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Research Paper

A DUAL ERGODIC THEOREM IN BANACH SPACES

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Abstract. For a sequence $\{x_n\}$, which represents the almost-orbit of the sequence $\{T_n\}$ of nonexpansive mappings, we establish a dual ergodic theorem. This result extends ergodic theorems related to the iterations of a nonexpansive mapping [5] and a sequence of nonexpansive mappings from Hilbert spaces to Banach spaces [1, 15]. Finally, we give some applications to fixed point iteration and zero problem for accretive operators.

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1. Introduction and Background

Let E be a real Banach space with norm $\|\cdot\|$ and dual space E^*. We denote the duality pairing on $E \times E^*$ by $\langle \cdot, \cdot \rangle$, and the unit sphere of E by S_E. The norm of E is said to be Gâteaux (resp. Fréchet) differentiable at $x \in S_E$ if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists for each point $y \in S_E$ (resp. uniformly for $y \in S_E$). A Banach space E is smooth (resp. uniformly smooth) if the norm is Gâteaux differentiable on S_E (resp. Fréchet differentiable for each $x \in S_E$). If E is smooth, then the duality mapping J from E to E^* defined by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ is single-valued. A Banach space E is uniformly convex if, for each $\epsilon > 0$, there is $\delta > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x - y\| > \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$. E is said to be strictly convex if, for each $x, y \in E$ with $\|x\| = \|y\| = 1$, we have $\|\frac{x+y}{2}\| < 1$ (see [14]).

A continuous linear functional μ on l^∞ is called a Banach limit if:
(L_1) $\|\mu\| = \mu(\mathbf{1}) = 1$,
(L_2) $\mu_n(x_n) = \mu_n(x_{n+1})$ for each $x = (x_1, x_2, \cdots) \in l^\infty$,
where $\mathbf{1} = (1, 1, \cdots) \in l^\infty$. In this paper, a Banach limit is denoted by Lim. A sequence $\{x_n\}$ of real numbers is called *almost convergent* to x if $\frac{1}{n}\sum_{i=k}^{n+k} x_i$ converges to x as $n \to \infty$ uniformly in k. In [10], it is proved that $\{x_n\}$ is almost convergent to x if and only if $\text{Lim} x_n = x$ for each Banach limit Lim.

Let C be a nonempty closed convex subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $\text{Fix}(T)$. If E is strictly convex, then $\text{Fix}(T)$ is closed and convex. Baillon

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[2] proved the first nonlinear ergodic theorem for nonexpansive mappings in Hilbert spaces. Define

$$z_n := \frac{1}{n}\sum_{k=1}^{n} T^{k-1}x$$

for each $n \in \mathbb{N}$ and $x \in C$, and suppose that $\mathrm{Fix}(T)$ is nonempty. By Baillon's ergodic theorem, the sequence $\{z_n\}$ converges weakly to some element of $\mathrm{Fix}(T)$. This result has been extended to Banach spaces by Reich, Bruck and Hirano and others (see [4, 5, 7, 8, 11, 12, 13]). They also extended the weak ergodic convergence to the weak almost convergence of the sequence $\{T^k x\}$ in Banach spaces. Another type of generalization of ergodic theorems from Hilbert spaces to Banach spaces is the dual ergodic theorem, which was first studied by Bruck and Reich [5]. These theorems establish the weak almost convergence to zero for $\{J(T^k x - p)\}$, where J is the duality mapping and p is an appropriate element of $\mathrm{Fix}(T)$.

Let $\{T_n\}$ be a sequence of nonexpansive self-mappings of C, and let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = T_n x_n, \quad x_1 = x \in C \tag{1.2}$$

and

$$z_n = \frac{1}{n}\sum_{k=1}^{n} x_k. \tag{1.3}$$

Akatsuka et al. [1] proved that in a Hilbert space, if the sequence $\{T_n\}$ converges pointwise to T and

$$\mathrm{Fix}(T) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset, \tag{1.4}$$

then the sequence $\{z_n\}$ given by (1.3) converges weakly to an element of $\mathrm{Fix}(T)$. This result is an extension of Baillon's nonlinear ergodic theorem for a sequence of nonexpansive mappings. These studies can be applied when examining iterative methods involving nonexpansive mappings, such as the Mann iteration and the proximal point algorithm. Recently, the authors [15] proved some ergodic theorems for sequences given by (1.2) and (1.3) in Hilbert spaces by replacing the assumption (1.4) with the following weaker condition:

$$\sum_{n=1}^{\infty} \|T_n q - q\| < \infty. \tag{1.5}$$

Kobayashi et al. [9] proved the almost convergence of the sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} \sup_{m\geq 1} \|x_{n+m+1} - T_{n+m}\cdots T_n x_n\| = 0 \tag{1.6}$$

to a fixed point of T if (1.5) holds and $T_n \to T$ uniformly on bounded subsets of E. The condition (1.6) is more general than (1.2). The sequence $\{x_n\}$ satisfying in (1.6) is called an almost-orbit of the sequence $\{T_n\}$. The notion of almost-orbit for iterations of a nonlinear mapping was first defined by Bruck [4]. An almost-orbit for a mapping T is defined as

$$\lim_{n\to\infty} \sup_{m\geq 1} \|x_{n+m} - T^m x_n\| = 0.$$

It is well known that the almost-orbit exhibits similar asymptotic behavior of the orbit of T. For a sequence of mappings, the concept of almost-orbit has been extended as described in (1.6) [9]. Here, we consider another definition of almost-ordit defined by

$$\lim_{n\to\infty}\sup_{m\geq 1}\|x_{n+m}-T_n^m x_n\|=0. \tag{1.7}$$

In the next section, we study the dual ergodic theorem for sequences $\{x_n\}$ satisfying (1.6) and (1.7). We also replace the condition (1.5) with one of the following conditions

$$\lim_{n\to\infty}\sup_{m\geq 1}\|T_{n+m}\cdots T_n q-q\|=0 \tag{1.8}$$

$$\lim_{n\to\infty}\sup_{m\geq 1}\|T_n^m q-q\|=0,\ \forall q\in \mathrm{Fix}(T). \tag{1.9}$$

Both conditions are weaker than (1.4), and Condition (1.8) is weaker than (1.5). In the next section, we present the main our theorem and in Section 3, we give some applications to iterative methods.

2. A Dual Ergodic Theorem

In [5], the authors proved an ergodic theorem for uniformly smooth Banach spaces as follows.

Theorem 2.1. *Let E be a uniformly smooth Banach space and C a nonempty closed convex subset of E. If T is a nonexpansive self-mapping of C with $\mathrm{Fix}(T)\neq\varnothing$, then there exists $p\in\mathrm{Fix}(T)$ such that for all $x\in C$, $\{J(T^n x-p)\}$ is weakly almost convergent to zero in E^*.*

The following theorem is the main result of the paper.

Theorem 2.2. *Let E be a uniformly smooth Banach space and $C\subset E$ a nonempty closed convex set. Suppose that $T_n : C\to C$ is a sequence of nonexpansive mappings that converges pointwise to T with $\mathrm{Fix}(T)\neq\varnothing$. If either*

(1) x_n is generated by (1.6) and condition (1.8) is satisfied, or

(2) x_n is given by (1.7) and the condition (1.9) is satisfied,

then there exists an element $v\in\mathrm{Fix}(T)$ such that $\{J(x_n-v)\}$ is weakly almost convergent to 0 in E^*.

Proof. Let $q\in\mathrm{Fix}(T)$. Assume that condition (1) is satisfied. By the triangular inequality, we have:

$$\begin{aligned}\|x_{n+m+1}-q\| &\leq \|x_{n+m+1}-T_{n+m}\cdots T_n x_n\|+\|T_{n+m}\cdots T_n x_n-T_{n+m}T_n q\|+\|T_{n+m}\cdots T_n q-q\|\\ &\leq \sup_{m\geq 1}\|x_{n+m+1}-T_{n+m}\cdots T_n x_n\|+\|x_n-q\|+\sup_{m\geq 1}\|T_{n+m}\cdots T_n q-q\|.\end{aligned}$$

Taking limsup as $m\to\infty$ and liminf as $n\to\infty$ implies that $\lim_{n\to\infty}\|x_n-q\|$ exists, and hence $\{x_n\}$ is bounded. Now, by assumption (2), we have:

$$\begin{aligned}\|x_{n+m}-q\| &\leq \|x_{n+m}-T_n^m x_n\|+\|T_n^m x_n-T_n^m q\|+\|T_n^m q-q\|\\ &\leq \|x_{n+m}-T_n^m x_n\|+\|x_n-q\|+\|T_n^m q-q\|.\end{aligned}$$

Taking limsup as $m \to \infty$, we obtain:

$$\limsup \|x_n - q\| \leq \sup_{m \geq 1} \|x_{n+m} - T_n^m x_n\| + \|x_n - q\| + \sup_{m \geq 1} \|T_n^m q - q\|.$$

Now, taking liminf as $n \to \infty$, we conclude that $\lim_{n\to\infty} \|x_n - q\|$ exists. Let Lim be any Banach limit and define $F : E \to \mathbb{R}^+$ by $F(z) = \mathrm{Lim}\|x_{n+1} - z\|^2$. The functional F is a continuous, convex, and coercive. Since E is reflexive, F attains its infimum over E. Therefore, the set M defined by

$$M = \{u \in E \ : \ \mathrm{Lim}\|x_n - u\|^2 = \inf_{z \in E} \mathrm{Lim}\|x_n - z\|^2\}$$

is a nonempty closed convex bounded set. Since Lim x_n = Lim x_{n+1} and T_n (for $n = 1, 2, \cdots$) are nonexpansive mappings such that $T_n \to T$, with both conditions (1) and (2) of the theorem, we have

$$\begin{aligned} F(Tu) &= \mathrm{Lim}\|x_n - Tu\|^2 \\ &= \mathrm{Lim}\|x_{n+1} - Tu\|^2 \\ &\leq \mathrm{Lim}(\|x_{n+1} - T_n x_n\| + \|T_n x_n - T_n u\| + \|T_n u - Tu\|)^2 \\ &\leq \mathrm{Lim}(\|x_n - u\| + \|T_n u - Tu\|)^2 \\ &= \mathrm{Lim}\|x_n - u\|^2 \\ &= F(u). \end{aligned} \tag{2.1}$$

Thus, we conclude that M is invariant under T and F is a Lyapunov function for T. Consequently, T has a fixed point $v \in M$, which is also a minimizer of F. Since $\{\|x_n - q\|\}$ converges for all $q \in \mathrm{Fix}(T)$, it follows that $\lim_{n\to\infty} \|x_n - v\|$ exists. This shows that the fixed point v is independent of the chosen Banach limit. Since v minimizes F, we have $F^{'}(v) = 0$. By the subdifferential inequality, we obtain:

$$\|x_n - (v + \lambda h)\|^2 - \|x_n - v\|^2 \geq 2\langle \lambda h, J(x_n - v)\rangle.$$

By taking Banach limit, we get:

$$\mathrm{Lim}\|x_n - (v + \lambda h)\|^2 - \mathrm{Lim}\|x_n - v\|^2 \geq 2\mathrm{Lim}\langle \lambda h, J(x_n - v)\rangle.$$

Dividing both sides of the last inequality by λ for $\lambda > 0$ and $\lambda < 0$, and letting $\lambda \to 0$, we arrive at:

$$\mathrm{Lim}\langle h, J(x_n - v)\rangle = \langle h, F'(v)\rangle = 0$$

for all $h \in E$ and any Banach limit Lim. By Lorentz's theorem [10], $\{\langle h, J(x_n - v)\rangle\}$ is almost convergent to 0, i.e. $\{J(x_n - v)\}$ is weakly almost convergent to 0 in E^*. □

3. Applications to some Iterative Methods

1. **Fixed point iteration.** Consider the sequence given by the Mann iteration

$$x_{n+1} = c_n x_n + (1 - c_n)Tx_n + e_n$$

where $T : C \to C$ is a nonexpansive mapping, $\{c_n\}$ is a positive real sequence and $\{e_n\}$ a sequence in a uniformly smooth Banach space E. It is easily seen that $\{x_n\}$ satisfies (1.6) with $T_n = c_n I + (1 - c_n)T$ (I is the identity mapping), if

$$\sum_{n=1}^{\infty} \|e_n\| < +\infty. \tag{3.1}$$

$T_n \to T$ if $c_n \to 0$ and $\mathrm{Fix}(T_n) = \mathrm{Fix}(T)$ implies condition (1.8). By Theorem 2.2, we can conclude that there exists $p \in \mathrm{Fix}(T)$ such that $\{J(x_n - p)\}$ is weakly almost convergent to the zero of E^*.

2. **Zeros of accretive operators.** A set-valued operator $A : E \to 2^E$ is called accretive if for all $x, y \in D(A) = \{x \in E : \quad Ax \neq \varnothing\}$ and for all $u \in Ax,\ v \in Ay$, we have:

$$\langle u - v, j\rangle \geq 0,$$

for all $j \in J(x-y)$, where $J : E \to 2^{E^*}$ is the duality mapping. An accretive operator A is m-accretive if $I + A$ is surjective, where I is the identity operator. It is well known that if A is an m-accretive operator, then for all $\lambda > 0$, $\mathcal{J}_\lambda : E \to E$ defined by $\mathcal{J}_\lambda := (I + \lambda A)^{-1}$ is a single-valued nonexpansive mapping, which is called the resolvent of A of order λ. The following identity holds for all $\lambda, \mu > 0$:

$$\mathcal{J}_\lambda x = \mathcal{J}_\mu\Big(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}\mathcal{J}_\lambda x\Big), \quad \forall x \in E \tag{3.2}$$

For accretive operators and their properties the reader can refer to [6, 14]. Consider the following proximal point algorithm for the m-accretive operator A

$$x_{n+1} = \mathcal{J}_{\lambda_n} x_n + e_n, \tag{3.3}$$

where $\lambda_n > 0$ and the $\{e_n\}$ is an error sequence in E. As an application of Theorem 2.2, we have the following result.

Corollary 3.1. *Let E be a uniformly smooth Banach space and let $A : E \to 2^E$ be an accretive operator with $A^{-1}(0) \neq \varnothing$. If $\{x_n\}$ is the sequence given by (3.3), $\sum_{n=1}^\infty \|e_n\| < +\infty$, and $\lambda_n \to \lambda > 0$, then there is an element $p \in A^{-1}(0)$ such that $\{J(x_n - p)\}$ is weakly almost convergent to the zero in E^*.*

Proof. Applied Part (i) of Theorem 2.2 with $T_n = \mathcal{J}_{\lambda_n}$, $T = \mathcal{J}_\lambda$. By (3.2) for all $x \in E$, we have

$$\begin{aligned}\|\mathcal{J}_{\lambda_n} x - \mathcal{J}_\lambda x\| &= \|\mathcal{J}_{\lambda_n} x - \mathcal{J}_{\lambda_n}\Big(\frac{\lambda_n}{\lambda}x + \frac{\lambda-\lambda_n}{\lambda}\mathcal{J}_\lambda x\Big)\| \\ &\leq \frac{|\lambda-\lambda_n|}{\lambda}\|x - \mathcal{J}_\lambda x\| \to 0,\end{aligned}$$

as $n \to \infty$. It is easily seen that sumability of $\{e_n\}$ implies (1.6). Condition (1.8) is also satisfied, because $\mathrm{Fix}(\mathcal{J}_\lambda) = A^{-1}(0)$ for all $\lambda > 0$. ☐

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