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Research Paper

FUZZY BCC-ALGEBRAS UNDER T-NORMS

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Abstract. In this paper, as using T-norms, we introduce and analyze the new classes of BCC-algebras as fuzzy subalgebras, fuzzy ideals, fuzzy left derivation ideals, fuzzy right derivation ideals and fuzzy derivation ideals and we obtain the relationships between them and classical concepts of BCC-algebras. Next we investigate the intersection and direct product of them and prove the algebraic structures of them. Finally, we investigate image and pre-image of them under homomorphisms of BCC-algebras.

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1. Introduction and Background

The concept of fuzzy sets was proposed by Zadeh [26]. Komori[9] introduced BCC-algebras such that are important class of logical algebras and many researchers [2, 3, 4, 6] investigated BCC-algebras. Dudek and Jun [5] introduced and investigated fuzzy BCC-ideals in BCC-algebras. Mostafa and Hassan [10] introduced fuzzy left (right) derivations BCC-ideals in BCC-algebras and investigated their basic properties. The triangular norm, T^n-norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T^n-norm. Later, Hohle [8], Alsina et al. [1] introduced the T^n-norm into fuzzy set theory and suggested that the T^n-norms be used for the intersection of fuzzy sets. Since then, many other researchers have presented various types of T^n-norms for particular purposes [7, 25]. In practice, Zadeh's conventional T^n-norm,$\bigwedge$, has been used in almost every design for fuzzy logic controllers and even in the modelling of other decision-making processes. However, some theoretical and experimental studies seem to indicate that other types of T^n-norms may work better in some situations, especially in the context of decisionmaking processes.The author by using norms, investigated some properties of fuzzy algebraic structures [11] - [24]. In this paper, using the triangular norm, we introduce some concepts such as fuzzy subalgebras, fuzzy ideals, fuzzy left derivations, fuzzy right derivations and fuzzy derivations of BCC-algebras, and discuss the relations among them, obtaining a series of conclusions of them. Also we describe them in the intersection and direct product and we prove some other properties of them. Finally, we investigate image and pre-image of them under homomorphisms of BCC-algebras.

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2. **Preliminaries**

In this section we cite the fundamental definitions and results that will be used in the sequel.

Definition 2.1. [5] A BCC-algebra $(\Psi, \diamond, 0)$ is a nonempty set Ψ with a constant 0 and a binary operation $\diamond$ such that:
(1) $((x \diamond y) \diamond (z \diamond y)) \diamond (x \diamond z) = 0$.
(2) $x \diamond 0 = x$.
(3) $x \diamond x = 0$.
(4) $0 \diamond x = 0$.
(5) $x \diamond y = y \diamond x = 0$ implies that $x = y$,
for all $x, y, z \in \Psi$.

Definition 2.2. [5] Assume $(\Upsilon, \diamond, 0)$ be a BCC-algebra. Define a binary relation $\leq$ on Υ as, $m \leq n$ if and only if $m \diamond n = 0$, this makes $(\Upsilon, \leq)$ as a partially ordered set.

Proposition 2.3. [5] *Let* $(\Phi, \diamond, 0)$ *be a* BCC*-algebra. Then*
(1) $(q \diamond w) \diamond q = 0$.
(2) $q \leq w$ *implies that* $q \diamond z \leq w \diamond z$.
(3) $q \leq w$ *implies that* $z \diamond w \leq z \diamond q$.
(4) $(q \diamond w) \diamond (z \diamond w) \leq q \diamond z$.
for all $q, w, z \in \Phi$. *For elements* q *and* w *of a* BCC*-algebra* $(\Phi, \diamond, 0)$ *denote* $q \wedge w = w \diamond (w \diamond q)$.

Lemma 2.4. [5] *As* $(\Omega, \diamond, 0)$ *be a* BCC*-algebra so*
(1) $0 \wedge e = 0$.
(2) $e \wedge y \leq y$.
for all $e, y \in \Omega$.

Definition 2.5. [5] Suppose $\Psi \neq \emptyset$ be a subset of a BCC-algebra Σ. We say Ψ is a subalgebra of Σ if $r \diamond y \in \Psi$ for all $r, y \in \Psi$.

Definition 2.6. [5] Let $\emptyset \neq \Delta \subseteq \Pi$ of a BCC-algebra Π is called an ideal of Π if
(1) $0 \in \Delta$,
(2) $t \diamond y \in \Delta$ and $y \in \Delta$ imply that $t \in \Delta$ for all $t, y \in \Pi$.

Definition 2.7. [5] A non-empty subset Φ of a BCC-algebra Π is called a BCC-ideal of Π if
(1) $0 \in \Phi$,
(2) $(u \diamond p) \diamond z \in \Phi$ and $p \in \Phi$ imply that $u \diamond z \in \Phi$ for all $u, p, z \in \Pi$.

Definition 2.8. [10] Let $(\Xi, \diamond, 0)$ be a BCC-algebra. A map $d : \Xi \to \Xi$ is called a left- right derivation (briefly (l, r)- derivation) of Ξ if $d(g \diamond h) = (d(g) \diamond h) \wedge (g \diamond d(h))$ for all $g, h \in \Xi$. Similarly, a map $d : \Xi \to \Xi$ is called a right- left derivation (briefly (r, l)-derivation) of Ξ if $d(g \diamond h) = (g \diamond d(h)) \wedge (d(g) \diamond h)$ for all $g, h \in \Xi$.
A map $d : \Xi \to \Xi$ is called a derivation of Ξ if d is both a (l, r)-derivation and a (r, l) -derivation of Ξ. Also we say that $d : \Xi \to \Xi$ is zero if $d(0) = 0$.

Lemma 2.9. [10] *A derivation* d *of* BCC*- algebra* Υ *is regular.*

Proposition 2.10. [10] *Let* $(\Upsilon, \diamond, 0)$ *be a* BCC*-algebra with partial order* $(\Upsilon, \diamond, 0)$, *and let* d *be a derivation of* Υ. *Hence*
(1) $d(k) \leq k$.
(2) $d(k \diamond y) \leq d(k) \diamond y$.

(3) $d(k \diamond y) \leq k \diamond d(y)$.
(4) $dd(k) \leq k$.
(5) $d(k \diamond d(k)) = 0$.
(6) $d^{-1}(0) = \{k \in \Upsilon; d(k) = 0\}$ *is a sub-algebra of* X,
for all $k, y \in \Upsilon$.

Proposition 2.11. [10] *Let* Φ *be a* BCC*-algebra and* d *be a derivation of* Φ*. Then* $Fix_d = \{b \in \Phi : d(b) = b\}$ *is a subalgebra of* Φ*.*

Definition 2.12. [10] Let $(\Psi, \diamond, 0)$ be a BCC-algebra and $d : \Psi \to \Psi$ be a self map. A non-empty subset A of a BCC-algebra Ψ is called left derivation BCC-ideal of Ψ if:
(1) $0 \in A$,
(2) $d(q \diamond y) \diamond u \in A$ and $d(y) \in A$ imply that $d(q \diamond u) \in A$ for all $q, y, u \in \Psi$.

Definition 2.13. [10] Let $(\Omega, \diamond, 0)$ be a BCC-algebra and $d : \Omega \to \Omega$ be a self map. A non-empty subset A of a BCC-algebra Ω is called right derivation BCC-ideal of Ω if:
(1) $0 \in A$,
(2) as $(a \diamond b) \diamond d(z) \in A$ and $d(b) \in A$ so $d(a \diamond z) \in A$ for all $a, b, z \in \Omega$.

Definition 2.14. [10] Let $(\Sigma, \diamond, 0)$ be a BCC-algebra and $d : \Sigma \to \Sigma$ be a self map. A non-empty subset A of a BCC-algebra Σ is called derivation BCC-ideal of Σ as:
(1) $0 \in A$,
(2)if $d((l \diamond y) \diamond z) \in A$ and $d(y) \in A$, then $d(l \diamond z) \in A$ for all $l, y, z \in \Sigma$.

Remark 2.15. [10] If $d : X \to X$ is fixed, the Definitions 2.12, 2.13 and 2.14 give the definition BCC-ideal.

Definition 2.16. [10] A mapping $\phi : \Phi \to \Psi$ of BCC-algebras is called a homomorphism if $\phi(x \diamond y) = \phi(x) \diamond \phi(y)$, for all $x, y \in \Phi$.

Definition 2.17. [11] Let Ω be an arbitrary set. A fuzzy subset of Ω, we mean a function from Ω into $[0,1]$. The set of all fuzzy subsets of Ω is called the $[0,1]$-power set of X and is denoted $[0,1]^{\Omega}$. For a fixed $p \in [0,1]$, the set $\pi_p = \{x \in \Omega : \pi(x) \geq p\}$ is called an upper level of π.

Definition 2.18. [11] Let ϱ be a function from set Ψ into set Ω such that $\pi \in [0,1]^X$ and $\varpi \in [0,1]^Y$. For all $x \in \Psi, y \in \Omega$, we define

$$\varrho(\pi)(y) = \sup\{\pi(x) \mid x \in \Psi, \varrho(x) = y\}$$

and

$$\varrho^{-1}(\varpi)(x) = \varpi(\varrho(x)).$$

Definition 2.19. [11] A t-norm T^n is a function $T^n : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(1) $T^n(t,1) = t$ (neutral element),
(2) $T^n(t,y) \leq T^n(t,z)$ if $y \leq z$ (monotonicity),
(3) $T^n(t,y) = T^n(y,t)$ (commutativity),
(4) $T^n(t, T^n(y,z)) = T^n(T^n(t,y), z)$ (associativity),
for all $t, y, z \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T^n(x_1, y_1) \geq T^n(x_2, y_2)$.

Example 2.20. [11] (1) $T_m^n(a,y) = \min\{a,y\}$.
(2) $T_b^n(a,y) = \max\{0, a+y-1\}$.
(3) $T_p^n(a,y) = ay$.
(4)

$$T_D^n(a,y) = \begin{cases} y & \text{if } a = 1 \\ a & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5)

$$T_{nM}^n(a,y) = \begin{cases} \min\{a,y\} & \text{if } a+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6)

$$T_{H_0}^n(a,y) = \begin{cases} 0 & \text{if } a = y = 0 \\ \frac{xy}{a+y-ay} & \text{otherwise.} \end{cases}$$

Note that we have $T_D(a,y) \leq T^n(a,y) \leq T_{\min}^n(a,y)$ for all $a, y \in [0,1]$.

We say that T^n be idempotent if for all $a \in [0,1]$ we have $T^n(a,a) = a$.

Definition 2.21. [11] Define function $T_m^n : \prod_{i=1}^m [0,1] \to [0,1]$ as

$$T_m^n(x_1, x_2, ..., x_m) = T^n(x_i, T_{m-1}^n(x_1, ..., x_{i-1}, x_{i+1}, x_{i+1}, ..., x_m))$$

for all $1 \leq i \leq m$, where $m \geq 2, T_2^n = T^n$ and $T_1^n = id$ (identity).

Lemma 2.22. [11] *For a t-norm T^n and every $x_i, y_i \in [0,1]$, where $1 \leq i \leq m$ and $m \geq 2$, we have*
$T_m^n(T^n(x_1, y_1,), T^n(x_2, y_2), ..., T^n(x_m, y_m)) = T^n(T_m^n(x_1, x_2, ..., x_m), T_m^n(y_1, y_2, ..., y_m))$.

Definition 2.23. [11] Assume $\gamma \in [0,1]^\Psi$ and $\delta \in [0,1]^\Omega$. Define the direct product of γ and δ is denoted by $\gamma \times \delta \in [0,1]^{\Psi \times \Omega}$ as

$$(\gamma \times \delta)(x,y) = T^n(\gamma(x), \delta(y))$$

for all $(x,y) \in \Psi \times \Omega$.

Lemma 2.24. [11] *Let T^n be a t-norm and let $\Upsilon = \prod_{i=1}^m \Upsilon_i$ be the direct product of sets $\{\Upsilon_i\}_{i=1}^n$ .If $\lambda_i \in [0,1]^{\Upsilon_i}$ where $1 \leq i \leq m$, then $\lambda = \prod_{i=1}^m \lambda_i$ defined by*

$$\lambda(x) = (\prod_{i=1}^m \lambda_i)(x_1, x_2, ..., x_m) = T_m^n(\lambda_1(x_1), \lambda_2(x_2), ..., \lambda_m(x_m))$$

for all $x = (x_1, x_2, ..., x_m) \in \Upsilon$.

Definition 2.25. [11] Let $\theta, \vartheta \in [0,1]^\Phi$ and define the intersection of θ and ϑ is denoted by $\theta \cap \vartheta \in [0,1]^\Phi$ as

$$(\theta \cap \vartheta)(x) = T^n(\theta(x), \vartheta(x))$$

for all $x \in \Phi$.

Lemma 2.26. [11] *Let T^n be a t-norm. Then*

$$T^n(T^n(x,y), T^n(w,z)) = T^n(T^n(x,w), T^n(y,z))$$

for all $x, y, w, z \in [0,1]$.

3. Fuzzy BCC-algebras under t-norms

Throughout this paper, M, N always mean two BCC-algebras unless otherwise specified.

Definition 3.1. $\kappa \in [0,1]^M$ is called a fuzzy subalgebra of M under t-norm T^n if

$$\kappa(m \diamond y) \geq T^n(\kappa(m), \kappa(y))$$

for all $m, y \in M$.
Denote by $FST^n(M)$, the set of all fuzzy subalgebras of M under t-norm T^n.

Example 3.2. Let $M = \{0,1,2,3\}$ be a set with:

$\diamond$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Then $(M, \diamond, 0)$ is a BCC-algebra.
Define fuzzy subset $\kappa : (M, \diamond, 0) \to [0,1]$ as

$$\kappa(m) = \begin{cases} 0.55 & \text{if } m = 0,1,3 \\ 0.15 & \text{if } m = 2 \end{cases}$$

let $T^n(a,b) = T^n_p(a,b) = ab$ for all $a, b \in [0,1]$ then $\kappa \in FST^n(M)$.

Proposition 3.3. *Suppose $\tau, \upsilon \in FST^n(M)$. Then $\tau \cap \upsilon \in FST^n(M)$.*

Proof. Let $f, y \in M$. Then

$$\begin{aligned}(\tau \cap \upsilon)(f \diamond y) &= T^n(\tau(f \diamond y), \upsilon(f \diamond y)) \\ &\geq T^n(T^n(\tau(f), \tau(y)), T^n(\upsilon(f), \upsilon(y))) \\ &= T^n(T^n(\tau(f), \upsilon(f)), T^n(\tau(y), \upsilon(y))) \\ &= T^n((\tau \cap \upsilon)(f), (\tau \cap \upsilon)(y))\end{aligned}$$

thus

$$(\tau \cap \upsilon)(f \diamond y) \geq T^n((\tau \cap \upsilon)(f), (\mu \cap \upsilon)(y)).$$

Then $\tau \cap \upsilon \in FST^n(M)$. □

Proposition 3.4. *Let $\sigma \in [0,1]^M$ such that T^n be idempotent t-norm. Then $\sigma \in FST^n(M)$ if and only if the set $\sigma_p = \{x \in M : \sigma(x) \geq p\} \neq \emptyset$ will be a subalgebra of M for all $p \in [0,1]$.*

Proof. Let $\sigma \in FST^n(M)$ and $a, b \in \sigma_p$. Then $\sigma(a \diamond by) \geq T^n(\sigma(a), \sigma(b)) \geq T^n(p,p) = p$ thus $a \diamond b \in \sigma_p$ and so σ_p will be a subalgebra of M for every $p \in [0,1]$.
Conversely, let $\sigma_p \neq \emptyset$ be a subalgebra of M for every $p \in [0,1]$. Let $p = T^n(\sigma(a), \sigma(b))$ and $a, b \in \sigma_p$. As σ_p is a subalgebra of M so $a \diamond b \in \sigma_p$ and thus

$$\sigma(a \diamond b) \geq p = T^n(\sigma(a), \sigma(b))$$

so $\sigma \in FST^n(M)$. □

Proposition 3.5. *Let $\varsigma \in FST^n(M)$ and T^n be idempotent t-norm. Then $\varsigma(0) \geq \varsigma(q)$ for all $q \in M$.*

Proof. Let $q \in M$. Then

$$\varsigma(0) = \varsigma(q \diamond q) \geq T^n(\varsigma(q), \mu(q)) = \varsigma(q)$$

thus $\varsigma(0) \geq \mu(q)$. □

Proposition 3.6. *If $\alpha \in FST^n(M)$ and $\beta \in FST^n(N)$, then $\alpha \times \beta \in FST^n(M \times N)$.*

Proof. Assume $(i_1, j_1), (i_2, j_2) \in M \times N$. Hence

$$\begin{aligned}(\alpha \times \beta)((i_1, j_1) \diamond (i_2, j_2)) &= (\alpha \times \beta)(i_1 \diamond i_2, j_1 \diamond j_2)\\ &= T^n(\alpha(i_1 \diamond i_2), \beta(j_1 \diamond j_2))\\ &\geq T^n(T^n(\alpha(i_1), \alpha(i_2)), T^n(\beta(j_1), \beta(j_2)))\\ &= T^n(T^n(\alpha(i_1), \beta(j_1)), T^n(\alpha(i_2), \beta(j_2)))\\ &= T^n((\alpha \times \beta)(i_1, j_1), (\alpha \times \beta)(i_2, j_2))\end{aligned}$$

so

$$(\alpha \times \beta)((i_1, j_1) \diamond (i_2, j_2)) \geq T^n((\alpha \times \beta)(i_1, j_1), (\alpha \times \beta)(i_2, j_2)).$$

Therefore $\alpha \times \beta \in FST^n(M \times N)$. □

Proposition 3.7. *Let $\eta_i \in FST^n(M_i)$ where $1 \leq i \leq m$, then $\eta = \prod_{i=1}^m \eta_i \in FST^n(\prod_{i=1}^m M_i)$.*

Proposition 3.8. *If $\rho \in FST^n(M)$ and $\omega : M \to N$ be a homomorphism of BCC-algebras, then $\omega(\rho) \in FST^n(N)$.*

Proof. Let $c_1, c_2 \in N$ and $b_1, b_2 \in M$ such that $\omega(b_1) = c_1$ and $\omega(b_2) = c_2$. Then

$$\begin{aligned}\omega(\rho)(c_1 \diamond c_2) &= \sup\{\rho(b_1 \diamond b_2) \mid b_1, b_2 \in M, \omega(b_1) = c_1, \omega(b_2) = c_2\}\\ &\geq \sup\{T^n(\rho(b_1), \rho(b_2) \mid b_1, b_2 \in M, \omega(b_1) = c_1, \omega(b_2) = c_2\}\\ &= T^n(\sup\{\rho(b_1) \mid b_1 \in M, \omega(b_1) = c_1\}, \sup\{\rho(b_2) \mid b_2 \in M, \omega(b_2) = c_2\})\\ &= T^n(\omega(\rho)(c_1), \omega(\rho)(c_2))\end{aligned}$$

thus

$$\omega(\rho)(c_1 \diamond c_2) \geq T^n(\omega(\rho)(c_1), \omega(\rho)(c_2)).$$

Thus $\omega(\rho) \in FST^n(N)$. □

Proposition 3.9. *If $\theta \in FST^n(N)$ and $\xi : M \to N$ be a homomorphism of BCC-algebras, then $\xi^{-1}(\theta) \in FST^n(M)$.*

Proof. Let $d_1, d_2 \in M$. Then

$$\begin{aligned}\xi^{-1}(\theta)(d_1 \diamond d_2) &= \theta(\xi(d_1 \diamond d_2))\\ &= \theta(\xi(d_1) \diamond \xi(d_2))\\ &\geq T^n(\theta(\xi(d_1)), \theta(\xi(d_2)))\\ &= T^n(\xi^{-1}(\theta)(d_1), \xi^{-1}(\theta)(d_2))\end{aligned}$$

then $\xi^{-1}(\theta) \in FST^n(M)$. □

Definition 3.10. Define $\chi \in [0,1]^M$ is a fuzzy BCC-ideal of M under t-norm T^n such that:
(1) $\chi(0) \geq \chi(m)$,
(2) $\chi(m \diamond z) \geq T^n(\chi((m \diamond y) \diamond z), \chi(y))$,
for all $m, y, z \in M$.
Denote by $FIT^n(M)$, the set of all fuzzy BCC-ideals of M under t-norm T^n.

Example 3.11. Let $M = \{0, 1, 2, 3, 4\}$ be a set given by :

$\diamond$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Then $(M, \diamond, 0)$ is a BCC-algebra. Define $\chi \in [0,1]^M$ as

$$\chi(m) = \begin{cases} 1 & \text{with } m = 0, 2 \\ t & \text{with } m = 1, 3, 4 \end{cases}$$

such that $t \in [0,1]$.
Let $T^n(a,b) = T^n_p(a,b) = ab$ for all $a, b \in [0,1]$ then $\chi \in FIT^n(M)$.

Proposition 3.12. *Let $\kappa \in [0,1]^M$ and T^n be idempotent t-norm. Then $\kappa \in FIT^n(M)$ if and only if the set $\kappa_q = \{x \in M : \kappa(x) \geq q\}$ is either empty or a BCC-ideal of M for every $q \in [0,1]$.*

Proof. Let $e, y, f \in M$. As $\kappa \in FIT^n(M)$ so $\kappa(0) \geq \kappa(e) \geq q$ so $0 \in \kappa_q$. If $(e \diamond y) \diamond f \in \kappa_q$ and $y \in \kappa_q$, then $\kappa(e \diamond f) \geq T^n(\kappa((e \diamond y) \diamond f), \kappa(y)) \geq T^n(q, q) = q$ then $e \diamond f \in \kappa_q$. So κ_q will be a BCC-ideal of M for every $q \in [0,1]$.
Conversely, let κ_q is either empty or a BCC-ideal of M for every $q \in [0,1]$. Let $q = T^n(\kappa((e \diamond y) \diamond f), \kappa(y))$ such that $(e \diamond y) \diamond f \in \kappa_q$ and $y \in \kappa_q$ then $e \diamond f \in \kappa_q$ so $\kappa(e \diamond f) \geq q = T^n(\kappa(e \diamond y), \kappa(y))$ thus $\kappa \in FIT^n(M)$. □

Proposition 3.13. *Let $\tau, \delta \in FIT^n(M)$. Then $\tau \cap \delta \in FIT^n(M)$.*

Proof. Let $g, y \in M$. Then
(1)

$$(\tau \cap \delta)(0) = T^n(\tau(0), \delta(0)) \geq T^n(\tau(g), \delta(g)) = (\tau \cap \delta)(g)$$

thus

$$(\tau \cap \delta)(0) \geq (\tau \cap \delta)(g).$$

(2)

$$\begin{aligned} (\tau \cap \delta)(g \diamond z) &= T^n(\tau(g \diamond z), \delta(g \diamond z)) \\ &\geq T^n(T^n(\tau((g \diamond y) \diamond z), \tau(y)), T^n(\delta((g \diamond y) \diamond z), \delta(y))) \\ &= T^n(T^n(\tau((g \diamond y) \diamond z), \delta((g \diamond y) \diamond z)), T^n(\tau(y), \delta(y))) \text{ (Lemma 2.26)} \\ &= T^n((\tau \cap \delta)((g \diamond y) \diamond z), (\tau \cap \delta)(y)) \end{aligned}$$

so

$$(\tau \cap \delta)(g \diamond z) \geq T^n((\tau \cap \delta)((x \diamond y) \diamond z), (\tau \cap \delta)(y)).$$

Then $\tau \cap \delta \in FIT^n(M)$. □

Proposition 3.14. *Let* $\varsigma \in FIT^n(M)$ *and* $\iota \in FIT^n(N)$. *Then* $\varsigma \times \iota \in FIT^n(M \times N)$.

Proof. Let $(w, y) \in M \times N$. Then

$$(\varsigma \times \iota)(0,0) = T^n(\varsigma(0), \iota(0)) \geq T^n(\varsigma(w), \iota(y)) = (\varsigma \times \nu)(w, y)$$

thus $(\varsigma \times \iota)(0,0) \geq (\varsigma \times \iota)(w, y)$.
Also let $w_i \in M$ and $y_i \in N$ for $i = 1, 2, 3$. Now

$$\begin{aligned}(\varsigma \times \iota)((w_1, y_1) \diamond (w_2, y_2)) &= (\varsigma \times \iota)(w_1 \diamond w_2, y_1 \diamond y_2)\\ &= T^n(\varsigma(x_1 \diamond w_2), \nu(y_1 \diamond y_2))\\ &\geq T^n(T^n(\varsigma((w_1 \diamond w_3) \diamond x_2), \varsigma(w_3)), T^n(\iota((y_1 \diamond y_3) \diamond y_2), \iota(y_3)))\\ &= T^n(T^n(\varsigma((w_1 \diamond w_3) \diamond w_2), \iota((y_1 \diamond y_3) \diamond y_2)), T^n(\varsigma(x_3), \iota(y_3)))\\ &= T^n((\varsigma \times \iota)((w_1 \diamond w_3) \diamond w_2, (y_1 \diamond y_3) \diamond y_2), (\varsigma \times \iota)(w_3, y_3))\\ &= T^n(((\varsigma \times \iota)((w_1, y_1) \diamond (w_3, y_3)) \diamond (w_2, y_2)), (\varsigma \times \iota)(w_3, y_3)).\end{aligned}$$

Therefore $\varsigma \times \iota \in FIT^n(M \times N)$. □

Proposition 3.15. *Let* $\kappa_i \in FIT^n(M_i)$ *where* $1 \leq i \leq m$, *then* $\kappa = \prod_{i=1}^m \kappa_i \in FIT^n(\prod_{i=1}^m M_i)$.

Proposition 3.16. *If* $\gamma \in FIT^n(M)$ *and* $\Phi : M \to N$ *be a homomorphism of* BCC*-algebras, then* $\Phi(\gamma) \in FIT^n(N)$.

Proof. As

$$\Phi(\gamma)(0) = \sup\{\gamma(0) \mid 0 \in M, \Phi(0) = 0\} \geq \sup\{\gamma(x) \mid m \in M, \Phi(m) = y\} = \Phi(\gamma)(y)$$

thus

$$\Phi(\gamma)(0) \geq \Phi(\gamma)(y).$$

Let $m_i \in M$ and $y_i \in N$ with $\Phi(m_i) = y_i$ for $i = 1, 2, 3$. Then

$$\Phi(\gamma)(y_1 \diamond y_2) = \sup\{\gamma(m_1 \diamond m_2) \mid \Phi(m_i) = y_i\}$$

$$\geq \sup\{T^n(\gamma((m_1 \diamond m_3) \diamond m_2), \gamma(m_3)) \mid \Phi(m_i) = y_i\}$$

$$= T^n(\sup\{\gamma((m_1 \diamond m_3) \diamond m_2) \mid \Phi((m_1 \diamond m_3) \diamond m_2) = (y_1 \diamond y_3) \diamond y_2\}, \sup\{\gamma(m_3) \mid \Phi(m_3) = y_3\})$$

$$= T^n(\Phi(\gamma)((y_1 \diamond y_3) \diamond y_2), \Phi(\gamma)(y_3))$$

therefore

$$\Phi(\gamma)(y_1 \diamond y_2) \geq T^n(\Phi(\gamma)((y_1 \diamond y_3) \diamond y_2), \Phi(\gamma)(y_3)).$$

Thus $\Phi(\gamma) \in FIT^n(N)$. □

Proposition 3.17. *If* $\sigma \in FIT^n(N)$ *and* $\Gamma : M \to N$ *be a homomorphism of* BCC*-algebras, then* $\Gamma^{-1}(\sigma) \in FIT^n(M)$.

Proof. Let $b \in M$. Then

$$\Gamma^{-1}(\sigma)(0) = \sigma(\Gamma(0)) \geq \sigma(\Gamma(b)) = \Gamma^{-1}(\sigma)(b).$$

Let $b_1, b_2, b_3 \in M$. As

$$\begin{aligned}\Gamma^{-1}(\sigma)(b_1\diamond b_2) &= \sigma(\Gamma(b_1\diamond b_2))\\ &= \sigma(\Gamma(b_1)\diamond\sigma(\Gamma(b_2))\\ &\geq T^n(\sigma(\Gamma(b_1)\diamond\Gamma(b_3))\diamond\Gamma(b_2)),\sigma(\Gamma(b_3)))\\ &= T^n(\sigma(\Gamma((b_1\diamond b_3))\diamond b_2),\sigma(\Gamma(b_3)))\\ &= T^n(\Gamma^{-1}(\sigma)((b_1\diamond b_3)\diamond b_2),\Gamma^{-1}(\sigma)(b_3))\end{aligned}$$

so

$$\Gamma^{-1}(\sigma)(b_1\diamond b_2)\geq T^n(\Gamma^{-1}(\sigma)((b_1\diamond b_3)\diamond b_2),\Gamma^{-1}(\sigma)(b_3)).$$

Then $\Gamma^{-1}(\sigma)\in FIT^n(M)$. □

Definition 3.18. Let $(M,\diamond,0)$ be a BCC-algebra and $d:M\to M$ be a self map. A fuzzy set $\lambda:M\to[0,1]$ in M is called a fuzzy left derivation BCC-ideal of M under t-norm T^n if it satisfies the following conditions:
(1) $\lambda(0)\geq\lambda(l)$,
(2) $\lambda(d(l\diamond u))\geq T^n(\lambda(d(l\diamond y)\diamond u),\lambda(d(y)))$
for all $l,y,u\in M$.
Denote by $FLDT^n(M)$, the set of all fuzzy left derivation BCC-ideals of M under t-norm T^n.

Proposition 3.19. *Let $\pi\in[0,1]^M$ and T^n be idempotent. Then $\pi\in FLDT^n(M)$ if and only if the set $\pi_k=\{a\in M:\pi(x)\geq k\}$ is either empty or a left derivation BCC-ideal of M for every $k\in[0,1]$.*

Proof. Let $\pi\in FLDT^n(M)$ and $a,y\in M$. Then $\pi(0)\geq\pi(a)\geq k$ and then $0\in\pi_k$. Also let $d(a\diamond y)\diamond z\in\pi_k$ and $d(y)\in\pi_k$. Then $\pi(d(a\diamond z))\geq T^n(\pi(d(a\diamond y)\diamond z),\pi(d(y)))\geq T^n(k,k)=k$ thus $d(a\diamond z)\in\pi_k$. Then π_k will be a left derivation BCC-ideal of M for every $k\in[0,1]$.
Conversely, let π_k is either empty or a left derivation BCC-ideal of M for every $k\in[0,1]$. Let $k=T^n(\pi(d(a\diamond y)\diamond z),\pi(d(y)))$ with $d(a\diamond y)\diamond z\in\pi_k$ and $d(y)\in\pi_k$. Then $d(a\diamond z)\in\pi_k$ thus

$$\pi(d(a\diamond z))\geq k=T^n(\pi(d(a\diamond y)\diamond z),\pi(d(y)))$$

so $\pi\in FLDT^n(M)$. □

Proposition 3.20. *Let $\varrho\in FLDT^n(M)$ and T^n be an idempotent t-norm. Then for all $h,y,z\in M$:*
(1) If $h\leq d(y)$, then $\varrho(d(h))\geq\varrho(d(y))$.
(2) If $h\diamond y\leq d(h)$, then $\varrho(d(h\diamond y))\geq\varrho(d(h))$.
(3) If $(h\diamond y)\diamond(z\diamond y)\leq d(h\diamond z)$, then $\varrho(d((h\diamond y)\diamond(z\diamond y)))\geq\varrho(d(h\diamond z))$.
(4) If $\varrho(d(h\diamond y))=\varrho(d(0))$, then $\varrho(d(h))\geq\varrho(d(y))$.

Proof. Let $h,y,z\in M$. Then
(1) Let $h\leq d(y)$ and since $d(y)\leq y$ (Proposition 2.10 part (1)) hence $h\leq y$ i.e. $h\diamond y=0$.

Thus

$$\begin{aligned}\varrho(d(h)) &= \varrho(d(h\diamond 0))\\ &\geq T^n(\varrho(d(h\diamond y)\diamond 0),\varrho(d(y)))\ as\ \ \varrho\in FLDT^n(M)\\ &= T^n(\varrho(d(h\diamond y),\varrho(d(y)))\\ &= T^n(\varrho(d(0),\varrho(d(y)))\\ &= \varrho(d(y))\end{aligned}$$

then $\varrho(d(h))\geq\varrho(d(y))$.
(2) As $h\diamond y\leq d(h)$ so from part(1) we get that $\varrho(d(h\diamond y))\geq\varrho(d(h))$.
(3) Let $(h\diamond y)\diamond(z\diamond y)\leq d(h\diamond z)$ then as part(1) we will have $\varrho(d((h\diamond y)\diamond(z\diamond y)))\geq\varrho(d(h\diamond z))$.
(4) Let $\varrho(d(h\diamond y))=\varrho(d(0))$. Then

$$\begin{aligned}\varrho(d(h)) &= \varrho(d(h\diamond 0))\\ &\geq T^n(\varrho(d(h\diamond y)\diamond 0),\varrho(d(y)))\ as\ \ \varrho\in FLDT^n(M)\\ &= T^n(\varrho(d(h\diamond y),\varrho(d(y)))\\ &= T^n(\varrho(d(0),\varrho(d(y)))\\ &= \varrho(d(y))\end{aligned}$$

then $\varrho(d(h))\geq\varrho(d(y))$. □

Proposition 3.21. *Let $\gamma,\delta\in FLDT^n(M)$. Then $\gamma\cap\delta\in FLDT^n(M)$.*

Proof. Let $j,y,z\in M$. Then
(1)

$$(\gamma\cap\delta)(0)=T^n(\gamma(0),\delta(0))\geq T^n(\gamma(j),\delta(j))=(\gamma\cap\delta)(j)$$

thus

$$(\gamma\cap\delta)(0)\geq(\gamma\cap\delta)(j).$$

(2)

$$\begin{aligned}(\gamma\cap\delta)(d(j\diamond z)) &= T^n(\gamma(d(j\diamond z)),\delta(d(j\diamond z)))\\ &\geq T^n(T^n(\gamma(d(j\diamond y)\diamond z),\gamma(d(y))),T^n(\delta(d(j\diamond y)\diamond z),\delta(d(y))))\\ &= T^n(T^n(\gamma(d(j\diamond y)\diamond z),\delta(d(j\diamond y)\diamond z)),T^n(\gamma(y),\delta(y)))\ (Lemma\ 2.26)\\ &= T^n((\gamma\cap\delta)(d(j\diamond y)\diamond z),(\gamma\cap\delta)(y))\end{aligned}$$

so

$$(\gamma\cap\delta)(j\diamond z)\geq T^n((\gamma\cap\delta)(d(j\diamond y)\diamond z),(\gamma\cap\delta)(y)).$$

Thus $\gamma\cap\delta\in FLDT^n(M)$. □

Proposition 3.22. *Let $\iota\in FLDT^n(M)$ and $\xi\in FLDT^n(N)$. Then $\iota\times\xi\in FLDT^n(M\times N)$.*

Proof. Let $(t,v)\in M\times N$. Then

$$(\iota\times\xi)(0,0)=T^n(\iota(0),\xi(0))\geq T^n(\iota(t),\xi(v))=(\iota\times\xi)(t,v)$$

thus $(\iota\times\xi)(0,0)\geq(\iota\times\nu)(t,v)$.
Also let $t_i\in M$ and $v_i\in N$ for $i=1,2,3$. Now

$$\begin{aligned}(\iota \times \xi)(d((t_1, v_1) \diamond (t_2, v_2))) &= (\iota \times \xi)(d(t_1 \diamond t_2, v_1 \diamond v_2)) \\ &= (\iota \times \xi)(d(t_1 \diamond t_2), d(v_1 \diamond v_2)) \\ &= T^n(\iota(d(t_1 \diamond t_2)), \xi(d(v_1 \diamond v_2))) \\ &\geq T^n(T^n(\iota(d(t_1 \diamond t_3) \diamond t_2), \iota(d(t_3))), T^n(\xi(d(v_1 \diamond v_3) \diamond v_2), \xi(d(v_3)))) \\ &= T^n(T^n(\iota(d(t_1 \diamond t_3) \diamond t_2), \xi(d(v_1 \diamond v_3) \diamond v_2)), T^n(\iota(d(t_3)), \xi(d(v_3)))) \\ &= T^n((\iota \times \xi)(d(t_1 \diamond t_3) \diamond t_2, d(v_1 \diamond v_3) \diamond v_2), (\iota \times \xi)(d(t_3), d(v_3))) \\ &= T^n((\iota \times \xi)(d((t_1, v_1) \diamond (t_3, v_3))) \diamond (t_2, v_2)), (\iota \times \xi)d(t_3, v_3)).\end{aligned}$$

Therefore $\iota \times \xi \in FLDT^n(M \times N)$. □

Proposition 3.23. *Let* $\Gamma_i \in FLDT^n(M_i)$ *where* $1 \leq i \leq m$, *then* $\Gamma = \prod_{i=1}^{m} \Gamma_i \in FLDT^n(\prod_{i=1}^{n} M_i)$.

Proposition 3.24. *If* $\phi \in FLDT^n(M)$ *and* $\Theta : M \to N$ *be a homomorphism of BCC-algebras, then* $\Theta(\phi) \in FLDT^n(N)$.

Proof. As

$$\Theta(\phi)(0) = \sup\{\phi(0) \mid 0 \in M, \Theta(0) = 0\} \geq \sup\{\phi(s) \mid s \in M, \Theta(s) = t\} = \Theta(\phi)(t)$$

thus

$$\Theta(\phi)(0) \geq \Theta(\phi)(t).$$

Let $s_i \in M$ and $t_i \in N$ with $\Theta(s_i) = t_i$ for $i = 1, 2, 3$. Now since

$$\begin{aligned}\Theta(\phi)(d(t_1 \diamond t_2)) &= \sup\{\phi(d(s_1 \diamond s_2)) \mid \Theta(s_i) = t_i\} \\ &\geq \sup\{T^n(\phi(d(s_1 \diamond s_3) \diamond s_2), \phi(d(s_3))) \mid \Theta(s_i) = t_i\} \\ &= T^n(\sup\{\phi(d(s_1 \diamond s_3) \diamond s_2) \mid \Theta(d(s_1 \diamond s_3) \diamond s_2) = d(t_1 \diamond t_3) \diamond t_2\}, \sup\{\phi(d(s_3)) \mid \Theta(d(s_3)) = d(t_3)\}) \\ &= T^n(\Theta(\phi)(d(t_1 \diamond t_3) \diamond t_2), \Theta(\phi)(d(t_3)))\end{aligned}$$

therefore

$$\Theta(\phi)(t_1 \diamond t_2) \geq T^n(\Theta(\phi)(d(t_1 \diamond t_3) \diamond t_2), \Theta(\phi)(d(t_3))).$$

Thus $\Theta(\phi) \in FLDT^n(N)$. □

Proposition 3.25. *If* $\sigma \in FLDT^n(N)$ *and* $\Upsilon : M \to N$ *be a homomorphism of BCC-algebras, then* $\Upsilon^{-1}(\sigma) \in FLDT^n(M)$.

Proof. Let $f \in M$. Then

$$\Upsilon^{-1}(\sigma)(0) = \sigma(\Upsilon(0)) \geq \sigma(\Upsilon(f)) = \Upsilon^{-1}(\sigma)(f).$$

Let $f_1, f_2, f_3 \in M$. Since

$$\begin{aligned}\Upsilon^{-1}(\sigma)(d(f_1 \diamond f_2)) &= \sigma(\Upsilon(d(f_1 \diamond f_2))) \\ &= \sigma(\Upsilon(d(f_1)) \diamond \Upsilon(d(f_2))) \\ &\geq T^n(\sigma(\Upsilon(d(f_1)) \diamond \Upsilon(d(f_3)))) \diamond \Upsilon(d(f_2)), \sigma(\Upsilon(d(f_3)))) \\ &= T^n(\sigma(\Upsilon(d(f_1 \diamond f_3)) \diamond f_2), \sigma(\Upsilon(d(f_3)))) \\ &= T^n(\Upsilon^{-1}(\sigma)(d(f_1 \diamond f_3) \diamond f_2), \Upsilon^{-1}(\sigma)(d(f_3)))\end{aligned}$$

so

$$\Upsilon^{-1}(\sigma)(d(f_1 \diamond f_2)) \geq T^n(\Upsilon^{-1}(\sigma)(d(f_1 \diamond f_3) \diamond f_2), \Upsilon^{-1}(\sigma)(d(f_3))).$$

Then $\Upsilon^{-1}(\sigma) \in FLDT^n(M)$. □

Definition 3.26. Let $(M, \diamond, 0)$ be a BCC-algebra and $d : M \to M$ be a self map. A fuzzy set $\eta : M \to [0, 1]$ in M is called a fuzzy right derivation BCC-ideal of M under t-norm T^n if it satisfies the following conditions:
(1) $\eta(0) \geq \eta(l)$,
(2) $\eta(d(l \diamond z)) \geq T^n(\eta((l \diamond y) \diamond d(z)), \eta(d(y)))$
for all $l, y, z \in M$.
Denote by $FRDT^n(M)$, the set of all fuzzy right derivation BCC-ideals of M under t-norm T^n.

Proposition 3.27. *Let $\delta \in [0, 1]^M$ and T^n be idempotent. Then $\delta \in FRDT^n(M)$ if and only if the set $\delta_a = \{c \in M : \delta(c) \geq a\}$ is either empty or a right derivation BCC-ideal of M for every $a \in [0, 1]$.*

Proof. Let $\delta \in FRDT^n(M)$ and $c, y \in M$. Then $\delta(0) \geq \delta(c) \geq a$ and then $0 \in \delta_a$. Also let $(c \diamond y) \diamond d(z) \in \delta_a$ and $d(y) \in \delta_a$. Thus $\delta(d(c \diamond z)) \geq T^n(\delta((c \diamond y) \diamond d(z)), \delta(d(y))) \geq T^n(a, a) = a$ so $d(c \diamond z) \in \delta_a$. Then δ_a will be a right derivation BCC-ideal of M for every $a \in [0, 1]$.
Conversely, let δ_a is either empty or a right derivation BCC-ideal of M for every $a \in [0, 1]$. Let $a = T^n(\delta((c \diamond y) \diamond d(z)), \delta(d(y)))$ with $(c \diamond y) \diamond d(z) \in \delta_a$ and $d(y) \in \delta_a$. Then $d(c \diamond z) \in \delta_a$ thus

$$\delta(d(c \diamond z)) \geq a = T^n(\delta((c \diamond y) \diamond d(z)), \delta(d(y)))$$

so $\delta \in FRDT^n(M)$. □

Proposition 3.28. *Let $\alpha, \tau \in FRDT^n(M)$. Then $\alpha \cap \tau \in FRDT^n(M)$.*

Proof. Let $w, e, z \in M$. Then
(1)

$$(\alpha \cap \tau)(0) = T^n(\alpha(0), \tau(0)) \geq T^n(\alpha(w), \tau(w)) = (\alpha \cap \tau)(w)$$

thus

$$(\alpha \cap \tau)(0) \geq (\alpha \cap \tau)(w).$$

(2)

$$\begin{aligned}(\alpha \cap \tau)(d(w \diamond z)) &= T^n(\alpha(d(w \diamond z)), \tau(d(w \diamond z))) \\ &\geq T^n(T^n(\alpha((w \diamond e) \diamond d(z)), \alpha(d(e))), T^n(\tau((w \diamond e) \diamond d(z)), \tau(d(e)))) \\ &= T^n(T^n(\alpha((w \diamond e) \diamond d(z)), \tau((w \diamond e) \diamond d(z))), T^n(\alpha(e), \tau(e))) \text{ (Lemma 2.26)} \\ &= T^n((\alpha \cap \tau)((w \diamond e) \diamond d(z)), (\alpha \cap \tau)(e))\end{aligned}$$

so

$$(\alpha \cap \tau)(w \diamond z) \geq T^n((\alpha \cap \tau)((x \diamond e) \diamond d(z)), (\alpha \cap \tau)(e)).$$

Thus $\alpha \cap \tau \in FRDT^n(M)$. □

Proposition 3.29. *Let $\zeta \in FRDT^n(M)$ and $\varpi \in FRDT^n(N)$. Then $\zeta \times \varpi \in FRDT^n(M \times N)$.*

Proof. Let $(g, k) \in M \times N$. Then

$$(\zeta \times \varpi)(0, 0) = T^n(\zeta(0), \varpi(0)) \geq T^n(\zeta(g), \varpi(k)) = (\zeta \times \varpi)(g, k)$$

thus $(\zeta \times \varpi)(0, 0) \geq (\zeta \times \varpi)(g, k)$.
Also let $g_i \in M$ and $k_i \in N$ for $i = 1, 2, 3$. Now

$$
\begin{aligned}
(\zeta \times \varpi)(d((g_1, k_1) \diamond (g_2, k_2))) &= (\zeta \times \varpi)(d(g_1 \diamond g_2, k_1 \diamond y_2)) \\
&= (\zeta \times \varpi)(d(g_1 \diamond g_2), d(k_1 \diamond k_2)) \\
&= T^n(\zeta(d(g_1 \diamond g_2)), \varpi(d(k_1 \diamond k_2))) \\
&\geq T^n(T^n(\zeta((g_1 \diamond g_3) \diamond d(g_2)), \zeta(d(g_3))), T^n(\varpi((k_1 \diamond k_3) \diamond d(k_2)), \varpi(d(k_3)))) \\
&= T^n(T^n(\zeta((g_1 \diamond g_3) \diamond d(g_2)), \nu((k_1 \diamond k_3) \diamond d(k_2))), T^n(\zeta(d(g_3)), \varpi(d(k_3)))) \\
&= T^n((\zeta \times \varpi)((g_1 \diamond g_3) \diamond d(g_2), (k_1 \diamond k_3) \diamond d(k_2)), (\zeta \times \varpi)(d(g_3), d(k_3))) \\
&= T^n((\zeta \times \varpi)(((g_1, k_1) \diamond (g_3, k_3))) \diamond (d(g_2), d(k_2))), (\zeta \times \varpi)(d(g_3), d(k_3))) \\
&= T^n((\zeta \times \varpi)(((g_1, k_1) \diamond (g_3, k_3))) \diamond d(g_2, k_2)), (\zeta \times \varpi)d(g_3, k_3))
\end{aligned}
$$

Therefore $\zeta \times \varpi \in FRDT^n(M \times N)$. □

Proposition 3.30. *Let* $\chi_i \in FRDT^n(M_i)$ *where* $1 \leq i \leq m$, *then* $\chi = \prod_{i=1}^{m} \chi_i \in FRDT^n(\prod_{i=1}^{m} M_i)$.

Proposition 3.31. *If* $\xi \in FRDT^n(M)$ *and* $\Theta : M \to N$ *be a homomorphism of BCC-algebras, then* $\Theta(\xi) \in FRDT^n(N)$.

Proof. As

$$\Theta(\xi)(0) = \sup\{\xi(0) \mid 0 \in M, \Theta(0) = 0\} \geq \sup\{\xi(t) \mid t \in M, \Theta(t) = e\} = \Theta(\xi)(e)$$

thus

$$\Theta(\xi)(0) \geq \Theta(\xi)(e).$$

Let $t_i \in M$ and $e_i \in N$ with $\Theta(t_i) = e_i$ for $i = 1, 2, 3$. Then

$$
\begin{aligned}
\Theta(\xi)(d(e_1 \diamond e_2)) &= \sup\{\xi(d(t_1 \diamond t_2)) \mid \Theta(x_i) = e_i\} \\
&\geq \sup\{T^n(\xi((t_1 \diamond t_3) \diamond d(t_2)), \xi(d(t_3))) \mid \Theta(t_i) = e_i\} \\
&= T^n(\sup\{\xi((t_1 \diamond t_3) \diamond d(t_2)) \mid \Theta((t_1 \diamond t_3) \diamond d(t_2)) = (e_1 \diamond e_3) \diamond d(e_2)\}, \sup\{\xi(d(t_3)) \mid \Theta(d(t_3)) = d(e_3)\}) \\
&= T^n(\Theta(\xi)((e_1 \diamond e_3) \diamond d(e_2)), \Theta(\xi)(d(e_3)))
\end{aligned}
$$

therefore

$$\Theta(\xi)(e_1 \diamond e_2) \geq T^n(\Theta(\xi)((y_1 \diamond e_3) \diamond d(e_2)), \Theta(\xi)(d(e_3))).$$

So $\Theta(\xi) \in FRDT^n(N)$. □

Proposition 3.32. *If* $\omega \in FRDT^n(N)$ *and* $\Delta : M \to N$ *be a homomorphism of BCC-algebras, then* $\Delta^{-1}(\omega) \in FRDT^n(M)$.

Proof. Let $u \in M$. Then

$$\Delta^{-1}(\omega)(0) = \omega(\Delta(0)) \geq \omega(\Delta(u)) = \Delta^{-1}(\omega)(x).$$

Let $u_1, u_2, u_3 \in M$. Since

$$
\begin{aligned}
\Delta^{-1}(\omega)(d(u_1 \diamond u_2)) &= \omega(\Delta(d(u_1 \diamond u_2))) \\
&\geq T^n(\omega(\Delta(u_1) \diamond \Delta(u_3)) \diamond \Delta(d(u_2)), \omega(\Delta(d(u_3)))) \\
&= T^n(\omega(\Delta((u_1 \diamond u_3)) \diamond d(u_2)), \omega(\Delta(d(u_3)))) \\
&= T^n(\Delta^{-1}(\omega)((u_1 \diamond u_3) \diamond d(u_2)), \Delta^{-1}(\omega)(d(u_3)))
\end{aligned}
$$

so

$$\Delta^{-1}(\omega)(d(u_1 \diamond u_2)) \geq T^n(\Delta^{-1}(\omega)((u_1 \diamond u_3) \diamond d(u_2)), \Delta^{-1}(\omega)(d(u_3))).$$

Then $\Delta^{-1}(\omega) \in FRDT^n(M)$. $\square$

Definition 3.33. Let $(M, \diamond, 0)$ be a BCC-algebra and $d : M \to M$ be a self map. A fuzzy set $\delta : M \to [0,1]$ in M is called a fuzzy derivation BCC-ideal of M under t-norm T^n if:
(1) $\delta(0) \geq \delta(k)$,
(2) $\delta(d(k \diamond z)) \geq T^n(\delta(d((k \diamond y) \diamond z)), \delta(d(y)))$
for all $k, y, z \in M$.
Denote by $FDT^n(M)$, the set of all fuzzy derivation BCC-ideals of M under t-norm T^n.

Example 3.34. Let $M = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra, in which the operation $\diamond$ is defined as follows:

$\diamond$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

define a map $d : M \to M$ by

$$d(m) = \begin{cases} 0 & \text{by } m = 0 \\ 2 & \text{by } m = 1 \\ 3 & \text{by } m = 2, 4 \\ 5 & \text{by } m = 3, 5. \end{cases}$$

Define a fuzzy set $\eta : d(m) \to [0,1]$ by

$$\eta(d(m)) = \begin{cases} 0.65 & \text{if } m = 0 \\ 0.5 & \text{if } m = 1, 2 \\ 0.45 & \text{if } m = 3, 4 \\ 0.3 & \text{if } m = 5. \end{cases}$$

Let $T^n(a, b) = T_p^n(a, b) = ab$ for all $a, b \in [0,1]$ then $\eta \in FDT^n(M)$.

Remark 3.35. If $d : M \to M$ is fixed, the Definitions 3.18, 3.26 and 3.33 give the definition of fuzzy BCC-ideal.

Proposition 3.36. *Let $\theta \in [0,1]^M$ and T^n be idempoten. Then $\theta \in FDT^n(M)$ if and only if the set $\theta_l = \{r \in M : \theta(r) \geq l\}$ is either empty or a derivation BCC-ideal of M for every $l \in [0,1]$.*

Proof. Since $\theta \in FDT^n(M)$ and $r, c \in M$ so $\theta(0) \geq \theta(r) \geq l$ and then $0 \in \theta_l$. Also let $d((r \diamond c) \diamond z) \in \theta_l$ and $d(c) \in \theta_l$. Then

$$\theta(d(r \diamond z)) \geq T^n(\theta(d((r \diamond c) \diamond z)), \theta(d(c))) \geq T^n(l, l) = l$$

thus $d(r \diamond z) \in \theta_l$. Then θ_l will be a derivation BCC-ideal of M for every $l \in [0,1]$.
Conversely, let θ_l is either empty or a derivation BCC-ideal of M for every $l \in [0,1]$. Let $l = T^n(\theta(d((r \diamond c) \diamond z)), \theta(d(c)))$ with $d((r \diamond c) \diamond z) \in \theta_l$ and $d(c) \in \theta_l$. Then $d(r \diamond z) \in \theta_l$ thus

$$\theta(d(r \diamond z)) \geq l = T^n(\theta(d((r \diamond y) \diamond z)), \theta(d(c)))$$

so $\theta \in FDT^n(M)$. $\square$

Proposition 3.37. *Let $\gamma, \varsigma \in FDT^n(M)$. Then $\gamma \cap \varsigma \in FDT^n(M)$.*

Proof. Let $r, j, z \in M$. Then
(1)

$$(\gamma \cap \varsigma)(0) = T^n(\gamma(0), \varsigma(0)) \geq T^n(\gamma(r), \varsigma(r)) = (\gamma \cap \varsigma)(r)$$

thus

$$(\gamma \cap \varsigma)(0) \geq (\gamma \cap \varsigma)(r).$$

(2)

$$\begin{aligned}
(\gamma \cap \varsigma)(d(r \diamond z)) &= T^n(\gamma(d(r \diamond z)), \varsigma(d(r \diamond z))) \\
&\geq T^n(T^n(\gamma(d((r \diamond j) \diamond z)), \gamma(d(j))), T^n(\varsigma((r \diamond j) \diamond d(z)), \varsigma(d(j)))) \\
&= T^n(T^n(\gamma(d((r \diamond j) \diamond z)), \varsigma(d((r \diamond j) \diamond z))), T^n(\gamma(j), \varsigma(j))) \text{ (Lemma 2.26)} \\
&= T^n((\gamma \cap \varsigma)(d((r \diamond j) \diamond z)), (\gamma \cap \varsigma)(j))
\end{aligned}$$

so

$$(\gamma \cap \varsigma)(r \diamond z) \geq T^n((\gamma \cap \varsigma)(d((r \diamond j) \diamond z)), (\gamma \cap \varsigma)(j)).$$

Thus $\gamma \cap \varsigma \in FDT^n(M)$. □

Proposition 3.38. *Let $\lambda \in FDT^n(M)$ and $\sigma \in FDT^n(N)$. Then $\lambda \times \sigma \in FDT^n(M \times N)$.*

Proof. Let $(p, q) \in M \times N$. Then

$$(\lambda \times \sigma)(0, 0) = T^n(\lambda(0), \sigma(0)) \geq T^n(\lambda(p), \sigma(q)) = (\lambda \times \sigma)(p, q)$$

thus $(\lambda \times \sigma)(0, 0) \geq (\lambda \times \sigma)(p, q)$.
Also let $p_i \in M$ and $q_i \in N$ for $i = 1, 2, 3$. Now

$$\begin{aligned}
(\lambda \times \sigma)(d((p_1, q_1) \diamond (p_2, q_2))) &= (\lambda \times \sigma)(d(p_1 \diamond p_2, q_1 \diamond q_2)) \\
&= (\lambda \times \sigma)(d(p_1 \diamond p_2), d(q_1 \diamond q_2)) \\
&= T^n(\lambda(d(p_1 \diamond p_2)), \sigma(d(q_1 \diamond q_2))) \\
&\geq T^n(T^n(\lambda(d(p_1 \diamond p_3) \diamond p_2), \lambda(d(p_3))), T^n(\sigma(d(q_1 \diamond q_3) \diamond q_2), \sigma(d(q_3)))) \\
&= T^n(T^n(\lambda(d(p_1 \diamond p_3) \diamond p_2), \sigma(d(q_1 \diamond q_3) \diamond q_2)), T^n(\lambda(d(p_3)), \nu(d(q_3)))) \\
&= T^n((\lambda \times \sigma)(d(p_1 \diamond p_3) \diamond p_2, d(q_1 \diamond q_3) \diamond q_2), (\lambda \times \sigma)(d(p_3), d(q_3))) \\
&= T^n((\lambda \times \sigma)(d((p_1, q_1) \diamond (p_3, q_3))) \diamond (p_2, q_2)), (\lambda \times \sigma)(d(p_3), d(q_3))) \\
&= T^n((\lambda \times \sigma)(d((p_1, q_1) \diamond (p_3, q_3))) \diamond (p_2, q_2), (\lambda \times \sigma)d(p_3, q_3))
\end{aligned}$$

thus $\lambda \times \sigma \in FDT^n(M \times N)$. □

Proposition 3.39. *Let $\Phi_i \in FDT^n(M_i)$ where $1 \leq i \leq m$, then $\Phi = \prod_{i=1}^m \Phi_i \in FDT^n(\prod_{i=1}^m M_i)$.*

Proposition 3.40. *If $\varrho \in FDT^n(M)$ and $\Upsilon : M \to N$ be a homomorphism of BCC-algebras, then $\Upsilon(\varrho) \in FDT^n(N)$.*

Proof. As

$$\Upsilon(\varrho)(0) = \sup\{\varrho(0) \mid 0 \in M, \Upsilon(0) = 0\} \geq \sup\{\varrho(s) \mid x \in M, \varphi(s) = y\} = \varphi(\varrho)(y)$$

thus

$$\varphi(\varrho)(0) \geq \Upsilon(\varrho)(y).$$

Let $s_i \in M$ and $y_i \in N$ with $\Upsilon(s_i) = y_i$ for $i = 1, 2, 3$. Then

$$\Upsilon(\varrho)(d(y_1 \diamond y_2)) = \sup\{\varrho(d(s_1 \diamond s_2)) \mid \Upsilon(s_i) = y_i\}$$

$$\geq \sup\{T^n(\varrho(d(s_1 \diamond s_3) \diamond s_2), \varrho(d(s_3))) \mid \Upsilon(s_i) = y_i\}$$

$$= T^n(\sup\{\varrho(d(s_1 \diamond s_3) \diamond s_2) \mid \Upsilon(d(s_1 \diamond s_3) \diamond s_2) = d(y_1 \diamond y_3) \diamond y_2\}, \sup\{\varrho(d(s_3)) \mid \Upsilon(d(s_3)) = d(y_3)\})$$

$$= T^n(\Upsilon(\varrho)(d(y_1 \diamond y_3) \diamond y_2)), \Upsilon(\varrho)(d(y_3)))$$

therefore

$$\Upsilon(\varrho)(y_1 \diamond y_2) \geq T^n(\Upsilon(\varrho)(d(y_1 \diamond y_3) \diamond y_2)), \Upsilon(\varrho)(d(y_3))).$$

Thus $\Upsilon(\varrho) \in FDT^n(N)$. □

Proposition 3.41. *If $\kappa \in FDT^n(N)$ and $\Omega : M \to N$ be a homomorphism of BCC-algebras, then $\Omega^{-1}(\kappa) \in FDT^n(M)$.*

Proof. Let $l \in M$. Then

$$\Omega^{-1}(\kappa)(0) = \kappa(\Omega(0)) \geq \kappa(\Omega(x)) = \Omega^{-1}(\kappa)(l).$$

Let $l_1, l_2, l_3 \in M$. Since

$$\begin{aligned}\Omega^{-1}(\kappa)(d(l_1 \diamond x_2)) &= \kappa(\Omega(d(l_1 \diamond l_2))) \\ &\geq T^n(\kappa(\Omega(d(l_1 \diamond l_3) \diamond l_2), \kappa(\Omega(d(l_3)))) \\ &= T^n(\Omega^{-1}(\kappa)(d(l_1 \diamond l_3) \diamond l_2), \Omega^{-1}(\kappa)(d(l_3)))\end{aligned}$$

so

$$\Omega^{-1}(\kappa)(d(l_1 \diamond l_2)) \geq T^n(\Omega^{-1}(\kappa)(d(l_1 \diamond l_3) \diamond l_2), \Omega^{-1}(\kappa)(d(l_3))).$$

Then $\Omega^{-1}(\kappa) \in FDT^n(M)$. □

Conclusion

In this paper, we introduce some concepts such as fuzzy subalgebras, fuzzy ideals, fuzzy left derivations, fuzzy right derivations and fuzzy derivations of BCC-algebras with respect to triangular norms we obtained some results of them. Now, as using triangular norms, one can introduce and investigate fuzzy subalgebras, fuzzy ideals, fuzzy left derivations, fuzzy right derivations and fuzzy derivations of BZ-algebras as we did with BCC-algebras and this can be an open problem for future works.

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