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Research Paper

NEW CONSTRUCTIONS OF SCALABLE K-FRAMES

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Abstract. The invariance of frames and their extensions under the operator perturbation is one of the most important problems in frame theory. In this paper, we focus on the stabilities of scalable K-frames under the operator perturbation and then we construct new scalable K-frames for Hilbert spaces by some operator theory tools. More precisely, we investigate several sufficient and/or conditions of the operator perturbation for a scalable K-frame by using certain operators with specific properties. Finally, since the finite sum of scalable K-frames may not be a scalable K-frames for the Hilbert space, we demonstrate that under some special conditions, the sum of two scalable K-frames remains a scalable K-frame.
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1. Introduction and Background

In the realm of functional analysis, frames extend the concept of orthonormal bases in Hilbert spaces, offering a more flexible and robust way to represent signals and data. While an orthonormal basis provides a unique and minimal representation, a frame allows for redundant (or "overcomplete") representations. This redundancy is a powerful feature that provides resilience against data loss and noise and enables more stable reconstruction. A sequence $\{f_i\}_{i\in\mathbb{I}}$ of elements in a Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1.1}$$

The numbers A, B are called frame bounds. If only the right inequality of (1.1) holds, $\{f_i\}_{i\in\mathbb{I}}$ is called a Bessel sequence. Frames are very important for applications, e.g. in physics [1, 8], signal processing [5, 4, 2] and acoustics [3, 12].

Various specialized types of frames have been developed to address the diverse needs of theoretical and applied mathematics. Among these, the K-frame stands out as a significant generalization. Introduced by L. Găvruţa [10], the concept of K-frames was initially developed to facilitate the study of atomic systems in relation to bounded linear operators. This framework offers greater flexibility compared to classical frames. Although the effective span or reconstruction capability of K-frames is primarily constrained to the range of the operator K, this very generality is what underscores their considerable practical importance in various applications.

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As mentioned, a standard frame $\{f_i\}_{i\in\mathbb{I}}$ for a Hilbert space $\mathcal{H}$ guarantees that any vector $f \in \mathcal{H}$ can be represented and reconstructed from its frame coefficients $\langle f, f_i\rangle$. However, in computational settings, the reconstruction process often involves the frame operator, which can be computationally complicated, especially for large and redundant systems. To address these practical considerations, the concept of scalable frames was introduced. Given a scalable frame, by rescaling its elements, it can be modified to become a Parseval frame. This rescaling obviates the need for computing the inverse of the frame operator, thereby simplifying reconstruction processes. This practical advantageis the main motivation of the theory of scalable frame theory. The concept of scalable frames was first introduced in [11] and has since been extensively developed by numerous researchers.

Parseval K-frames represent one of the most significant relevant types of K-frames, finding important applications, particularly in areas like signal communication. Their importance stems from their simplified reconstruction properties and enhanced numerical stability compared to general K-frames. Given their desirable characteristics, it is indeed a very natural and crucial question to ask: How can one construct a Parseval K-frame from a given general K-frame? This motivation led to the definition of scalable K-frames. Ramesan and Ravindran [13] proposed the concept of scalable K-frames and investigated some properties of them.

As scalable K-frames have advantages in practical applications over other frames, it has become an important topic to make full use of various conditions to construct a new scalable K-frames from a given scalable frames or Bessel sequences. This paper focuses on constructing scalable K-frames by the ways which are relatively simple and different from previous methods with certain special operators. Moreover, we discuss the sum of scalable K-frames. In fact, we provide some conditions under which, the sum of two scalable K-frames is a scalable K-frame.

2. Notation and definitions

In this section, we collect the basic notation and some preliminary results. Throughout the paper, $\mathcal{H}$ is a separable Hilbert space and $\mathbb{I}$ is an at most countable index set. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded and linear operators on $\mathcal{H}$. For $U \in \mathcal{B}(\mathcal{H})$, the notations U^*, $R(U)$ and $N(U)$ denote respectively the adjoint operator, the range and the null space of U. It is well known from [7] that if $U \in \mathcal{B}(\mathcal{H})$, then $N(U^*) = R(U)^\perp$. Also, if U is closed range, then $R(U) = N(U^*)^\perp$. The operator $U \in \mathcal{B}(\mathcal{H})$ is called isometry if $U^*U = I$ and is called co-isometry if $UU^* = I$. Furthermore, $U \in \mathcal{B}(\mathcal{H})$ is called unitary if $U^* = U^{-1}$. For non-invertible operators, various types of generalized inverses exist in the literature. One of the most importand of these generalized inverses is called the pseudo-inverse operator. In the next lemma, we collect some properties of this operator.

Lemma 2.1 ([6])**.** *Suppose that U is a bounded and invertible operator on $\mathcal{B}(\mathcal{H})$ with closed range $R(U)$. Then, there exists a unique operator $U^\dagger \in \mathcal{B}(\mathcal{H})$, called the pseudo-inverse of U, satisfying*

$$\begin{aligned} UU^\dagger U &= U, & U^\dagger U U^\dagger &= U^\dagger, \\ \left(UU^\dagger\right)^* &= UU^\dagger, & \left(U^\dagger U\right)^* &= U^\dagger U, \\ N\left(U^\dagger\right) = (R(U))^\perp &= N\left(U^*\right), & R\left(U^\dagger\right) = (N(U))^\perp &= R\left(U^*\right). \end{aligned}$$

The following useful result, so-called Douglas's theorem, will be used in the sequel.

Theorem 2.2. [9] *Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$. The following statements are equivalent:*

(1) $R(T) \subseteq R(S)$.
(2) *There exists* $\lambda > 0$ *such that* $TT^* \leq \lambda SS^*$.
(3) *There exists* $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $T = SU$.

K-frames which recently introduced by Găvruţa [10] are generalization of frames, in the meaning that the lower frame bound only holds for that admits to reconstruct from the range of a linear and bounded operator K in a Hilbert space.

Definition 2.3. A sequence $\{f_i\}_{i\in\mathbb{I}}$ of elements in $\mathcal{H}$ is called a K-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}).$$

In particular, if $A\|K^*f\|^2 = \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2$, for all $f \in \mathcal{H}$, the K-frame is said to be a tight K-frame for $\mathcal{H}$. If $A = 1$, then it is called a Parseval K-frame for $\mathcal{H}$.

Recent studies highlight the utility of K-frames in addressing data loss in signal communication. Specifically, Parseval K-frames and tight K-frames are particularly effective for this purpose. Given their importance, there's a strong interest in methods that can transform an existing K-frame into either a Parseval K-frame or a tight K-frame. The most straightforward approach to achieving this modification is by scaling the vectors of the given K-frame. This concept so called scalable K-frame was first introduced in [13] as follows.

Definition 2.4. A K-frame $\{f_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ is said to be scalable K-frame for $\mathcal{H}$ if there exist non-negative scalars $\{a_i\}_{i\in\mathbb{I}}$ such that $\{a_if_i\}_{i\in\mathbb{I}}$ is a Parseval K-frame for $\mathcal{H}$.

Before examining the results, it is worth noting that a sequence $\{a_i\}_{i\in\mathbb{I}}$ is called semi-normalized if there are bounds $0 < c \leq d < \infty$, such that $c \leq |a_i| \leq d$, for each $i \in \mathbb{I}$.

3. Main Results

In this section, we construct new scalable K-frames for Hilbert spaces. In fact, we discuss the operator perturbation of scalable K-frame. Let us first present a lemma needed in the sequel.

Lemma 3.1. *Let* $F = \{f_i\}_{i\in\mathbb{I}}$ *be a Bessel sequence with upper bound* B *and* $a = \{a_i\}_{i\in\mathbb{I}}$ *be a semi-normalized sequence of non negative scalars. Then the operator*

$$T_{aF} : \ell^2(\mathbb{I}) \to \mathcal{H}, \quad T_{aF}(\{c_i\}_{i\in\mathbb{I}}) := \sum_{i\in\mathbb{I}} a_ic_if_i,$$

is a bounded and linear operator on $\mathcal{H}$.

Proof. Since $F = \{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence with upper bound B, so by [6, Theorem 3.2.3], the synthesis operator

$$T_F : \ell^2(\mathbb{I}) \to \mathcal{H}, \quad T_F(\{c_i\}_{i\in\mathbb{I}}) := \sum_{i\in\mathbb{I}} c_if_i, \quad \left(\{c_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})\right),$$

is bounded and $\|T_F\| \leq \sqrt{B}$. So, for a semi-normalized sequence $\{a_i\}_{i\in\mathbb{I}}$ with upper bound d, we have

$$\|T_{aF}(\{c_i\}_{i\in\mathbb{I}})\| = \left\|\sum_{i\in\mathbb{I}} a_ic_if_i\right\| \leq d\left\|\sum_{i\in\mathbb{I}} c_if_i\right\|,$$

which implies that $\|T_{aF}\| \leq c\sqrt{B}$ and so T_{aF} is bounded. The linearity of T_{aF} is clear. □

In the next two propositions, we construct scalable K-frames by using the corresponding synthesis operators of the given Bessel sequences.

Proposition 3.2. *Suppose that $F = \{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence for $\mathcal{H}$ and $a = \{a_i\}_{i\in\mathbb{I}}$ is a semi-normalized sequence of non negative scalars. Then, $F = \{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$ if and only if $R(K) = R(T_{aF})$.*

Proof. Let $R(K) = R(T_{aF})$. Then, by Theorem 2.2, there exists $\lambda > 0$ such that $KK^* = \lambda T_{aF}T_{aF}^*$. So,

$$KK^*f = \lambda T_{aF}T_{aF}^* = \lambda \sum_{i\in\mathbb{I}} \langle f, a_i f_i\rangle a_i f_i = \sum_{i\in\mathbb{I}} \left\langle f, \sqrt{\lambda} a_i f_i \right\rangle \sqrt{\lambda} a_i f_i,$$

which shows that $F = \{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame with the scaling sequence $\{\sqrt{\lambda} a_i\}_{i\in\mathbb{I}}$. Conversely, assume that $F = \{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame with the scaling sequence $\{\sqrt{\lambda} a_i\}_{i\in\mathbb{I}}$, for some $\lambda > 0$. Then, for every $f \in \mathcal{H}$,

$$KK^*f = \sum_{i\in\mathbb{I}} \left\langle f, \sqrt{\lambda} a_i f_i \right\rangle \sqrt{\lambda} a_i f_i = \lambda \sum_{i\in\mathbb{I}} \langle f, a_i f_i\rangle a_i f_i = \lambda T_{aF}T_{aF}^* f.$$

So, by Theorem 2.2, we have $R(K) = R(T_{aF})$. □

The next result is an immediate consequence of Proposition 3.2.

Corollary 3.3. *Suppose that $F = \{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence for $\mathcal{H}$ and $a = \{a_i\}_{i\in\mathbb{I}}$ is a semi-normalized sequence of non negative scalars. If there exists an isometry operator $U : \mathcal{H} \to \ell^2(\mathbb{I})$ such that $T_{aF} = KU^*$, then $F = \{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$.*

Proof. For each $f \in \mathcal{H}$,

$$T_{aF}T_{aF}^* f = (KU^*)(UK^*)f = KK^*f.$$

So, by Theorem 2.2, $R(T_{aF}) = R(K)$ and hence by Proposition 3.2, $F = \{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$. □

Proposition 3.4. *Let $F = \{f_i\}_{i\in\mathbb{I}}$ be a scalable K-frame with the scaling sequence $\{a_i\}_{i\in\mathbb{I}}$. Moreover, let $G = \{g_i\}_{i\in\mathbb{I}}$ be a Bessel sequence and $b = \{b_i\}_{i\in\mathbb{I}}$ be a semi-normalized sequence of non negative scalars. If there exists a co-isometry operator $U \in \mathcal{B}(\ell^2(\mathbb{I}))$ such that $T_{aF} = T_{bG}U$, then $G = \{g_i\}_{i\in\mathbb{I}}$ is a scalable K-frame with the scaling sequence $b = \{b_i\}_{i\in\mathbb{I}}$.*

Proof. For each $f \in \mathcal{H}$,

$$\|K^*f\|^2 = \sum_{i\in\mathbb{I}} |\langle f, a_i f_i\rangle|^2 = \|T_{aF}^* f\|^2 = \|U^*T_{bG}^* f\|^2 = \|T_{bG}^* f\|^2 = \sum_{i\in\mathbb{I}} |\langle f, b_i g_i\rangle|^2,$$

so the proof is complete. □

The following result provides some equivalent conditions under which a sequence $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable K-frame, where U is a co-isometry operator.

Proposition 3.5. *Let $U \in \mathcal{B}(\mathcal{H})$ be a co-isometry operator such that $UK = KU$. Then, $\{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $R(U^*)$ if and only if $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$.*

Proof. First, assume that $\{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $R(U^*)$. Then, for every $f \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{I}} |\langle f, a_i U f_i\rangle|^2 = \sum_{i\in\mathbb{I}} |\langle U^*f, a_i f_i\rangle|^2 = \|K^*U^*f\|^2 = \|U^*K^*f\|^2 = \|K^*f\|^2,$$

so $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$. Conversely, let $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$ and $g \in R(U^*)$. Then, there exists $f \in \mathcal{H}$ such that $g = U^*f$. We have,

$$\begin{aligned}\|K^*g\|^2 = \|K^*U^*f\|^2 = \|U^*K^*f\|^2 = \|K^*f\|^2 &= \sum_{i\in\mathbb{I}} |\langle f, a_iUf_i\rangle|^2 \\ &= \sum_{i\in\mathbb{I}} |\langle U^*f, a_if_i\rangle|^2 \\ &= \sum_{i\in\mathbb{I}} |\langle g, a_if_i\rangle|^2,\end{aligned}$$

hence $\{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $R(U^*)$. □

In the following, we give a proposition which can construct scalable K-frames on subspace $R(U)$ rather than $\mathcal{H}$.

Proposition 3.6. *Let $\{f_i\}_{i\in\mathbb{I}}$ be a scalable K-frame for $\mathcal{H}$ and $U \in \mathcal{B}(\mathcal{H})$ is a closed range operator. Then, the sequence $\{UU^\dagger f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $R(U)$.*

Proof. Assume that $\{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$ with scaling sequence $\{a_i\}_{i\in\mathbb{I}}$. Then, for every $f \in R(U)$,

$$\sum_{i\in\mathbb{I}} \left|\langle f, a_iUU^\dagger f_i\rangle\right|^2 = \sum_{i\in\mathbb{I}} \left|\langle UU^\dagger f, a_if_i\rangle\right|^2 = \sum_{i\in\mathbb{I}} |\langle f, a_if_i\rangle|^2 = \|K^*f\|^2.$$

□

Next, necessary and sufficient conditions for scalable K-frames are given.

Proposition 3.7. *Let $K \in \mathcal{B}(\mathcal{H})$ be a surjective operator and $\{f_i\}_{i\in\mathbb{I}}$ be a scalable K-frame for $\mathcal{H}$. Moreover, suppose that $U \in \mathcal{B}(\mathcal{H})$ has closed range. Then, U is surjective if and only if the sequence $\{UU^\dagger f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$.*

Proof. Assume that $U \in \mathcal{B}(\mathcal{H})$ is surjective. Then, by Proposition 3.6, the sequence $\{UU^\dagger f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $R(U) = \mathcal{H}$. Conversely, let $\{UU^\dagger f_i\}_{i\in\mathbb{I}}$ be a scalable K-frame for $\mathcal{H}$. Then, for every $f \in \mathcal{H}$,

$$\sum_{i\in\mathbb{I}} \left|\langle f, a_iUU^\dagger f_i\rangle\right|^2 = \|K^*f\|^2. \tag{3.1}$$

Now, if $UU^\dagger f = 0$, for some $f \in \mathcal{H}$, then by (3.1), we have $K^*f = 0$. Thus, the surjectivity of K implies that $f = 0$, that is concluded that $UU^\dagger$ and so $U^\dagger$ is injective. Since $N(U^\dagger) = N(U^*)$, we deduce that U is surjective. □

Now, we show that a scalable K-frame can construct a scalable W-frame by the perturbation of some co-isometry bounded linear operators.

Proposition 3.8. *Suppose that $\{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$ and $W \in \mathcal{B}(\mathcal{H})$. Moreover, suppose that there exists a co-isometry operator $U \in \mathcal{B}(\mathcal{H})$ with $WU = UK$. Then, $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable W-frame for $\mathcal{H}$.*

Proof. For every $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} |\langle f, a_i U f_i\rangle|^2 = \sum_{i\in\mathbb{I}} |\langle U^* f, a_i f_i\rangle|^2 = \|K^*U^*f\|^2 &= \|(UK)^*f\|^2 \\ &= \|(WU)^*f\|^2 \\ &= \|U^*W^*f\|^2 \\ &= \|W^*f\|^2.\end{aligned}$$

□

The next proposition gives an equivalent characterization of the operator perturbation, related to an unitary operator, for a scalable K-frame.

Proposition 3.9. *Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator such that $UK = KU$. Then, $\{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$ if and only if $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$.*

Proof. Since U is a unitary operator, so $UU^* = U^*U = I$. Now, assume that $\{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$. Then

$$\sum_{i\in\mathbb{I}} |\langle f, a_i U f_i\rangle|^2 = \sum_{i\in\mathbb{I}} |\langle U^* f, a_i f_i\rangle|^2 = \|K^*U^*f\|^2 = \|U^*K^*f\|^2 = \|K^*f\|^2,$$

which shows that $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$. Conversely, suppose that $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$. Clearly, U^* is also a unitary operator and $U^*K = KU^*$. Hence, $\{U^*(Uf_i)\}_{i\in\mathbb{I}} = \{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$. □

Proposition 3.10. *Suppose that $K \in \mathcal{B}(\mathcal{H})$ is a surjective operator and $U \in \mathcal{B}(\mathcal{H})$ is a self-adjoint and closed range operator. Moreovere, let $\{Uf_i\}_{i\in\mathbb{I}}$ be a scalable K-frame for $\mathcal{H}$. Then, U is invertible and $\{f_i\}_{i\in\mathbb{I}}$ is a scalable $U^{-1}K$-frame for $\mathcal{H}$.*

Proof. First, we show that U^* is injective. Since, $\{Uf_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$, so for every $f \in \mathcal{H}$,

$$\|K^*f\|^2 = \sum_{i\in\mathbb{I}} |\langle f, a_i U f_i\rangle|^2 = \sum_{i\in\mathbb{I}} |\langle U^* f, a_i f_i\rangle|^2 .$$

So, it is concluded that $N(U^*) \subseteq N(K^*)$. Now, surjectivity of K implies that K^* and hence U^* is injective. Since, U^* is injective, self adjoint and closed range, so it is surjective and hence an invertible operator. On the other hand, for every $f \in \mathcal{H}$, there exists some $g \in \mathcal{H}$ such that $f = Ug$. So,

$$\begin{aligned}\sum_{i\in\mathbb{I}} |\langle f, a_i f_i\rangle|^2 &= \sum_{i\in\mathbb{I}} |\langle Ug, a_i f_i\rangle|^2 \\ &= \sum_{i\in\mathbb{I}} |\langle g, a_i U f_i\rangle|^2 \\ &= \|K^*g\|^2 = \|K^*U^{-1}f\|^2 = \|(U^{-1}K)^*f\|^2,\end{aligned}$$

which yields that $\{f_i\}_{i\in\mathbb{I}}$ is a scalable $U^{-1}K$-frame for $\mathcal{H}$. □

4. Sum of scalable K-frames

This section examines the sum of scalable K-frames, establishing sufficient conditions for when the sum, as well as their operator perturbations, qualifies as a scalable K-frame.

Proposition 4.1. *Suppose that $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ are two scalable K-frames for $\mathcal{H}$ with scaling sequences $\{a_i\}_{i\in\mathbb{I}}$ and $\{b_i\}_{i\in\mathbb{I}}$, respectively. If $T_{bG}T^*_{aF} = 0$, then $\{a_i f_i + b_i g_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$.*

Proof. Before proceding, we recall from Lemma 3.1 that the operator $T_{aF} : \ell^2(\mathbb{I}) \to \mathcal{H}$ is bounded and linear and it's adjoint is defined as follows.

$$T^*_{aF} : \mathcal{H} \to \ell^2(\mathbb{I}), \quad T^*_{aF}(f) = \{\langle f, a_i f_i\rangle\}_{i\in\mathbb{I}}, \ \ (f \in \mathcal{H}).$$

Put $c_i = \frac{1}{\sqrt{2}}$, for each $i \in \mathbb{I}$. Now, for every $f \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in\mathbb{I}} |\langle f, c_i(a_i f_i + b_i g_i)\rangle|^2 &= \sum_{i\in\mathbb{I}} \left|\left\langle f, \frac{a_i}{\sqrt{2}} f_i\right\rangle + \left\langle f, \frac{b_i}{\sqrt{2}} g_i\right\rangle\right|^2 \\
&= \frac{1}{2}\sum_{i\in\mathbb{I}} |\langle f, a_i f_i\rangle|^2 + \frac{1}{2}\sum_{i\in\mathbb{I}} |\langle f, b_i g_i\rangle|^2 + Re\left(\sum_{i\in\mathbb{I}} \langle f, a_i f_i\rangle\langle b_i g_i, f\rangle\right) \\
&= \frac{1}{2}\|K^* f\|^2 + \frac{1}{2}\|K^* f\|^2 + Re\langle T_{bG}T^*_{aF} f, f\rangle \\
&= \|K^* f\|^2.
\end{aligned}$$

□

Proposition 4.2. *Suppose that $F = \{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$ with scaling sequences $\{a_i\}_{i\in\mathbb{I}}$ and $U \in \mathcal{B}(\mathcal{H})$ is a co-isometry operator with $UK = KU$. If $US_{aF} = 0$, then the sequence $\{a_i f_i + a_i U f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$, where $S_{aF} = T_{aF}T^*_{aF}$ is the frame operator.*

Proof. Put $c_i = \frac{1}{\sqrt{2}}$, for each $i \in \mathbb{I}$. Now, for every $f \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in\mathbb{I}} |\langle f, c_i(a_i f_i + a_i U f_i)\rangle|^2 &= \sum_{i\in\mathbb{I}} \left|\left\langle f, \frac{a_i}{\sqrt{2}} f_i\right\rangle + \left\langle f, \frac{a_i}{\sqrt{2}} U f_i\right\rangle\right|^2 \\
&= \frac{1}{2}\sum_{i\in\mathbb{I}} |\langle f, a_i f_i\rangle|^2 + \frac{1}{2}\sum_{i\in\mathbb{I}} |\langle U^* f, a_i f_i\rangle|^2 \\
&+ Re\left(\sum_{i\in\mathbb{I}} \langle f, a_i f_i\rangle\langle a_i f_i, U^* f\rangle\right) \\
&= \frac{1}{2}\|K^* f\|^2 + \frac{1}{2}\|K^* U^* f\|^2 + Re\langle US_{aF} f, f\rangle \\
&= \frac{1}{2}\|K^* f\|^2 + \frac{1}{2}\|U^* K^* f\|^2 \\
&= \|K^* f\|^2.
\end{aligned}$$

□

Corollary 4.3. *Suppose that $F = \{f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$ with scaling sequences $\{a_i\}_{i\in\mathbb{I}}$ and $U \in \mathcal{B}(\mathcal{H})$ is a co-isometry operator with $UK = KU$. If $US_{aF} = 0$, then for any natural number n, the sequence $\{a_i f_i + a_i U^n f_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$.*

Proposition 4.4. *Suppose that $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ are two scalable K-frames for $\mathcal{H}$ with scaling sequences $\{a_i\}_{i\in\mathbb{I}}$ and $\{b_i\}_{i\in\mathbb{I}}$, respectively, and $U_1, U_2 \in \mathcal{B}(\mathcal{H})$ are two co-isometry operators with $U_1K = KU_1$ and $U_2K = KU_2$. If $U_2T_{bG}(U_1T_{aF})^* = 0$, then $\{a_iU_1f_i + b_iU_2g_i\}_{i\in\mathbb{I}}$ is a scalable K-frame for $\mathcal{H}$.*

Proof. Put $c_i = \frac{1}{\sqrt{2}}$, for each $i \in \mathbb{I}$. Now, for every $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} |\langle f, c_i(a_iU_1f_i + b_iU_2g_i)\rangle|^2 &= \sum_{i\in\mathbb{I}} \left|\left\langle f, \frac{a_i}{\sqrt{2}}U_1f_i\right\rangle + \left\langle f, \frac{b_i}{\sqrt{2}}U_2g_i\right\rangle\right|^2 \\ &= \frac{1}{2}\sum_{i\in\mathbb{I}} |\langle U_1^*f, a_if_i\rangle|^2 + \frac{1}{2}\sum_{i\in\mathbb{I}} |\langle U_2^*f, b_ig_i\rangle|^2 \\ &+ Re\left(\sum_{i\in\mathbb{I}}\langle U_1^*f, a_if_i\rangle\langle b_ig_i, U_2^*f\rangle\right) \\ &= \frac{1}{2}\|K^*U_1^*f\|^2 + \frac{1}{2}\|K^*U_2^*f\|^2 + Re\langle U_2T_{bG}(U_1T_{aF})^*f, f\rangle \\ &= \frac{1}{2}\|U_1^*K^*f\|^2 + \frac{1}{2}\|U_2^*K^*f\|^2 \\ &= \|K^*f\|^2.\end{aligned}$$

□

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