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Research Paper

SOME PROPERTIES OF THE α-CHEBYSHEV WAVELETS APPROXIMATION WITH HÖLDER FUNCTION

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Abstract. In this paper, we consider $\alpha-$Chebyshev polynomials, $\alpha-$Chebyshev wavelet approximation and hölder of order l. We estimate $\alpha-$Chebyshev-wavelet approximation of a function f having satisfy condition hölder where f is expanded in terms of $\alpha-$Chebyshev wavelet polynomials.
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1. Introduction and Background

There are several subjects in which these functions take a significant position in modern developments including Chebyshev wavelets, wavelet approximation, numerical integration, and spectral methods for partial differential equations.
It is said pseudo-Chebyshev if $\alpha = \frac{1}{2}$. (see [1-5])

The $\alpha-$Chebyshev functions for $|x| \leq 1$ are

$Kinds$	$\alpha - Chebyshev\ Functions$
$First - Kind\ \alpha\ Chebyshev$	$T_n^\alpha(x) = cos(n+\alpha)\theta$
$Second - Kind\ \alpha\ Chebyshevl$	$U_n^\alpha(x) = \frac{sin(n+\alpha)\theta}{sin\theta}$
$Third - Kind\ \alpha\ Chebyshev$	$V_n^\alpha(x) = \frac{cos(n+\alpha)\theta}{sin\theta}$
$Fourth\ Kind\ \alpha\ Chebyshev$	$W_n^\alpha(x) = sin(n+\alpha)\theta$

Theorem 1.1. *The first kind $\alpha-$Chebyshev function where $|x| \leq 1$ is solution for first kind $\alpha-$Chebyshev differential equation* $(1-x^2)y'' - xy' + (n+\alpha)^2 y = 0$,

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Proof. Suppose $y = cos(n+\alpha)(\theta) = cos(n+\alpha)(cos^{-1}x)$, then

$$\begin{array}{rcl} \frac{dy}{dx} & = & (n+\alpha)(1-x^2)^{-\frac{1}{2}}sin(n+\alpha)(cos^{-1}x) \\ \frac{d^2y}{dx^2} & = & -(n+\alpha)^2(1-x^2)^{-1}cos(n+\alpha)(cos^{-1}x) \\ & + & x(1-x^2)^{\frac{-3}{2}}(n+\alpha)sin(n+\alpha)(cos^{-1}x), \\ & Then & (1-x^2)y" - xy' + (n+\alpha+1)^2y \\ & = & (1-x^2)[-(n+\alpha)^2(1-x^2)^{-1}cos(n+\alpha)(cos^{-1}x) \\ & + & x(1-x^2)^{\frac{-3}{2}}(n+\alpha)sin(n+\alpha)(cos^{-1}x)] \\ & - & x[(n+\alpha)(1-x^2)^{-\frac{1}{2}}sin(n+\alpha)(cos^{-1}x)] \\ & + & (n+\alpha)^2cos(n+\alpha)(cos^{-1}x) \\ & = & -(n+\alpha)^2cos(n+\alpha)(cos^{-1}x) \\ & + & x(1-x^2)^{\frac{-1}{2}}(n+\alpha)sin(n+\alpha)(cos^{-1}x) \\ & - & x(n+\alpha)(1-x^2)^{-\frac{1}{2}}sin(n+\alpha)(cos^{-1}x) \\ & + & (n+\alpha)^2cos(n+\alpha)(cos^{-1}x) \\ & = & 0. \end{array}$$

□

We definition α-Chebyshev wavelets. Suppose $k \in \mathbb{N}$ (degree of multiresolution), $m \geq 0$, $n = 0, 1, 2, \cdots, 2^{k-1}$ (see [5-7]).

$\alpha - Chebyshev\ wavelets$	$Sequences\ \alpha - Chebyshev\ function$
$T^{\alpha}_{n,m}(t)$	$\begin{cases} \sqrt{\frac{2^{k+1}}{n}}T^{\alpha}_m(2^kt-2n+1)) & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}] \\ 0 & \text{otherwise} \end{cases}.$
$U^{\alpha}_{n,m}(t)$	$\begin{cases} \sqrt{\frac{2^{k+1}}{n}}U^{\alpha}_m(2^kt-2n+1) & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}] \\ 0 & \text{otherwise} \end{cases}.$
$V^{\alpha}_{n,m}(t)$	$\begin{cases} \sqrt{\frac{2^{k+1}}{n}}V^{\alpha}_m(2^kt-2n+1) & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}] \\ 0 & \text{otherwise} \end{cases}.$
$W^{\alpha}_{n,m}(t)$	$\begin{cases} \sqrt{\frac{2^{k+1}}{n}}W^{\alpha}_m(2^kt-2n+1) & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}] \\ 0 & \text{otherwise} \end{cases}$

In the following, we suppose $\Psi^{\alpha}_{n,m} = T^{\alpha}_{n,m}\ or\ \Psi^{\alpha}_{n,m} = U^{\alpha}_{n,m},\ or\ \Psi^{\alpha}_{n,m} = V^{\alpha}_{n,m},\ or\ \Psi^{\alpha}_{n,m} = W^{\alpha}_{n,m}$.

A function $f \in L^2[-1, 1)$ is expanded by α-Chebyshev wavelet series as

$$f(t) = \sum_{n=0}^{2^k}\sum_{m=0}^{\infty} c_{n,m}\Psi^{\alpha}_{n,m}(t),$$

where

$$c_{n,m} = \int_{-1}^{1} f(t)\Psi^{\alpha}_{n,m}(t)\omega^{\alpha}_{n,m}(t)dt,$$

and $\omega^{\alpha}_{n,m}$ is the weight function of α-Chebyshev wavelets. Also

$$\int_{-1}^{1} \Psi^{\alpha}_{n,m}(t)\Psi^{\alpha}_{n,m}(y)\omega^{\alpha}_{n,m}(t)dx = L,$$

$$L = \begin{cases} \frac{\pi}{2} & m = n, \Psi = T^{\alpha},\ or\ \Psi = W^{\alpha} \\ 0 & \text{m} \neq n, \Psi = T^{\alpha},\ or\ W^{\alpha}, or\ U^{\alpha}, or\ \Psi = V^{\alpha}, \\ \pi & m = n, \Psi = U^{\alpha},\ \Psi = V^{\alpha} \end{cases},$$

$$\omega^{\alpha}_{n,m}(t) = \begin{cases} \frac{1}{\sqrt{1-t^2}} & \Psi = T^{\alpha},\ or\ \Psi = W^{\alpha} \\ \sqrt{1-t^2} & \Psi = U^{\alpha}, or\ \Psi = V^{\alpha} \end{cases},$$

It is necessary to study multiresolution analysis and Mallat's Theorem for wavelet approximation (see [7-9]).

Definition 1.2. Multiresolution Analysis: An **MRA** with scaling function ϕ is a collection of closed subspaces $\{V_j\}_{j\in Z}$ of $L^2(\mathbb{R})$, such that

(i) $V_j \subset V_{j+1}$;
(ii) $f(x) \in V_j \iff f(2x) \in V_{j+1}$;
(iii) $\overline{\cup V_j} = L^2(\mathbb{R})$,
(iv) $\cap V_j = 0$;
(v) There exists a function $\phi \in V_0$ such that the collection $\{\phi(xk) :\ k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

The sequence of wavelet subspaces W_j of $L^2(\mathbb{R})$ is such that $V_j \perp W_j$, for all$j$ and $V_{j+1} = V_j \bigoplus W_j$. Closure of $\bigoplus W_j$ is dense in $L^2(\mathbb{R})$ for L^2 norm.

Now we state Mallat's theorem which guarantees that in the presence of an orthogonal **MRA**, an orthonormal basis for $L^2(\mathbb{R})$ exists. These basis functions are fundamental in the theory of wavelets which helps us to develop advanced computational techniques.

Lemma 1.3. (Mallat's Theorem) *Given an orthogonal MRA with scaling function ϕ, there is a wavelet $\psi \in L^2 * (\mathbb{R})$ such that for each $j \in \mathbb{Z}$, the family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j. Hence the family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Definition 1.4. (i) Let $P_n(f)$ be the orthogonal projection of $L^2([-1,1])$ onto V_n. Then

$$P_n(f) = \sum_{-\infty}^{\infty} < f, \phi_{n,k} > \phi_{n,k},\ n = 1, 2, 3, \cdots$$

(ii) The wavelet approximation of the Chebyshev polynomial is defined by

$$E_n(f) = \|fP_n(f)\|_2 = \int_{-1}^{1} |f(t)P_n(f)(t)|^2 dt = o(\phi(n)).$$

Theorem 1.5. *Let $f \in L^2([-1,1])$ be a continuous function and $f(t) = \sum_{n=0}^{2^k}\sum_{m=0}^{\infty} c^{\alpha}_{n,m}\Psi^{\alpha}_{n,m}(t)$ and the series $\sum_{n=0}^{2^k}\sum_{m=0}^{\infty} |c^{\alpha}_{n,m}|^2$ be convergent. Then α-Chebyshev wavelet approximation f, for every M is the partial sums*

$$s_{2^k,M-1}(t) = \sum_{n=0}^{2^k}\sum_{m=0}^{M-1} c^{\alpha}_{n,m}\Psi^{\alpha}_{n,m}(t),$$

and

$$E_{2^k,M-1}(f)=o((L\sum_{n=0}^{2^k}\sum_{m=M}^{\infty}|c_{n,m}^{\alpha}|^2)).$$

Proof. We have

$$\begin{aligned}
\| & \quad f-s_{2^k,M-1}\|_2^2 \\
= & \quad \|\int_{-1}^{1}(\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}c_{n,m}^{\alpha}\Psi_{n,m}^{\alpha}(t)-\sum_{n=0}^{2^k}\sum_{m=0}^{M-1}c_{n,m}^{\alpha}\Psi_{n,m}^{\alpha}(t))^2\omega_{n,m}^{\alpha}(t)dt\| \\
\leq & \quad \int_{-1}^{1}\|(\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}c_{n,m}^{\alpha}\Psi_{n,m}^{\alpha}(t)-\sum_{n=0}^{2^k}\sum_{m=0}^{M-1}c_{n,m}^{\alpha}\Psi_{n,m}^{\alpha}(t))^2\|\omega_{n,m}^{\alpha}(t)dt \\
= & \quad \int_{-1}^{1}\|\sum_{n=0}^{2^k}\sum_{m=M}^{\infty}c_{n,m}^{\alpha}\Psi_{n,m}^{\alpha}(t)\|^2\omega_{n,m}^{\alpha}(t)dt \\
\leq & \quad \sum_{n=0}^{2^k}\sum_{m=M}^{\infty}|c_{n,m}^{\alpha}|^2\int_{-1}^{1}|\Psi_{n,m}^{\alpha}(t)|^2\omega_{n,m}^{\alpha}(t)dt \\
= & \quad L\sum_{n=0}^{2^k}\sum_{m=M}^{\infty}|c_{n,m}^{\alpha}|^2
\end{aligned}$$

Therefore $\|f-s_{M-1}\|_2\leq(L\sum_{m=M}^{\infty}|T_{n,m}^{\alpha}|^2)^{\alpha}$. That is

$$E_{2^k,M-1}(f)=o((L\sum_{n=0}^{2^k}\sum_{m=M}^{\infty}|c_{n,m}^{\alpha}|^2)^{\alpha}),$$

□

2. Wavelet approximation with Hölder condition of order $0<l\leq 1$

In this section, we estimate wavelet approximation of a function f having satistisfy condition hölder as $0<l\leq 1$.

Definition 2.1. Suppose $D\subset\mathbb{C}$ is open bounded. A function $f:\ D\to\mathbb{C}$ satisfies the Hölder condition of order $0<l\leq 1$ if there is a constant $C>0$ for which $|f(x)-f(y)|\leq C|x-y|^l$ for all $x,y\in D$.

Theorem 2.2. *Let $f:[0,1]\to\mathbb{R}$ be a function satisfying condition hölder with order $0<l\leq 1$.*

i) If $f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}T_{n,m}^{\alpha}(t)$ is expanded in terms of wavelet α-Chebyshev polynomial of the first kind. Then the $\alpha-$Chebyshev wavelet approximation $E_{2^k,l}(t)$ of f by $(2^k,l)$ is th partial sum:

$$s_{2^k-1,l}=\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=o}^{l}c_{n,m}T_{n,m}^{\alpha},$$

and

$$lim_{l\to\infty}E_{2^k,l}(f)=lim_{l\to\infty}\|f-s_{2^k,l}\|_2=0$$

(ii) If $f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}U_{n,m}^{\alpha}(t)$ *is expanded in terms of the wavelet* α*- Chebyshev polynomial of the second kind. Then* $\alpha-$*Chebyshev wavelet approximation* $E_{2^k,l}(t)$ *of* f *by* $(2^k,l)$ *is th partial sum*

$$s_{2^k-1,l}=\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}c_{n,m}U_{n,m}^{\alpha},$$

and

$$lim_{l\to\infty}E_{2^k,l}(f)=lim_{l\to\infty}\|f-s_{2^k,l}\|_2=0$$

(iii) If $f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}V_{n,m}^{\alpha}(t)$ *is expanded in terms of the wavelet* α*- Chebyshev polynomial of the third kind. Then the* $\alpha-$*Chebyshev wavelet approximation* $E_{2^k,l}(t)$ *of* f *by* $(2^k,l)$ *is th partial sum*

$$s_{2^k-1,l}=\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}c_{n,m}V_{n,m}^{\alpha},$$

and

$$lim_{l\to\infty}E^2_{2^k,l}(f)=lim_{l\to\infty}\|f-s_{2^k,l}\|_2=0$$

(iv) If $f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}W_{n,m}^{\alpha}(t)$ *is expanded in terms of wavelet* α*-Chebyshev polynomial of the fourth kind. Then the* $\alpha-$*Chebyshev wavelet approximation* $E_{2^k,l}(t)$ *of* f *by* $(2^k,l)$ *is th partial sum*

$$s_{2^k-1,l}=\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}c_{n,m}W_{n,m}^{\alpha},$$

and

$$lim_{l\to\infty}E_{2^k-1,l}(f)=lim_{l\to\infty}\|f-s_{2^{k-1},l}\|_2=0$$

Proof. (i) Since $f:[0,1]\to\mathbb{R}$ be a function satisfying condition höder with order l.

Note that if $0\le\theta\le\pi$, then

$$\begin{array}{rcl} -1\le & cos\theta & \le 1\\ \dfrac{2n-2}{2^k}\le & \frac{cos\theta+2n-1}{2^k} & \le\dfrac{2n}{2^k}\\ 0\le\dfrac{n-1}{2^{k-1}}\le & \frac{cos\theta+2n-1}{2^k} & \le\dfrac{n}{2^{k-1}}\le 1. \end{array}$$

Case 1; $l = 1$
Since

$$\begin{aligned}
c_{n,m} &= < f(t), T^{\alpha}_{n,m}(t) >_w \\
&= 2^{\frac{k}{2}} \int_0^1 f(t) T^{\alpha}_m (2^k t - 2n + 1) w^{\alpha}(2^k t - 2n + 1) dt \\
&= 2^{\frac{k}{2}} \int_0^{\pi} f(\frac{cos\theta + 2n - 1}{2^k}) T^{\alpha}_m (cos\theta) \frac{1}{\sqrt{1 - cos^2\theta}} sin\theta d\theta \\
&\leq \alpha 2^{\frac{k}{2}} \int_0^{\pi} (\frac{cos\theta + 2n - 1}{2^k})^l cos(m+\alpha)\theta \frac{1}{\sqrt{1 - cos^2\theta}} sin\theta d\theta \\
&\leq C 2^{\frac{k}{2}} \int_0^{\pi} (\frac{cos\theta + 2n - 1}{2^k}) cos(m+\alpha)\theta \frac{1}{\sqrt{1 - cos^2\theta}} sin\theta d\theta \\
&\leq C 2^{-\frac{k}{2}} \int_0^{\pi} ((cos\theta) cos((m+\alpha)\theta) + (2n-1) cos(m+\alpha)\theta)) d\theta \\
&\leq C 2^{-\frac{k}{2}} \int_0^{\pi} cos\theta cos(m+\alpha)\theta d\theta \\
&= C 2^{-\frac{k}{2}-1} \int_0^{\pi} [cos(m-\alpha)\theta + cos(m+\frac{3}{2})\theta] d\theta \\
&= C 2^{-\frac{k}{2}-1} [\frac{sin(m-\alpha)\theta}{m-\alpha} + \frac{sin(m+\frac{3}{2})\theta}{m+\frac{3}{2}}]_0^{\pi} \\
&= C 2^{-\frac{k}{2}-1} [\frac{1}{m-\alpha} - \frac{1}{m+\frac{3}{2}}] \\
&\leq \frac{C 2^{-\frac{k}{2}-1}}{m-\alpha},
\end{aligned}$$

therefore

$$|c_{n,m}| \leq \frac{C}{m-\alpha}.$$

For $\|T^{\alpha}_{n,m}\|_2$

$$\begin{aligned}
\|T^{\alpha}_{n,m}\|_2^2 &= 2^k \int_0^1 |T^{\alpha}_m (2^k t - 2n + 1)|^2 w^{\alpha}(2^k t - 2n + 1) dt \\
&\leq 2^k \int_{-1}^1 T^{\alpha}_m (2^k t - 2n + 1)^2 w^{\alpha}(2^k t - 2n + 1) dt \\
&= \frac{\pi}{2} 2^k \\
&= \pi 2^{k-1},
\end{aligned}$$

and

$$\begin{aligned}
E^2_{2^k-1,l}(f) &= \|f - s_{2^k-1,l}\|_2^2 \\
&= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}T^{\alpha}_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m}T_{n,m+\alpha}\|_2^2 \\
&= \|\sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty} c_{n,m}T^{\alpha}_{n,m}\|_2^2 \\
&\leq \sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty} (\frac{2^{-\frac{k}{2}-1}C}{m-\alpha})^2(\pi 2^{k-1}) \\
&= 2^{-k-2}C^2\pi\sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty}(\frac{1}{m-\alpha})^2 \\
&= 2^{-k-2}(2^k-1)C^2\pi\sum_{m=l+1}^{\infty}(\frac{1}{m-\alpha})^2
\end{aligned}$$

Since the series $\sum_{m=0}^{\infty}(\frac{1}{m-\alpha})^2$ is convergence. Then

$$lim_{l\to\infty}E_{2^k-1,l}(f) = 0.$$

Case 2; $0 < l < 1$
Since

$$\begin{aligned}
c_{n,m} &= < f(t), T^{\alpha}_{n,m}(t) >_w \\
&= 2^{\frac{k}{2}}\int_0^1 f(t)T^{\alpha}_m(2^kt-2n+1)w^{\alpha}(2^kt-2n+1)dt \\
&= 2^{\frac{k}{2}}\int_0^{\pi} f(\frac{cos\theta+2n-1}{2^k})T^{\alpha}_m(cos\theta)\frac{1}{\sqrt{1-cos^2\theta}}sin\theta d\theta \\
&\leq \alpha 2^{\frac{k}{2}}\int_0^{\pi} cos(m+\alpha)\theta\frac{1}{\sqrt{1-cos^2\theta}}sin\theta d\theta \\
&\leq C2^{-\frac{k}{2}}\int_0^{\pi} cos(m+\alpha)\theta d\theta \\
&= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
E^2_{2^k-1,l}(f) &= \|f - s_{2^k-1,l}\|_2^2 \\
&= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}T^{\alpha}_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m}T_{n,m+\alpha}\|_2^2 \\
&= 0,
\end{aligned}$$

$$lim_{l\to\infty}E_{2^k-1,l}(f) = 0.$$

(ii) We have

$$\begin{aligned}
\|U^{\alpha}_{n,m}\|^2_2 &= 2^k \int_0^1 |U^{\alpha}_m(2^k t-2n+1)|^2 w^{\alpha}(2^k t-2n+1)dt \\
&\leq 2^k \int_{-1}^1 |U^{\alpha}_m(2^k t-2n+1)|^2 w^{\alpha}(2^k t-2n+1)dt \\
&= 2^k \int_{-1}^1 U^{\alpha}_m(2^k t-2n+1)^2 w^{\alpha}(2^k t-2n+1)dt \\
&= 2^k \pi.
\end{aligned}$$

Case 1. $l=1$.

$$\begin{aligned}
c_{n,m} &= < f(t), U^{\alpha}_{n,m}(t) >_w \\
&= 2^{\frac{k}{2}} \int_0^1 f(t) U^{\alpha}_m(2^k t-2n+1) w^{\alpha}(2^k t-2n+1)dt \\
&= 2^{\frac{k}{2}} \int_0^{\pi} f(\frac{cos\theta+2n-1}{2^k}) U^{\alpha}_m(cos\theta) sin\theta sin\theta d\theta \\
&= C2^{\frac{k}{2}} \int_0^{\pi} (\frac{cos\theta+2n-1}{2^k})^l U^{\alpha}_m(cos\theta) sin\theta sin\theta d\theta \\
&= C2^{\frac{k}{2}} \int_0^{\pi} (\frac{cos\theta+2n-1}{2^k}) U^{\alpha}_m(cos\theta) sin\theta sin\theta d\theta \\
&\leq C2^{-\frac{k}{2}} \int_0^{\pi} (cos\theta+2n-1) U^{\alpha}_m(cos\theta) sin\theta sin\theta d\theta \\
&= C2^{-\frac{k}{2}} \int_0^{\pi} (cos\theta+2n-1) \frac{sin(m+\frac{3}{2})\theta}{sin\theta} sin^2\theta d\theta \\
&= C2^{-\frac{k}{2}} \int_0^{\pi} (cos\theta+2n-1) sin(m+\frac{3}{2})\theta sin\theta d\theta \\
&= C2^{-\frac{k}{2}-1} \int_0^{\pi} (cos\theta+2n-1)[cos(m+\frac{5}{2})\theta - cos(m+\frac{3}{2})\theta] d\theta \\
&= C2^{-\frac{k}{2}-1} \int_0^{\pi} (cos\theta cos(m+\frac{5}{2})\theta + cos\theta cos(m+\frac{3}{2})\theta) d\theta \\
&= C2^{-\frac{k}{2}-2} \int_0^{\pi} (cos(m+\frac{7}{2})\theta + cos(m+\frac{3}{2})\theta + cos(m+\frac{5}{2})\theta + cos(m+\alpha)\theta) d\theta \\
&= C2^{-\frac{k}{2}-2} [\frac{1}{m+\frac{7}{2}} + \frac{1}{m+\frac{3}{2}} + \frac{1}{m+\frac{5}{2}} + \frac{1}{m+\alpha}] \\
&\leq \frac{4C2^{-\frac{k}{2}-2}}{m+\frac{7}{2}} \\
&= \frac{C2^{-\frac{k}{2}}}{m+\frac{7}{2}}.
\end{aligned}$$

Therefore,

$$|c_{n,m}| \leq \frac{C2^{-\frac{k}{2}}}{m+\frac{7}{2}}.$$

and

$$\begin{aligned}
E^2_{2^k-1,l}(f) &= \|f - s_{2^k-1,l}\|_2^2 \\
&= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}U^{\alpha}_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m}U^{\alpha}_{n,m}\|_2^2 \\
&= \|\sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty} c_{n,m}U^{\alpha}_{n,m}\|_2^2 \\
&\leq \sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty} |c_{n,m}|^2\|U^{\alpha}_{n,m}\|_2^2 \\
&= \pi C^2\sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty}(\frac{1}{m+\alpha})^2 \\
&= 2^{k-1}\pi C^2\sum_{m=l+1}^{\infty}(\frac{1}{m+\alpha})^2.
\end{aligned}$$

Since- the series $\sum_{m=0}^{\infty}(\frac{1}{m+\alpha})^2$ is convergence. Then

$$lim_{l\to\infty}E_{2^k-1,l}(f) = 0.$$

Case 2; $0 < l < 1$
Since

$$\begin{aligned}
c_{n,m} &= < f(t), U^{\alpha}_{n,m}(t) >_w \\
&= 2^{\frac{k}{2}}\int_0^1 f(t)U^{\alpha}_m(2^kt-2n+1)w^{\alpha}(2^kt-2n+1)dt \\
&= 2^{\frac{k}{2}}\int_0^{\pi} f(\frac{cos\theta+2n-1}{2^k})U^{\alpha}_m(cos\theta)\frac{1}{\sqrt{1-cos^2\theta}}sin\theta d\theta \\
&\leq \alpha 2^{\frac{k}{2}}\int_0^{\pi} cos(m+\alpha)\theta\frac{1}{\sqrt{1-cos^2\theta}}sin\theta d\theta \\
&\leq C2^{-\frac{k}{2}}\int_0^{\pi} cos(m+\alpha)\theta d\theta \\
&= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
E^2_{2^k-1,l}(f) &= \|f - s_{2^k-1,l}\|_2^2 \\
&= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}U^{\alpha}_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m}U_{n,m+\alpha}\|_2^2 \\
&= 0,
\end{aligned}$$

$$lim_{l\to\infty}E_{2^k-1,l}(f)=0.$$

(iii) We have
case 1. $l=1$.

$$\begin{aligned}
c_{n,m} &= <f(t),V^{\alpha}_{n,m}(t)>_w\\
&= 2^{\frac{k}{2}}\int_0^1 f(t)V^{\alpha}_m(2^kt-2n+1)w^{\alpha}(2^kt-2n+1)dt\\
&= 2^{\frac{k}{2}}\int_0^{\pi} f(\frac{cos\theta+2n-1}{2^k})\frac{cos(n+\alpha)\theta}{sin\theta}sin\theta sin\theta d\theta\\
&\leq C2^{-\frac{k}{2}}\int_0^{\pi}(\frac{cos\theta+2n-1}{2^k})^l cos(m+\alpha)\theta sin\theta d\theta\\
&\leq C2^{-\frac{k}{2}}\int_0^{\pi}(\frac{cos\theta+2n-1}{2^k})cos(m+\alpha)\theta sin\theta d\theta\\
&\leq C2^{-\frac{k}{2}}\int_0^{\pi}(cos\theta+2n-1)cos(m+\alpha)\theta sin\theta d\theta\\
&\leq C2^{-\frac{k}{2}-1}\int_0^{\pi}(cos\theta+2n-1)(sin(m+\frac{3}{2})\theta+sin(m+\alpha)\theta)d\theta\\
&= C2^{-\frac{k}{2}-1}\int_0^{\pi}(cos\theta sin(m+\frac{3}{2})\theta+cos\theta sin(m+\alpha)d\theta\\
&+ C2^{-\frac{k}{2}-1}(2n-1)\int_0^{\pi}(sin(m+\frac{3}{2})\theta+sin(m+\alpha)\theta)d\theta\\
&= C2^{-\frac{k}{2}-2}\int_0^{\pi}(sin(m+\frac{5}{2})\theta+sin(m+\alpha)\theta)d\theta\\
&+ C2^{-\frac{k}{2}-2}\int_0^{\pi}(sin(m+\frac{3}{2})\theta+sin(m+\alpha)\theta)d\theta\\
&+ C2^{-\frac{k}{2}}(2n-1)\int_0^{\pi}(sin(m+\frac{3}{2})\theta+sin(m+\alpha)\theta)d\theta\\
&= C2^{-\frac{k}{2}-2}[\frac{cos(m+\frac{5}{2})\pi-1}{m+\frac{5}{2}}+\frac{cos(m+\alpha)\pi-1}{m+\alpha}\\
&+ \frac{cos(m+\frac{3}{2})\pi-1}{m+\frac{3}{2}}+\frac{cos(m+\alpha)\pi-1}{m+\alpha}]\\
&= -C2^{-\frac{k}{2}-2}[\frac{1}{m+\frac{5}{2}}+\frac{2}{m+\alpha}+\frac{1}{m+\frac{3}{2}}]
\end{aligned}$$

and

$$\begin{aligned}
|c_{n,m}| &= C2^{-\frac{k}{2}-2}[\frac{1}{m+\frac{5}{2}}+\frac{2}{m+\alpha}+\frac{1}{m+\frac{3}{2}}]\\
&\leq \frac{C2^{-\frac{k}{2}}}{m+\alpha}.
\end{aligned}$$

For $V^{\alpha}_{n,m}$

$$\begin{aligned}\|V^{\alpha}_{n,m}\|^2_2 &= 2^{-\frac{k}{2}}\int_0^1 |V^{\alpha}_m(2^kt-n)|^2 w^{\alpha}(t)dt\\ &\leq 2^{-\frac{k}{2}}\int_{-1}^1 V^{\alpha}_m(2^kt-n)^2\sqrt{\frac{1+t}{1-t}}dt\\ &= 2^{-\frac{k}{2}}\pi\end{aligned}$$

Therefore

$$\begin{aligned}E^2_{2^k-1,l}(f) &= \|f-s_{2^k-1,l}\|^2_2\\ &= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}V^{\alpha}_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l}c_{n,m}V^{\alpha}_{n,m}\|^2_2\\ &= \|\sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty}c_{n,m}V^{\alpha}_{n,m}\|^2_2\\ &\leq \sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty}|c_{n,m}|^2\|V^{\alpha}_{n,m}\|^2_2\\ &= \pi C^2\sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty}(\frac{1}{m+\frac{7}{2}})^2\\ &= 2^{k-1}\pi C^2\sum_{m=l+1}^{\infty}(\frac{1}{m+\frac{7}{2}})^2.\end{aligned}$$

Since the series $\sum_{m=0}^{\infty}(\frac{1}{m+\frac{7}{2}})^2$ is convergence. Then

$$lim_{l\to\infty}E_{2^k-1,l}(f)=0.$$

Case 2; $0<l<1$
Since

$$\begin{aligned}c_{n,m} &= <f(t),V^{\alpha}_{n,m}(t)>_w\\ &= 2^{\frac{k}{2}}\int_0^1 f(t)V^{\alpha}_m(2^kt-2n+1)w^{\alpha}(2^kt-2n+1)dt\\ &= 2^{\frac{k}{2}}\int_0^{\pi} f(\frac{cos\theta+2n-1}{2^k})V^{\alpha}_m(cos\theta)\frac{1}{\sqrt{1-cos^2\theta}}sin\theta d\theta\\ &\leq \alpha 2^{\frac{k}{2}}\int_0^{\pi} cos(m+\alpha)\theta\frac{1}{\sqrt{1-cos^2\theta}}sin\theta d\theta\\ &\leq C2^{-\frac{k}{2}}\int_0^{\pi}cos(m+\alpha)\theta d\theta\\ &= 0.\end{aligned}$$

Therefore

$$\begin{aligned} E^2_{2^k-1,l}(f) &= \|f - s_{2^k-1,l}\|_2^2 \\ &= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}V^{\alpha}_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m}V_{n,m+\alpha}\|_2^2 \\ &= 0, \end{aligned}$$

$$lim_{l\to\infty}E_{2^k-1,l}(f) = 0.$$

(iv) We have
case 1. $l = 1$.

$$\begin{aligned} c_{n,m} &= < f(t), W^{\alpha}_{n,m} >_w \\ &= 2^{\frac{k}{2}}\int_0^1 f(t)W^{\alpha}_m(2^kt - 2n + 1)w^{\alpha}(2^kt - 2n + 1)dt \\ &= 2^{\frac{k}{2}}\int_0^{\pi} f(\frac{cos\theta + 2n - 1}{2^k})sin(m + \alpha)\theta\frac{1}{\sqrt{1 - cos^2\theta}}sin\theta d\theta \\ &\leq C2^{\frac{k}{2}}\int_0^{\pi}(\frac{cos\theta + 2n - 1}{2^k})^l sin(m + \alpha)\theta d\theta \\ &\leq C2^{\frac{k}{2}}\int_0^{\pi}(\frac{cos\theta + 2n - 1}{2^k})sin(m + \alpha)\theta d\theta \\ &\leq C2^{-\frac{k}{2}}\int_0^{\pi}(cos\theta + 2n - 1)sin(m + \alpha)\theta d\theta \\ &= C2^{-\frac{k}{2}}\int_0^{\pi}[cos\theta sin(m + \alpha)\theta + (2n - 1)sin(m + \alpha)\theta]d\theta \\ &= C2^{-\frac{k}{2}-1}\int_0^{\pi}(sin(m + \frac{3}{2}\theta + sin(m + \alpha) + \frac{2n - 1}{m + \alpha}[cos(m + \alpha)\pi - 1]d\theta \\ &= C2^{-\frac{k}{2}-1}\frac{1}{m + \frac{3}{2}}[cos(m + \frac{3}{2})\pi - 1] + \frac{1}{m + \alpha}[cos(m + \alpha)\pi - 1] \\ &+ \frac{2n - 1}{m + \alpha}[cos(m + \alpha)\pi - 1] \\ &= C2^{-\frac{k}{2}-1}[\frac{1}{m + \frac{3}{2}} + \frac{1}{m + \alpha} + \frac{2n - 1}{m + \alpha}] \end{aligned}$$

Therefore,

$$|c_{n,m}| \leq \frac{C2^{-\frac{k}{2}-1}(2n + 1)}{m + \alpha},$$

For $\|W_m^\alpha\|_2$

$$\begin{aligned}
\|W_m^\alpha\|_2^2 &= 2^k \int_0^1 |W_{m+\alpha}^\alpha(2^k t - 2n + 1)|^2 w^\alpha(2^k t - 2n + 1)dt \\
&\leq 2^k \int_{-1}^1 W_m^\alpha(2^k t - 2n + 1)^2 w^\alpha(2^k t - 2n + 1)dt \\
&= \frac{\pi}{2} 2^k \\
&= \pi 2^{k-1},
\end{aligned}$$

$$\begin{aligned}
E_{2^k-1,l}^2 &= \|f - s_{2^k-1,l}\|_2^2 \\
&= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m} W_{n,m}^\alpha - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m} W_{n,m}^\alpha\|_2^2 \\
&\leq \sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty} c_{n,m} W_{n,m}^\alpha\|_2^2 \\
&= \sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty} |c_{n,m}|^2 \|W_{n,m}^\alpha\|_2^2 \\
&= C^2 2^{2k-2} \sum_{n=0}^{2^k-1} (2n+1)^2 \sum_{m=l+1}^{\infty} \frac{1}{(m+\alpha)^2}
\end{aligned}$$

Since the series $\sum_{m=0}^{\infty}(\frac{1}{m+\alpha})^2$ is convergence. Then

$$lim_{l\to\infty} E_{2^k-1,l}(f) = 0.$$

□

Case 2; $0 < l < 1$
Since

$$\begin{aligned}
c_{n,m} &= < f(t), W_{n,m}^\alpha(t) >_w \\
&= 2^{\frac{k}{2}} \int_0^1 f(t) W_m^\alpha(2^k t - 2n + 1) w^\alpha(2^k t - 2n + 1)dt \\
&= 2^{\frac{k}{2}} \int_0^\pi f(\frac{cos\theta + 2n - 1}{2^k}) W_m^\alpha(cos\theta) \frac{1}{\sqrt{1 - cos^2\theta}} sin\theta d\theta \\
&\leq \alpha 2^{\frac{k}{2}} \int_0^\pi cos(m+\alpha)\theta \frac{1}{\sqrt{1 - cos^2\theta}} sin\theta d\theta \\
&\leq C 2^{-\frac{k}{2}} \int_0^\pi cos(m+\alpha)\theta d\theta \\
&= 0.
\end{aligned}$$

Therefore

$$\begin{aligned} E^2_{2^k-1,l}(f) &= \|f - s_{2^k-1,l}\|_2^2 \\ &= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}W^{\alpha}_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m}W_{n,m+\alpha}\|_2^2 \\ &= 0, \end{aligned}$$

$$lim_{l\to\infty} E_{2^k-1,l}(f) = 0.$$

Conflict of Interest and Authorship Conformation Form

o All authors have participated in (a) conception and design, or analysis and interpretation of the data; (b) drafting the article or revising it critically for important intellectual content; and (c) approval of the final version.

o This manuscript has not been submitted to, nor is under review at, another journal or other publishing venue.

o The authors have no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript

Declarations

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volume~30 of {\em Frontiers in Applied Mathematics}.
\newblock Society for Industrial and Applied Mathematics (SIAM),
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PA, 2005.

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P. Hartman and G. Stampacchia,
\newblock On some non-linear elliptic differential-functional equations.
\newblock {\em Acta Mathematica}, {\bf 115}(1):271--310, 1966.


\bibitem{toivanenkou}
J. Toivanen,
\newblock Numerical valuation of European and American options under Kou's
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jump-diffusion model.
\newblock {\em SIAM Journal on Scientific Computing}, {\bf
↪  30}(4):1949--1970, 2008.

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Conclusion

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Acknowledgments

The unnumbered section acknowledgments is usually placed before the references to thank all those who have helped in carrying out the research.

Appendix A. **Appendices**

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```
\appendix
\section{\bf This is the title of the first appendix}
\section{\bf This is the title of the second appendix}
```

Subsections, equations, theorems, figures, tables, etc. within appendices will then be automatically numbered as appropriate.

References

[1] Brandi, Ricci, P. E., *Some properties of the pseudo-Chebyshev polynomials of half-integer degree* Tbilisi Math. J. 12 (2019), no. 4, 111-121

[2] Çakmak, M. Uslu, K. A. *generalization of Chebyshev polynomials with well-known kinds and transition relations.* Acta Univ. Apulensis Math. Inform. No. 57 (2019), 19-30.

[3] Dehghan, M. Eslahchi, M. R. *Best uniform polynomial approximation of some rational functions.* Comput. Math. Appl. 59 (2010), no. 1, 382–390.

[4] Eslahchi, M. R. Dehghan, M. *The best uniform polynomial approximation to the class of the form* $\frac{1}{(a^2 \pm x^2)}$. Nonlinear Anal. 71 (2009), no. 3-4, 740-750.

[5] Kumar, L. S., Mishra, S., Awasthi, S. K., *Error bounds of a function related to generalized Lipschitz class via the pseudo-Chebyshev wavelet and its applications in the approximation of functions.* Carpathian Math. Publ. 14 (2022), no. 1, 29-48.
[6] Nigam, H. K. Mohapatra, R. N. Murari, K. *Wavelet approximation of a function using Chebyshev wavelets.* Thai J. Math. (2020), 197-208.
[7] Jesmani, S. M. Mazaheri, H. S. M. and Shojaeian, S. *Wavelet approximation with Chebyshev.* Iranian Journal of Numerical Analysis and Optimization. 14 (2024), no.1, 315-329..
[8] Mason, J. C. Handscomb, D. C. *Chebyshev polynomials.* Chapman and Hall/CRC, Boca Raton, FL, 2003.
[9] Rivlin, T. J. *An introduction to the approximation of functions.* Corrected reprint of the original. Dover Books on Advanced Mathematics. Dover Publications, Inc., New York, 1981.

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