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Research Paper

PARALLEL DYNAMICAL SYSTEMS:CHARACTERIZATION, CHAOS PRESERVATION, AND SYNCHRONIZATION APPLICATIONS

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Abstract. This paper introduces and rigorously analyzes the concept of parallel dynamical systems, a novel framework for understanding and achieving synchronization in chaotic systems. For any given dynamical system defined by ordinary differential equations, we construct a corresponding parallel system through a scaling transformation of phase space variables. We prove rigorously that the parallel system preserves all fundamental chaotic properties of the original system, including sensitive dependence on initial conditions, topological transitivity, and density of periodic orbits. This theoretical foundation enables a controller-free synchronization method where systems naturally synchronize through appropriate initial condition scaling. Numerical simulations of the Lorenz system validate our theoretical predictions, demonstrating perfect synchronization and practical applicability.

1. Introduction

The study of dynamical systems and chaos theory has fundamentally transformed our understanding of complex nonlinear phenomena across diverse scientific disciplines including physics, biology, engineering, and economics [6, 7]. Chaotic systems exhibit exquisite sensitivity to initial conditions—popularly known as the "butterfly effect"—which results in long-term unpredictability despite deterministic short-term evolution [3].

Among various chaotic phenomena, synchronization—where multiple systems adjust their dynamics to achieve coordinated behavior—has attracted substantial research interest [4]. Applications span secure communications [9], parallel computing, biological rhythm modeling, and coupled oscillator networks. Traditional synchronization approaches typically require designing sophisticated feedback controllers or coupling schemes to force response (slave) systems to track drive (master) system dynamics [5].

This paper presents a fundamentally geometric approach to synchronization through parallel dynamical systems. Our core contribution involves generating new systems from existing ones via simple phase space scaling transformations. We demonstrate that these parallel systems are topologically equivalent to their originals [1] and crucially inherit all chaotic properties. This intrinsic equivalence provides a powerful synchronization mechanism: by appropriate initial condition scaling, parallel system trajectories perfectly align with original system trajectories without active control.

The paper is structured as follows: Section 2 reviews essential dynamical systems and chaos theory concepts, formally defining parallel dynamical systems and their fundamental properties. Section 3 presents our main theoretical results proving chaos preservation under

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parallel transformations. Section 4 discusses practical applications and numerical validation. Section 5 concludes with future research directions.

2. Preliminaries and Definitions

2.1. **Dynamical Systems and Flows.** Consider a nonlinear system of ordinary differential equations (ODEs):

$$\dot{x}(t) = f(x(t)), \qquad x \in \mathbb{R}^n, \quad t \in [t_0, t_1] \subset \mathbb{R} \tag{2.1}$$

Definition 2.1 (Dynamical System and Flow [1])**.** Let $N \subseteq \mathbb{R}^n$ be an open subset and $f \in C^1(N)$. For $x_0 \in N$, let $\phi(t, x_0)$ be the solution to the initial value problem:

$$\dot{x}(t) = f(x(t)) \tag{2.2}$$

$$x(0) = x_0 \tag{2.3}$$

defined on its maximal interval of existence $I(x_0) \subset \mathbb{R}$. The set of mappings ϕ_t defined by $\phi_t(x) = \phi(t, x)$ is called the *flow* of the differential equation (2.1). The triple $(f, \mathbb{R}^n, \phi_t(\cdot))$ constitutes a dynamical system derived from the ODE (2.1).

The C^1-map $\phi : \mathbb{R} \times N \to N$ satisfies:

- $\phi_0(x) = x$ for all $x \in N$
- $\phi(s, \phi(t, x)) = \phi_{t+s}(x)$ for all $s, t \in \mathbb{R}$ and $x \in N$
- For fixed $t \in \mathbb{R}$, the map $\phi_t : N \to N$ is a C^1-diffeomorphism with inverse ϕ_{-t}

Remark 2.2 ([1])**.** If $\phi(t, x)$ is a flow on $N \subset \mathbb{R}^n$, then the function

$$f(x) = \frac{d}{dt}\phi(t, x)\bigg|_{t=0}$$

defines a C^1-vector field on N.

Solutions of (2.1) are called *trajectories*, and the set

$$\mathcal{O}(x_0) = \{x(t) \in \mathbb{R}^n : x(0) = x_0, t \in \mathbb{R}\}$$

is called the *orbit* through $x_0 \in \mathbb{R}^n$.

Definition 2.3 (Topological Equivalence [8])**.** Let N and $\tilde{N}$ be open subsets of $\mathbb{R}^n$ with $f \in C^1(N)$ and $g \in C^1(\tilde{N})$. The systems

$$\dot{x} = f(x) \tag{2.4}$$

$$\dot{y} = g(y) \tag{2.5}$$

are *topologically equivalent* if there exists a homeomorphism $H : N \to \tilde{N}$ that maps trajectories of (2.4) onto trajectories of (2.5) while preserving time orientation.

Example 2.4 ([1])**.** Consider the system with:

$$f(x) = \begin{cases} -x_1 \\ x_1^2 + x_2 \end{cases}$$

The flow with initial condition $x(0) = c$ is:

$$\phi_t(c) = \left(c_1 e^{-t},\ c_2 e^t + \frac{c_1^2}{3}(e^t - e^{-2t})\right) \tag{2.6}$$

2.2. **Chaotic Dynamical Systems.**
Chaotic systems exhibit three fundamental characteristics:

Definition 2.5 (Sensitive Dependence on Initial Conditions)**.** Let Λ be a compact invariant set under $\phi_t(x)$ (i.e., $\phi_t(\Lambda) \subseteq \Lambda$ for all $t \in \mathbb{R}$). The system exhibits *sensitive dependence* on Λ if there exists $\epsilon > 0$ such that for any $x \in \Lambda$ and neighborhood U of x, there exists $y \in U$ and $t \geq 0$ such that:

$$\|\phi_t(x) - \phi_t(y)\| > \epsilon$$

Definition 2.6 (Topological Transitivity)**.** The system is *topologically transitive* on Λ if for any pair of open sets $U, V \subseteq \Lambda$, there exists $t \in \mathbb{R}$ such that $\phi_t(U) \cap V \neq \emptyset$.

Definition 2.7 (Periodic Orbit)**.** A point $p \in N$ is *periodic* with period $T > 0$ if $\phi_T(p) = p$ and $\phi_t(p) \neq p$ for $0 < t < T$. The orbit $\mathcal{O}(p)$ is called a *periodic orbit.*

Definition 2.8 (Chaotic System [10])**.** A system is *chaotic* on $\Lambda subseteq \mathbb{R}^n$ if:
1. It exhibits sensitive dependence on initial conditions
2. It is topologically transitive
3. Periodic orbits are dense in Λ

2.3. **Parallel Dynamical Systems.**

Definition 2.9 (Parallel Dynamical System [2])**.** Let systems (2.4) and (2.5) define flows $\phi(t, x)$ and $\psi(t, x)$ respectively. The dynamical system $\psi(t, x)$ is a *parallel system* of $\phi(t, x)$ if there exists $r \in \mathbb{R}^+$ such that:

$$r\phi(t, x) = \psi(t, rx) \tag{2.7}$$

The systems are topologically equivalent via homeomorphism $H(x) = rx$, which maps trajectories as:

$$H(f(x)) = rf(x) = r\frac{d}{dt}\phi(t, x) = \frac{d}{dt}\psi(t, rx) = g(rx) = g(H(x)) \tag{2.8}$$

Time orientation is preserved since $r > 0$.

Example 2.10. For the system in Example (2.6) with $r = \frac{1}{2}$, the parallel system is:

$$\psi_t(c) = \left(2c_1 e^{-t},\ 2c_2 e^t + \frac{4c_1^2}{3}(e^t - e^{-2t})\right)$$

Example 2.11 (Lorenz System Parallel Transformation)**.** The Lorenz system:

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1) \\ \dot{x}_2 &= \gamma x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 \end{aligned}$$

with $\alpha = 10$, $\beta = 8/3$, $\gamma = 28$. For $r = \frac{1}{2}$, its parallel system becomes:

$$\begin{aligned} \dot{z}_1 &= \alpha(z_2 - z_1) \\ \dot{z}_2 &= \gamma z_1 - z_2 - 2z_1 z_3 \\ \dot{z}_3 &= -\beta z_3 + 2z_1 z_2 \end{aligned}$$

Henceforth, let $\psi(t, z)$ denote the parallel system of $\phi(t, x)$ with $N \subseteq \mathbb{R}^n$ open and $\tilde{N} := rN = \{rx : x \in N\}$.

3. Theoretical Results: Chaos Preservation

3.1. Density of Periodic Orbits.

Lemma 3.1. *If $p \in Per(\phi(t,x))$ is a periodic point of $\phi(t,x)$, then $rp \in \tilde{N}$ is a periodic point of $\psi(t,z)$.*

Proof. We consider three cases:

Case 1: Equilibrium point. If p is an equilibrium, $f(p) = 0$. Then:

$$g(rp) = \frac{d}{dt}\psi(t,rp)\bigg|_{t=0} = \frac{d}{dt}[r\phi(t,p)]\bigg|_{t=0} = rf(p) = 0$$

Thus rp is an equilibrium of $\psi(t,z)$.

Case 2: Fixed point. If $\phi(t,p) = p$, then:

$$\psi(t,rp) = r\phi(t,p) = rp$$

Hence rp is a fixed point of $\psi(t,z)$.

Case 3: Periodic point. If $\phi_T(p) = p$ for period $T > 0$, then:

$$\psi_T(rp) = r\phi_T(p) = rp$$

Therefore rp is periodic with period T for $\psi(t,z)$. □

Proposition 3.2. *If periodic points of $\phi(t,x)$ are dense in N, then periodic points of $\psi(t,z)$ are dense in $\tilde{N}$.*

Proof. Let $\tilde{U} \subseteq \tilde{N}$ be open. Then $U = \frac{1}{r}\tilde{U}$ is open in N. By density, there exists a periodic point $x \in U$ of $\phi(t,x)$. By Lemma 3.1, rx is periodic for $\psi(t,z)$. Since $x \in U$, we have $rx \in \tilde{U}$. Thus every open set in $\tilde{N}$ contains a periodic point of $\psi(t,z)$. □

3.2. Topological Transitivity.

Proposition 3.3. *If $\phi(t,x)$ is topologically transitive on N, then $\psi(t,x)$ is topologically transitive on $\tilde{N}$.*

Proof. Let $\tilde{U}, \tilde{V} \subseteq \tilde{N}$ be open sets. Then $U = \frac{1}{r}\tilde{U}$ and $V = \frac{1}{r}\tilde{V}$ are open in N. By topological transitivity of $\phi(t,x)$, there exists $t \in \mathbb{R}$ such that:

$$\phi_t(U) \cap V \neq \emptyset$$

Applying the parallel transformation:

$$\psi_t(\tilde{U}) = r\phi_t\left(\frac{1}{r}\tilde{U}\right) = r\phi_t(U)$$

Thus:

$$\psi_t(\tilde{U}) \cap \tilde{V} = r[\phi_t(U) \cap V] \neq \emptyset$$

Hence $\psi(t,x)$ is topologically transitive. □

3.3. Sensitive Dependence on Initial Conditions.

Proposition 3.4. *If $\phi(t,x)$ has sensitive dependence on initial conditions on N, then so does $\psi(t,x)$ on $\tilde{N}$.*

Proof. Let $\epsilon > 0$ be the sensitivity constant for $\phi(t,x)$. For any $y \in \tilde{N}$ and neighborhood $\tilde{U}$ of y, let $x = \frac{1}{r}y$ and $U = \frac{1}{r}\tilde{U}$. By sensitivity, there exists $z \in U$ and $t \geq 0$ such that:

$$\|\phi_t(x) - \phi_t(z)\| > \epsilon$$

Let $w = rz \in \tilde{U}$. Then:

$$\|\psi_t(y) - \psi_t(w)\| = \|r\phi_t(x) - r\phi_t(z)\| = r\|\phi_t(x) - \phi_t(z)\| > r\epsilon$$

Thus $\psi(t,x)$ exhibits sensitive dependence with constant $\delta = r\epsilon$. □

Theorem 3.5 (Main Result)**.** *For any chaotic system on $N \subseteq \mathbb{R}^n$, its parallel system is chaotic on $\tilde{N}$.*

Proof. Immediate from Propositions 3.2, 3.3, and 3.4, which establish that $\psi(t,x)$ preserves all three defining properties of chaos. □

4. Applications and Numerical Validation

4.1. Robustness Analysis. Practical implementations must consider disturbances and uncertainties. Suppose the actual parallel system experiences bounded disturbance $\delta(t)$:

$$\dot{z} = g(z) + \delta(t), \quad \|\delta(t)\| \leq \Delta \tag{4.1}$$

The synchronization error $e = z - rx$ evolves as:

$$\dot{e} = \dot{z} - r\dot{x} = g(z) - g(rx) + \delta(t) = g(rx + e) - g(rx) + \delta(t) \tag{4.2}$$

If g is Lipschitz continuous with constant L_g, then:

$$\|\dot{e}\| \leq L_g\|e\| + \Delta \tag{4.3}$$

This implies ultimate boundedness: $\|e(t)\| \leq \Delta/L_g$ for sufficiently large t, demonstrating inherent robustness.

4.2. Numerical Simulations. We validate our theoretical results with the Lorenz system. Figure 1 demonstrates perfect synchronization between the original and parallel systems with $r = 0.5$.

The scaling factor r provides additional flexibility: $r < 1$ enables amplitude scaling useful in secure communications for signal masking, while $r > 1$ allows signal amplification.

5. Conclusion and Future Work

This paper has established parallel dynamical systems as a powerful framework for understanding and achieving chaos synchronization. Our key contributions include:

1. **Rigorous Foundation**: We proved that parallel systems preserve all chaotic properties—sensitive dependence, topological transitivity, and density of periodic orbits—under scaling transformations.
2. **Controller-Free Synchronization**: We developed a novel synchronization method that requires no active control, leveraging natural system dynamics through initial condition scaling.

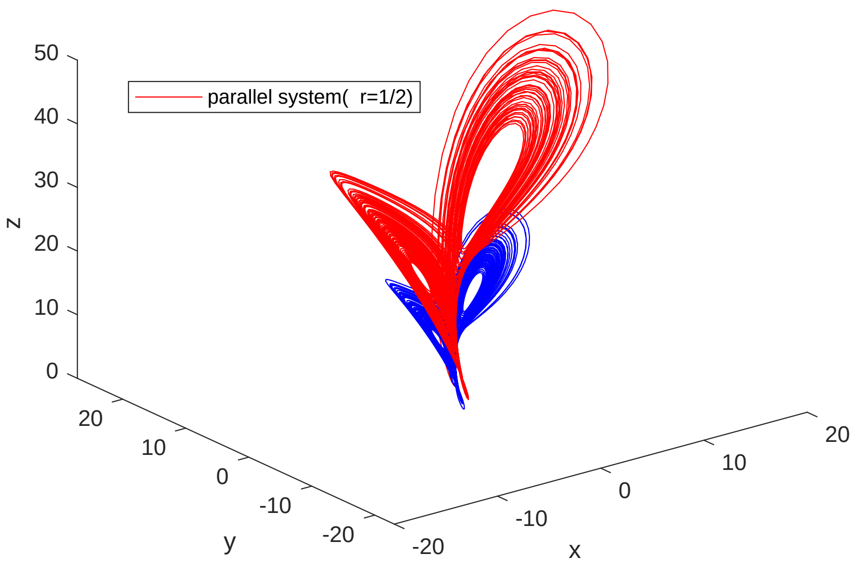

FIGURE 1. Synchronization between original Lorenz system (x) and parallel system (z) with scaling factor $r = 0.5$

3. **Practical Robustness**: We demonstrated inherent robustness to bounded disturbances, ensuring practical applicability.

4. **Numerical Validation**: Lorenz system simulations confirmed theoretical predictions and practical feasibility.

Future research directions include: - Extending the framework to fractional-order systems - Investigating networks of parallel systems - Exploring applications in secure communications where r serves as an encryption key - Developing adaptive scaling strategies for time-varying systems - Investigating quantum analogues of parallel dynamical systems

The parallel systems approach represents a significant contribution to nonlinear dynamics, offering both theoretical elegance and practical utility for chaos synchronization and beyond.

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