&\leq 2^{\frac{k}{2}-1} P(\frac{1}{m} + \frac{1}{m+2} + \frac{1}{m} + \frac{1}{m+2}) \\
&- 2\frac{L^2 Q}{2^k}(\frac{1}{m} + \frac{1}{m+1}) \\
&\leq k(\frac{1}{m} + \frac{1}{m+1})
\end{aligned}$$

therefore

$$|c_{n,m}|^2 \leq k^2 (\frac{1}{m} + \frac{1}{m+1})^2,$$

and

$$\begin{aligned}
\|U_{n,m}\|_2^2 &= 2^{\frac{k}{2}}\int_0^L |U_{n,m}(t)|^2\sqrt{1-t^2}dt| \\
&= \sum_{n=1}^{2^{k-1}}\int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} |U_{n,m}(t)|^2\sqrt{1-t^2}dt| \\
&= \sum_{n=1}^{2^{k-1}}\int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} |U_m(p,q,e^{i(\frac{2^k}{L}t-2n+1)})|^2\sqrt{1-(\frac{2^k}{L}t-2n+1)^2}dt \\
put \quad & cosu=\frac{2^k}{L}t-2n+1 \\
&= 2^{\frac{k}{2}}\sum_{n=1}^{2^k}\int_0^1 U_m(u)^2\sqrt{1-u^2}du \\
&\leq \frac{\pi}{2}2^{\frac{k}{2}}\sum_{n=1}^{2^k}\|U_m(t)\|_2^2 \\
&= \pi^2 2^{\frac{k}{2}-1}.
\end{aligned}$$

with respect to any set W

$$\begin{aligned}
\| \quad & p-s_{2^k,l-1}\|_2-\|f-p\|_2 \\
\leq \quad & \|f-s_{2^k,l-1}\|_2 \\
= \quad & \|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}U_{n,m}-\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}c_{n,m}U_{n,m}\|_2^2 \\
= \quad & \|\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}U_{n,m}\|_2^2 \\
\leq \quad & k(\frac{1}{m}+\frac{1}{m+1})^2
\end{aligned}$$

It follows that

$$F_f(p)=o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(\frac{1}{m}+\frac{1}{m+1})^2),$$

and

$$lim_{l\to\infty}\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(\frac{1}{m}+\frac{1}{m+1})^2=0.$$

□

Corollary 2.2. *Let $f\in L^2[0,L]$ be a continuous function and there is a $Q>0$ such that $|f'(t)|\leq Q$.*

(i) If $f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}t_{n,m}V_{n,m}(t)$*. Then wavelet coapproximation* f*, for every* l *with respect to any set* W *is the partial sums* $s_{2^k,l-1}(t)=\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}t_{n,m}V_{n,m}(t)$*. and*

$$F_f(p)=o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_2(\frac{1}{m-1}+\frac{1}{m})^2)),$$

(ii) If $f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}t_{n,m}W_{n,m}(t)$*. Then wavelet approximation* f*, for every* l *with respect to any set* W *is the partial sums* $s_{2^k,l-1}(t)=\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}t_{n,m}W_{n,m}(t)$*. and*

$$F_f(p)=o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_2(\frac{1}{m-1}+\frac{1}{m})^2)),$$

Proof. (i) We have

$$\begin{aligned}
\| & \quad p-s_{2^k,l-1}\|_2-\|f-p\|_2\\
\leq & \quad \|f-s_{2^k,l-1}\|_2\\
= & \quad \|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty}c_{n,m}V_{n,m}-\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}c_{n,m}V_{n,m}\|_2\\
= & \quad \|\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}V_{n,m}\|_2\\
\leq & \quad \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|V_{n,m}\|_2\\
\leq & \quad \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|U_{n,m}-U_{n,m-1}\|_2\\
\leq & \quad \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|U_{n,m}\|_2+\|U_{n,m-1}\|_2\\
\leq & \quad \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_1(\frac{1}{m-1}+\frac{1}{m})^2)
\end{aligned}$$

It follows that

$$F_f(p)=o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_2(\frac{1}{m-1}+\frac{1}{m})^2)),$$

(ii) We have

$$\begin{aligned}
\| & \ p-s_{2^k,l-1}\|_2-\|f-p\|_2 \\
\leq & \ \|f-s_{2^k,l-1}\|_2 \\
= & \ \|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty}c_{n,m}W_{n,m}-\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}c_{n,m}W_{n,m}\|_2 \\
= & \ \|\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}W_{n,m}\|_2 \\
\leq & \ \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|W_{n,m}\|_2 \\
\leq & \ \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|U_{n,m}+U_{n,m-1}\|_2 \\
\leq & \ \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|U_{n,m}\|_2+U_{n,m-1}\|_2 \\
\leq & \ \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_1(\frac{1}{m-1}+\frac{1}{m})^2)
\end{aligned}$$

It follows that

$$F_f(p)=o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_2(\frac{1}{m-1}+\frac{1}{m})^2)),$$

□

We put

$$W_1=\{\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}c_{n,m}{\psi_{n,m}}^+:\ c_{n,m}\in\mathbb{R}\},$$

$$W_2=\{\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}c_{n,m}{\psi_{n,m}}^-:\ c_{n,m}\in\mathbb{R}\}$$

and

$$W_3=\{\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}:\ c_{n,m}\in\mathbb{R}\}.$$

Theorem 2.3. *Let $f\in L^2([0,L])$ be a continuous function and $f(t)=\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}t_{n,m}{\psi_{n,m}}^+(t)$ be expanded in terms of Chebyshev wavelets. The series $\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}|t_{n,m}|^2L_{n,m}$ is convergent, then for every $l>0$ complex (p,q)-extension wavelet coapproximation f with respect to W_1, is the partial sums of:*

$$s_{2^k,l-1}=\sum_{n=0}^{2^k}\sum_{m=0}^{l-1}t_{n,m}{\psi_{n,m}}^+.$$

and

$$F_f(p) = o(\sum_{n=0}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m}).$$

for every $p \in W_1$.

Proof. Suppose $p(t) = \sum_{n=0}^{2^k}\sum_{m=0}^{\infty} c_{n,m}{\psi_{n,m}^{\alpha}}^{+}(t) \in W_1$.

$$\begin{aligned}
\| \quad & p - s_{2^k,l-1}\|_2 - \|f-p\|_2 \\
\leq \quad & \|f - s_{2^k,l-1}\|_2 \\
= \quad & \int_0^L |\sum_{n=1}^{2^k}\sum_{m=l}^{\infty} t_{n,m}{\psi_{n,m}}^{+}(t)|^2 \omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} \int_0^1 |t_{n,m}|^2 |{\psi_{n,m}}^{+}(t)|^2 \omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 \int_0^1 |\psi_{n,m}(t)|^2 \omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 \int_{-1}^1 |\psi_{n,m}(t)|^2 \omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 \|\psi_{n,m}\|_2^2. \\
= \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m},
\end{aligned}$$

Therefore $\|p - s_{M-1}\|_2 - \|f-p\|_2 \leq \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m}$. That is

$$F_f(p) = o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m}),$$

for every $p \in W_1$ where,

$$lim_{l\to\infty}\sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m} = 0.$$

□

Corollary 2.4. *Let* $f(t) = \sum_{n=0}^{2^{k-1}}\sum_{m=0}^{\infty} t_{n,m}{\psi_{n,m}}^{-}(t)$ *be expanded in wavelet coapproximation. The series* $\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}$ *is convergent, then for every* $l > 0$ *Chebyshev wavelet coapproximation* f *with respect to* W_2*, is the partial sums of:*

$$s_{2^k,l-1} = \sum_{n=1}^{2^k}\sum_{m=0}^{l-1} t_{n,m}{\psi_{n,m}}^{-}.$$

and

$$F_f(p) = o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m}).$$

for every $p \in W_2$.

Theorem 2.5. *Let* $f \in L^2([0, L])$ *be a continuous function and* $f(t) = \sum_{n=0}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\psi_{n,m}(t)$ *be expanded in terms of Chebyshev wavelets. If the series* $\sum_{n=0}^{2^k}\sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}$ *is convergent, then for every* $l > 0$ *Chebyshev wavelet coapproximation* f *with respect to* W_1, *is the partial sums of:*

$$s_{2^k,l-1} = \sum_{n=0}^{2^k}\sum_{m=0}^{l-1} t_{n,m}\psi_{n,m}.$$

and

$$F_f(p) = o(\sum_{n=0}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m}).$$

for every $p \in W_3$.

Proof. Suppose $p = \sum_{n=1}^{2^k}\sum_{m=0}^{\infty} c_m\psi_{n,m} \in W_3$.

$$\begin{aligned}
\| \quad & p - s_{2^k,l-1}\|_2 - \|f - p\|_2 \\
\leq \quad & \|f - s_{2^k,l-1}\|_2 \\
= \quad & \int_0^L |\sum_{n=1}^{2^k}\sum_{m=l}^{\infty} t_{n,m}\psi_{n,m}(x)|^2 \omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} \int_0^1 |t_{n,m}|^2 |\psi_{n,m}(x)|^2 \omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 \int_0^1 |\psi_{n,m}(x)|^2 \omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 \int_{-1}^1 |\psi_{n,m}(x)|^2 \omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 \|\psi_{n,m}\|_2^2. \\
= \quad & \sum_{m=0}^{k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m},
\end{aligned}$$

Therefore $\|p - s_{M-1}\|_2 - \|f - p\|_2 \leq \sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m}$. That is

$$F_f(p) = o(\sum_{n=0}^{2^k}\sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m}),$$

for every $p \in W_3$ where,

$$lim_{l\to\infty} \sum_{n=0}^{2^k} \sum_{m=l}^{\infty} |t_{n,m}|^2 L_{n,m} = 0.$$

□

Suppose $1 \leq M \leq 2^k$ is a fixed number and $f(t) = \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} t_{n,m}\psi_{n,m}(t)$, we put

$$f_1(t) = \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} t_{n,m}\psi_{n,m}{}^{-}(t),$$

and

$$f_2(t) = \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} t_{n,m}\psi_{n,m}{}^{+}(t),$$

then

$$f = f_1 - f_2.$$

Theorem 2.6. *Let $f(t) = \sum_{n=l}^{\infty} \sum_{m=0}^{\infty} t_{n,m}\psi_{n,,m}(t)$ be expanded in terms of Chebyshev wavelets. If the series $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}$ is convergent, Then Chebyshev wavelet coapproximation of f_2 is f_1 with respect to any set $W \subseteq L^2[0, L]$*
and

$$F_f(p) = o(\sum_{n=l}^{\infty} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}),$$

for every $p \in W$.

Proof. Suppose $p \in W$. We have

$$\begin{aligned}
\|p - f_1\|_2 &= \|p - f_1 + f_2 - f_2\|_2 \\
&= \|p - f_2 - f\|_2 \\
&\leq \|f_2 - p\|_2 + \|f\|_2
\end{aligned}$$

and

$$\begin{aligned}
F_f(p) &= \|p - f_1\|_2 - \|f_2 - p\|_2 \\
&\leq \|f\|_2 \\
&= \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m} \\
&\leq \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}
\end{aligned}$$

□

Corollary 2.7. *Let* $f(t) = \sum_{n=l}^{\infty}\sum_{m=0}^{\infty} t_{n,m}\psi_{n,m}(t)$ *be expanded in terms of Chebyshev wavelets. If the series* $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}$ *is convergent. Then Chebyshev wavelet coapproximation of* f_1 *is* f_2 *with respect to any set* $W \subseteq L^2[0,L]$ *and*

$$F_f(p) = o(\sum_{n=M}^{\infty}\sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}),$$

for every $p \in W$.

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