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Research Paper

RADO'S THEOREM ON VECTOR SPACE

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Abstract. In this paper, we investigate partition regularity of matrices over vector spaces where the matrix entries consist of linear transformations on the vector space. The paper also examines the existence of monochromatic solutions for matrix entries of linear transformations, which generalizes Rado's theorem.Our main results establish that matrices satisfying these conditions are not only partition regular but also preserve solutions within large combinatorial sets, such as central sets. Furthermore, we present a counterexample involving affine maps, demonstrating a limitation of these results and highlighting the necessity of the linearity assumption. This work bridges combinatorial number theory and linear algebra, offering a functional-analytic perspective on Ramsey-type problems.

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1. **Introduction**

Two central themes have shaped the development of the combinatorial theory of matrices: one concerns the behavior of matrix images over large subsets of algebraic structures, and the other involves structural conditions that ensure the existence of monochromatic solutions under finite colorings. The former focuses on how matrices transform "large" sets—such as IP, central, or piecewise syndetic sets—into sets retaining combinatorial largeness, as explored in [1, 6, 7] and formalized in Theorem 2.2. The latter investigates which matrices, when applied to finite colorings of domains such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, or more general commutative semigroups, necessarily yield monochromatic images—see [2, 3, 5].

At the heart of this inquiry lies the concept of *partition regularity*, a fundamental notion in Ramsey theory and additive combinatorics. In its classical form, a matrix $A \in \mathbb{Z}^{u\times v}$ is said to be *kernel partition regular* (KPR) over a semigroup S if, for every finite coloring of S, there exists a monochromatic vector $\vec{x} \in S^v$ such that $A\vec{x} = \vec{0}$. The iconic result in this direction is Rado's theorem [9], which characterizes such matrices via the so-called *columns condition*.

These notions form a powerful algebraic-combinatorial framework for analyzing solution sets of linear systems within partitioned algebraic structures. The conditions under which matrices exhibit KPR or IPR often relate to structural properties such as the first-entries condition and the columns condition. Furthermore, the framework allows not only for the existence of monochromatic solutions but also guarantees that such solutions occur within every sufficiently "large" subset—most notably within central sets, which play a pivotal role in the

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algebraic-topological study of partition regularity through the Stone–Čech compactification βS.

A classical example is the matrix $A = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$, corresponding to the equation $x+y=z$. Schur's Theorem asserts that this matrix is KPR over $\mathbb{N}$, ensuring that in any finite coloring of the naturals, one can find a monochromatic triple satisfying the equation. This insight connects partition regularity to a broader family of Ramsey-theoretic phenomena.

Recent advances have shifted attention toward generalizing these properties beyond matrices with numeric entries. In particular, there is growing interest in studying matrices whose entries are linear or affine transformations on infinite-dimensional vector spaces. This opens new directions for exploring the combinatorial behavior of function-valued matrices under algebraic and topological constraints.

In this paper, we develop a theory of partition regularity for matrices whose entries are linear or affine maps acting on infinite-dimensional vector spaces over a field $\mathbb{F}$. We provide extensions of classical notions such as the first-entries condition and the columns condition to operator-valued settings, investigate their implications for KPR and IPR, and examine how these properties interact with large combinatorial sets such as central and C^*-sets. Our results not only generalize Rado-type theorems to functional settings but also clarify limitations and counterexamples that arise when translating scalar-based results to operator contexts.

2. **Preliminaries**

Let S be any set. A filter on S is a nonempty set $\mathcal{U}$ of subsets of S with the following properties:

(a) If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.
(b) If $A \in \mathcal{U}$ and $A \subseteq B \subseteq S$, then $B \in \mathcal{U}$.
(c) $\emptyset \notin \mathcal{U}$.

An ultrafilter on S is a filter on S which is not properly contained in any other filter on S. We define $\beta S = \{p \mid p \text{ is an ultrafilter on } S\}$, identifying the principal ultrafilters with the points of S, so that we may consider $S \subseteq \beta S$. For each subset $A \subseteq S$, we define $\overline{A} = \{p \in \beta S : A \in p\}$. The collection $\{\overline{A} : A \subseteq S\}$ then forms a basis for the topology on βS, and each $\overline{A}$ is the closure of A in βS.

The operation $+$ on S extends to an operation, also denoted by $+$, on βS such that $(\beta S, +)$ becomes a right topological semigroup, and S is contained in the topological center of βS,that topological center of S is the set $\Lambda(S) := \{x \in S \mid \lambda_x \text{ is continuous}\}$. That is, for each $p \in \beta S$, the right translation map $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous, and for each $x \in S$, the left translation map $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x + q$ is continuous.

Despite using the same symbol $+$, the extended operation on βS is generally not commutative, even when S itself is commutative. In fact, when $S = \mathbb{N}$ or $S = \mathbb{Z}$, the topological center of βS coincides with S; that is, for any $p \in S^* := \beta S \setminus S$, the left translation λ_p is not continuous.

Given $p, q \in \beta S$ and a subset $A \subseteq S$, we have

$$A \in p + q \iff \{x \in S : -x + A \in q\} \in p,$$

where $-x + A = \{y \in S : x + y \in A\}$.

As is the case for any compact Hausdorff right topological semigroup, βS contains idempotent elements and has a smallest two-sided ideal, denoted by $K(\beta S)$. This ideal is both the union of all minimal left ideals and the union of all minimal right ideals of βS.

An idempotent in βS belongs to $K(\beta S)$ if and only if it is minimal with respect to the natural partial ordering on idempotents, defined by $p \leq q$ if and only if $p+q=q+p=p$. Such idempotents are simply called minimal.

Each minimal left ideal of βS is closed. Moreover, the intersection of any minimal left ideal with any minimal right ideal is a group, and all such groups are isomorphic.

Given a subset $X \subseteq \beta S$, we define

$$E(X)=\{p \in X : p+p=p\},$$

that is, the set of all idempotents contained in X.

It was shown in [9] that the columns condition implies kernel partition regularity (KPR). A matrix $A \in M_{u\times v}(\mathbb{Q})$ is said to satisfy the **columns condition** over $\mathbb{Q}$ if, letting $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_v$ denote the columns of A, there exists a partition $\{I_1, \dots, I_m\}$ of $\{1, \dots, v\}$ such that:

(a) $\sum_{i\in I_1} \vec{c}_i = \vec{0}$.

(b) For each $t \geq 2$, the sum $\sum_{i\in I_t} \vec{c}_i$ is a $\mathbb{Q}$-linear combination of the set $\{\vec{c}_j : j \in \bigcup_{s=1}^{t-1} I_s\}$.

As a consequence, Rado proved that a matrix $A \in M_{u\times v}(\mathbb{Q})$ is kernel partition regular over $\mathbb{N}$ if and only if it satisfies the columns condition over $\mathbb{Q}$.

A matrix $A \in M_{u\times v}(\mathbb{Q})$ is said to be *image partition regular* (IPR) over a semigroup S if, for every finite coloring of $S \setminus \{0\}$, there exists a vector $\vec{x} \in (S\setminus\{0\})^v$ such that all entries of $A\vec{x}$ are monochromatic.

We note that to ensure these concepts are well defined, the semigroup from which solutions are drawn must be suitably structured. However, in many cases this requirement is automatically satisfied when the entries of the matrix lie in an appropriate algebraic setting.

A matrix A is said to be **appropriate** for a semigroup S under the following conditions: if S is cancellative, then the entries of A lie in $\mathbb{Z}$ and A has no zero rows; otherwise, the entries lie in $\mathbb{N}_0$ (the set of natural numbers including zero), again with no zero rows.

There exists a strong connection between the notion of image partition regularity and central sets; see [4] for details. For a matrix $A \in M_{u\times v}(\mathbb{Q})$, the following statements are equivalent:

(a) A is image partition regular (IPR) over $\mathbb{N}$.

(b) For every central set $C \subseteq \mathbb{N}$, the set $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is central.

(c) For every central set $C \subseteq \mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.

The concept of **first entries matrices** is also relevant in the study of image partition regularity. A matrix $A \in M_{u\times v}(\mathbb{Q})$ is called a first entries matrix if no row is identically zero, and the first nonzero entry in each row is positive and occurs in the same column across all such rows.

Hindman [4] showed that every central set in $\mathbb{N}$ contains a solution to any first entries matrix. For example, the matrix

$$A=\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & k \end{pmatrix}$$

has a solution in every central set, for any $k \in \mathbb{N}$.

Definition 2.1 ([7])**.** Let $(S,+)$ be a commutative semigroup, and let $A \subseteq S$.

(a) A is a *Q set* whenever there exists a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that whenever $m<n$, we have $x_n \in x_m + A$.

(b) A is a *P set* whenever for every $k \in \mathbb{N}$, there exist $a, d \in S$ such that

$$\{a, a+d, \ldots, a+kd\} \subseteq A.$$

(c) A is an *IP set* whenever there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that

$$\mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A,$$

where

$$\mathrm{FS}(\langle x_n \rangle) = \left\{ \sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \right\}$$

(that $\mathcal{P}_f(\mathbb{N})$ is finite power sets of $\mathbb{N}$). Equivalently, A is an IP set if and only if there exists an idempotent $p \in \beta S$ such that $A \in p$.

(d) A is a *J set* whenever for every $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$,

$$a + \sum_{n \in H} f(n) \in A.$$

(e) A is a *C set* if and only if there exists an idempotent $p \in \beta S$ such that $p \in \overline{A} \cap \mathrm{J}(S)$, where

$$\mathrm{J}(S) = \{p \in \beta S : \forall A \in p,\ A \text{ is a J set}\}.$$

(f) A is *piecewise syndetic* (or a *PS set*) if and only if there exists $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$, there exists $x \in S$ with

$$F + x \subseteq \bigcup_{t \in G} (-t + A).$$

Equivalently, A is piecewise syndetic if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.

(g) A is a *QC set* if and only if there exists an idempotent in $\overline{A} \cap \overline{K(\beta S)}$.

(h) A is *central* if and only if there exists an idempotent in $\overline{A} \cap K(\beta S)$.

(i) A is *syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that

$$S = \bigcup_{t \in G} (-t + A).$$

Equivalently, A is syndetic if and only if for every left ideal $L \subseteq \beta S$, we have $\overline{A} \cap L \neq \emptyset$.

(j) A is an *SC set* if and only if for every left ideal $L \subseteq \beta S$, there exists an idempotent in $\overline{A} \cap L$.

(k) A is *thick* if and only if for every $F \in \mathcal{P}_f(S)$, there exists $x \in S$ such that

$$F + x \subseteq A.$$

Equivalently, A is thick if and only if there exists a left ideal $L \subseteq \beta S$ such that $L \subseteq \overline{A}$.

Given a property ϕ of subsets of S, we define its **dual property** ϕ^* as follows: a set $A \subseteq S$ is said to have property ϕ^* if and only if $A \cap B \neq \emptyset$ for every subset $B \subseteq S$ with property ϕ.

A matrix is said to be image partition regular over $\mathbb{N}$ (denoted IPR/$\mathbb{N}$) if it has a solution in every "large" subset of $\mathbb{N}$ of type Ψ, for appropriate combinations of (Ψ, X), as given in the following theorem.

Theorem 2.2 ([7])**.** *Let A be a $u \times v$ matrix with entries from a set X, and suppose that A has property Y. If $B \subseteq \mathbb{N}$ is a set with Ψ-Property, then the solution set*

$$\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$$

is also a Ψ-set in $\mathbb{N}^v$.

Theorem	$\boldsymbol{\Psi}$	$\boldsymbol{X}$	$\boldsymbol{Y}$
4.1	*C, central, SC**	$\mathbb{Q}$	*IPR/*$\mathbb{N}$
4.2	*QC*	$\mathbb{Q}$	*IPR/*$\mathbb{N}$, $\text{rank}(A) = u$
4.3	*SC*	$\mathbb{Q}^+$	*IPR/*$\mathbb{N}$, $\text{rank}(A) = u$
4.4	*thick*	$\mathbb{Z}$	*IPR/*$\mathbb{N}$
4.5	*PS**	$\mathbb{N}_0$	*IPR/*$\mathbb{N}$

Proof. See Theorem 1.4. in [7] □

In this theorem study relations related to matrices, image partition regularity, and several notions of largeness in a semigroup. The primary goal of this theorem is to show how certain subsets of the natural numbers with a Ψ-property are preserved under a linear transformation, given specific conditions.

3. Rado's Theorem on Vector Spaces

Let $\mathbb{F}$ be a field, and let V be a vector space over $\mathbb{F}$. The set of all linear transformations from V to itself is denoted by $\text{Hom}_{\mathbb{F}}(V)$. Equipped with pointwise addition and scalar multiplication, $\text{Hom}_{\mathbb{F}}(V)$ is itself a vector space over $\mathbb{F}$.

Definition 3.1. Let $u, v \in \mathbb{N}$, and let $A = [\varphi_{ij}]_{u \times v}$ be a $u \times v$ matrix with entries in $\text{Hom}_{\mathbb{F}}(V)$. We say that A satisfies the **first entries condition** if:

(a) For each $i \in \{1, \ldots, u\}$, there exists $t \in \{1, \ldots, v\}$ such that $\varphi_{it} \neq \mathbf{0}$, where $\mathbf{0} \colon V \to V$ denotes the zero map.

(b) For all $i, j \in \{1, \ldots, u\}$, if

$$k_i = \min\{t : \varphi_{it} \neq \mathbf{0}\} = \min\{t : \varphi_{jt} \neq \mathbf{0}\} = k_j,$$

then $\varphi_{i,k_i} = \varphi_{j,k_j}$ as linear transformations.

An element $\psi \in \text{Hom}_{\mathbb{F}}(V)$ is called a *first entry* of A if there exists $i \in \{1, \ldots, u\}$ such that

$$\psi = \varphi_{i,k} \quad \text{where} \quad k = \min\{t : \varphi_{it} \neq \mathbf{0}\}.$$

Theorem 3.2 ([8])**.** *Let $\mathbb{F}$ be a field, and let V be an infinite-dimensional vector space over $\mathbb{F}$. Let $u, v \in \mathbb{N}$, and let $A = [\varphi_{ij}]_{u \times v}$ be a matrix in $\text{Hom}_{\mathbb{F}}(V)^{u \times v}$ satisfying the first entries condition.*

If $C \subseteq V$ is a central set in the additive semigroup $(V, +)$, then there exist sequences $\{x_{i,n}\}_{n=1}^{\infty} \subseteq V$ for $i = 1, \ldots, v$ such that for every nonempty finite subset $G \subseteq \mathbb{N}$, the following hold:

(a) The vector

$$\vec{x}_G = \left(\sum_{n \in G} x_{1,n}, \sum_{n \in G} x_{2,n}, \ldots, \sum_{n \in G} x_{v,n} \right)^{\top}$$

lies in $(V \setminus \{0\})^v$.

(b) The image under A satisfies

$$A\vec{x}_G \in C^u,$$

where the matrix-vector product is defined component-wise by

$$(A\vec{x}_G)_i = \sum_{j=1}^{v} \varphi_{ij} \left(\sum_{n \in G} x_{j,n} \right), \quad \text{for each } i = 1, \dots, u.$$

Proof. See Theorem 2.6 in [8] □

Corollary 3.3. *Let $\mathbb{F}$ be a field, V be an infinite-dimensional vector space over $\mathbb{F}$, and $u, v \in \mathbb{N}$. Let A be a $u \times v$ matrix with entries in $\mathrm{Hom}_F(V)$ satisfying the first entries condition. Then A is* ***strongly image partition regular*** *over V. That is, for any $r \in \mathbb{N}$ and partition $V \setminus \{0\} = \bigcup_{i=1}^{r} E_i$, there exist $i \in \{1, \dots, r\}$ and $\vec{x} \in (V \setminus \{0\})^v$ such that:*

$$A\vec{x} \in (E_i)^u,$$

where $A\vec{x} = \left(\sum_{j=1}^{v} \varphi_{1j}(x_j), \dots, \sum_{j=1}^{v} \varphi_{uj}(x_j) \right)^\top$.

Proof. The set $\{0\}$ is not central in $(V, +)$. By Corollary 4.33 [4], $V^* = \beta V \setminus V$ is an ideal of $(\beta V, +)$, so all minimal idempotents lie in V^*. Thus, $\{0\}$ contains no minimal idempotent.

Since $V \setminus \{0\} = \bigcup_{i=1}^{r} E_i$, one E_i is central in $(V, +)$ by the partition properties of central sets. Apply Theorem3.2 with $C = E_i$. There exist sequences $\langle x_{j,n} \rangle_{n=1}^{\infty}$ $(1 \leq j \leq v)$ such that for any finite $G \subseteq \mathbb{N}$,

$$\vec{x}_G = \left(\sum_{n \in G} x_{1,n}, \dots, \sum_{n \in G} x_{v,n} \right)^\top \in (V \setminus \{0\})^v \quad \text{and} \quad A\vec{x}_G \in (E_i)^u.$$

For $G = \{1\}$, let $\vec{x} = (x_{1,1}, \dots, x_{v,1})^\top$. Then $\vec{x} \in (V \setminus \{0\})^v$ and $A\vec{x} \in (E_i)^u$. Thus, A is strongly image partition regular. □

Definition 3.4 (Columns Condition over a Vector Space)**.** Let V be a vector space over a field $\mathbb{F}$, and let $u, v \in \mathbb{N}$. Let $C \in \mathrm{Hom}_{\mathbb{F}}(V)^{u \times v}$ be a $u \times v$ matrix with entries in $\mathrm{Hom}_{\mathbb{F}}(V)$, and let $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_v$ denote the columns of C.

We say that C satisfies the **columns condition** over $\mathbb{F}$ if there exists $m \in \mathbb{N}$ and a partition $\{I_1, I_2, \dots, I_m\}$ of $\{1, 2, \dots, v\}$ such that:

(a) $\sum_{i \in I_1} \vec{c}_i = \vec{0}$, where the sum is interpreted entrywise in $\mathrm{Hom}_{\mathbb{F}}(V)^u$.

(b) For each $t \in \{2, 3, \dots, m\}$, let $J_t = \bigcup_{k=1}^{t-1} I_k$. Then there exists a family of scalars $\{\delta_{t,i}\}_{i \in J_t} \subseteq \mathbb{F}$ such that:

$$\sum_{i \in I_t} \vec{c}_i = \sum_{i \in J_t} \delta_{t,i} \vec{c}_i.$$

Theorem 3.5. *Let $\mathbb{F}$ be a field, V an infinite-dimensional vector space over $\mathbb{F}$, and let $u, v \in \mathbb{N}$. Let $C \in \mathrm{Hom}_{\mathbb{F}}(V)^{u \times v}$ be a matrix satisfying the columns condition over $\mathbb{F}$. Then C is* ***kernel partition regular*** *over V. That is, for every $r \in \mathbb{N}$ and every partition*

$$V \setminus \{0\} = \bigcup_{i=1}^{r} E_i,$$

there exist $i \in \{1, \dots, r\}$ and $\vec{x} \in (E_i)^v \cap (V \setminus \{0\})^v$ such that $C\vec{x} = \vec{0}$.

Proof. Let $\{I_1,\dots,I_m\}$ be the partition of $\{1,\dots,v\}$ and $\{\delta_{t,i}\}_{i\in J_t}\subseteq\mathbb{F}$ (for $t\geq 2$) be the scalars guaranteed by the columns condition, where $J_t=\bigcup_{k=1}^{t-1}I_k$.

Define a $v\times m$ matrix $B=[b_{i,t}]$ over $\mathbb{F}$ by

$$b_{i,t}=\begin{cases}\delta_{t,i} & \text{if } i\in J_t,\\ 1 & \text{if } i\in I_t,\\ 0 & \text{otherwise,}\end{cases}\qquad \text{for all } (i,t)\in\{1,\dots,v\}\times\{1,\dots,m\}.$$

Note that B satisfies the first entries condition: in each row, the first nonzero entry is 1 and occurs in a fixed column depending on the partition.

Moreover, we have $CB=\mathbf{0}$. Indeed, for each $j\in\{1,\dots,u\}$ and each $t\in\{1,\dots,m\}$:
- If $t=1$, then $\sum_{i\in I_1}c_{j,i}=\vec{0}$ by condition (a) of the columns condition.
- If $t\geq 2$, then by condition (b),

$$\sum_{i\in I_t}c_{j,i}=\sum_{i\in J_t}\delta_{t,i}c_{j,i}\quad\Rightarrow\quad\sum_{i\in I_t}c_{j,i}+\sum_{i\in J_t}(-\delta_{t,i})c_{j,i}=\vec{0}.$$

So, column-wise, $CB=\mathbf{0}\in\operatorname{Hom}_{\mathbb{F}}(V)^{u\times m}$.

Now, by Corollary 3.3 (presumably stating that first-entry matrices over $\mathbb{F}$ are kernel partition regular), there exists $i\in\{1,\dots,r\}$ and $\vec{y}\in(V\setminus\{0\})^m$ such that

$$\vec{x}:=B\vec{y}\in(E_i)^v\cap(V\setminus\{0\})^v.$$

Then:

$$C\vec{x}=C(B\vec{y})=(CB)\vec{y}=\mathbf{0},$$

as required. □

In the following example, we show that Theorem 3.5 does not hold for a matrix whose entries are affine maps.

Example 3.6. Let $\mathbb{F}=\mathbb{Q}$, $V=\mathbb{Q}$, and let $A=\big(f\big)$, where $f\colon V\to V$ is defined by $f(x)=x+1$. Let $C=2\mathbb{Z}$, the set of even integers.

- Since A has only one nonzero row, it trivially satisfies the first entries condition.
- The image $f[V]=\mathbb{Q}$, because f is surjective: for any $y\in\mathbb{Q}$, we have $f(y-1)=y$. Since $\mathbb{Q}\cap D=D\neq\emptyset$ for any C-set $D\subseteq\mathbb{Q}$, it follows that $\mathbb{Q}$ is a C^*-set.
- The set $C=2\mathbb{Z}$ contains the IP-set

$$\mathrm{FS}(\langle 2\rangle_{n=1}^{\infty})=\left\{\sum_{n\in G}2 : G\subset\mathbb{N}\text{ finite, } G\neq\emptyset\right\},$$

  and therefore C is a C-set.

Now suppose, toward contradiction, that there exists a sequence $\langle x_{1,n}\rangle_{n=1}^{\infty}\subset\mathbb{Q}$ such that for every finite nonempty $G\subset\mathbb{N}$,

$$\vec{x}_G:=\sum_{n\in G}x_{1,n}\neq 0,\quad\text{and}\quad A\vec{x}_G=f(\vec{x}_G)\in C.$$

Now examine small values of G:

- For $G=\{n\}$, we have $f(x_{1,n})=x_{1,n}+1\in 2\mathbb{Z}$, so $x_{1,n}\in 2\mathbb{Z}-1$, i.e., $x_{1,n}$ is odd.

- For $G = \{1, 2\}$, we get:

$$\begin{aligned} f(x_{1,1} + x_{1,2}) &= (x_{1,1} + x_{1,2}) + 1 \\ &= (\text{odd} + \text{odd}) + 1 \\ &= (\text{even}) + 1 = \text{odd}, \end{aligned}$$

  which is not in $C = 2\mathbb{Z}$.

Hence, such a sequence cannot exist, and this shows that even when $f[V]$ is a C^*-set and C is a C-set, the conclusion may fail unless stronger structural assumptions are made.

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