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Research Paper

FIRST MODULE COHOMOLOGY OF TRIANGULAR BANACH ALGEBRAS ON INDUCED SEMIGROUP ALGEBRAS

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Abstract. Let S be a discrete semigroup and T be a left multiplier operator on S. The semigroup S, equipped with the new operation defined by $T, (s \circ t := sT(t))$, is called the induced semigroup and is denoted by S_T. We consider the semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$, as well as the triangular Banach algebras:

$$\mathcal{T} = \begin{bmatrix} \ell^1(S) & M_{\delta_S} \\ & \ell^1(S) \end{bmatrix} \qquad \text{and} \qquad \mathcal{T}_T = \begin{bmatrix} \ell^1(S_T) & M_{\delta_{S_T}} \\ & \ell^1(S_T) \end{bmatrix}.$$

In this paper, we show that the first module cohomology groups of these triangular Banach algebras, $\mathcal{H}^1_{\mathfrak{I}}(\mathcal{T}, \mathcal{T}^*)$ and $\mathcal{H}^1_{\mathfrak{I}_T}(\mathcal{T}_T, \mathcal{T}_T^*)$, are equal, where

$$\mathfrak{I} = \left\{ \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix}, \alpha \in \ell^1(E) \right\} \qquad , \qquad \mathfrak{I}_T = \left\{ \begin{bmatrix} \beta & \\ & \beta \end{bmatrix}, \beta \in \ell^1(E_T) \right\}.$$

Here, E and E_T denote the sets of idempotent elements in S and S_T, respectively. This result implies that, in a particular case, $\mathcal{T}$ is weakly $\mathfrak{I}$-module amenable if and only if $\mathcal{T}_T$ is weakly $\mathfrak{I}_T$-module amenable. Furthermore, M_{δ_s} and $M_{\delta_{S_T}}$ denote the canonical left modules over $\ell^1(S)$ and $\ell^1(S_T)$, respectively.

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1. **Introduction and Background**

The concept of amenability was first introduced for locally compact groups. Later, the notion of an amenable Banach algebra was defined and studied by Johnson in [5], who proved that a discrete group G is amenable if and only if its group algebra $\ell^1(G)$ is amenable. Building on the definitions of derivation and inner derivation, the concept of amenability was extended from group algebras to all Banach algebras. Subsequently, several generalizations such as the amenability of semigroups and semigroup algebras were developed.

Amini, in [1], introduced the notion of *module amenability* for Banach algebras as an extension of Johnson's amenability. He showed that for each inverse semigroup S with subsemigroup $E = E_S$ (the set of idempotent elements), the $\ell^1(E)$-module amenability of $\ell^1(S)$ is equivalent to the amenability of S, where the module action is defined as

$$\delta_e \cdot \delta_s = \delta_s, \qquad \delta_s \cdot \delta_e = \delta_{se} \qquad (e \in E,\ s \in S), \tag{1.1}$$

and δ_s, δ_e denote the point mass functions.

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Amini and Bagha, in [2], introduced the concept of *weak module amenability* for Banach algebras. They proved that for every commutative inverse semigroup S, the semigroup algebra $\ell^1(S)$, with the module action

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e = \delta_{es} \qquad (e \in E,\, s \in S), \tag{1.2}$$

is weakly $\ell^1(E)$-module amenable. Equivalently, they showed that the first module cohomology group of $\ell^1(S)$ satisfies

$$\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) = 0.$$

The study of higher-order module cohomology groups was continued by Nasrabadi and Pourabbas in [10, 11], who proved that the first and second module cohomology groups of $\ell^1(S)$ (relative to $\ell^1(E)$) are 0 and a Banach space, respectively.

Laali, in [6], used a left multiplier operator T on a Banach algebra A to define the concept of the *induced algebra* A_T and investigated its structural similarities and differences with A.

By considering the induced semigroup algebra $\ell^1(S_T)$, the present author, together with Miri and Nasrabadi, showed in [7] that if T is a bijective multiplier on a discrete semigroup S, then

$$\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \qquad \text{and} \qquad \mathcal{H}^1_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T))$$

are isomorphic. Consequently, $\ell^1(S)$ is weakly $\ell^1(E)$-module amenable if and only if $\ell^1(S_T)$ is weakly $\ell^1(E_T)$-module amenable, where E and E_T denote the subsemigroups of idempotent elements in S and S_T, respectively.

Furthermore, they proved in [8] that under the same assumptions,

$$\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \simeq \mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^\infty(S_T)),$$

and that for every odd $n \in \mathbb{N}$, the cohomology group

$$\mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^1(S_T)^{(n)})$$

is a Banach space whenever S is a commutative inverse semigroup.

In a related study, the author and Nasrabadi in [9] investigated the induced semigroup algebra $\ell^1(S_T)$ and established that, for suitable multiplier T and when S is a monoid semigroup, the first cyclic cohomology groups satisfy

$$\mathcal{HC}^1(\ell^1(S), \ell^\infty(S)) \cong \mathcal{HC}^1(\ell^1(S_T), \ell^\infty(S_T)).$$

Furthermore, they proved that if S is completely regular, then $\ell^1(S_T)$ is cyclically amenable. In the present paper, we extend these ideas by investigating the first module cohomology group and weak module amenability of triangular Banach algebras associated with the semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$. We begin by defining the triangular Banach algebra constructed from an induced semigroup and study its fundamental properties. We then analyze its algebraic and homological structure, focusing on its relationship with the corresponding triangular Banach algebra arising from the original semigroup. Finally, an illustrative example is presented to demonstrate the validity of the results and to highlight the significance of triangular Banach algebras in the broader contexts of functional analysis and abstract algebra.

We now recall some basic notions and definitions. Let A and $\mathfrak{A}$ be Banach algebras such that A is an $\mathfrak{A}$-module with compatible actions, and let X be a Banach A-bimodule and a Banach

$\mathfrak{A}$-bimodule with compatible actions (see [1]). The module X is called a *commutative* (or *bi-commutative*) Banach A–$\mathfrak{A}$-module if

$$\alpha \cdot x = x \cdot \alpha \quad (a \cdot x = x \cdot a) \qquad (\text{for all } \alpha \in \mathfrak{A},\ a \in A,\ x \in X). \tag{1.3}$$

If X is a (commutative) Banach A–$\mathfrak{A}$-module, then its dual X^* is also a (commutative) Banach A–$\mathfrak{A}$-module, where the actions of A and $\mathfrak{A}$ on X^* are defined by

$$(\alpha \cdot f)(x) = f(x \cdot \alpha), \qquad (a \cdot f)(x) = f(x \cdot a), \tag{1.4}$$

for all $\alpha \in \mathfrak{A}$, $a \in A$, $x \in X$, and similarly for the right actions. In particular, if A is a commutative Banach $\mathfrak{A}$-module, then both A and A^* are commutative Banach A–$\mathfrak{A}$-modules.

Definition 1.1. A bounded map $D : A \to X$ is called an $\mathfrak{A}$-module derivation if

$$D(a \pm b) = D(a) \pm D(b), \qquad D(ab) = D(a) \cdot b + a \cdot D(b), \tag{1.5}$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \qquad D(a \cdot \alpha) = D(a) \cdot \alpha, \tag{1.6}$$

for all $a, b \in A$ and $\alpha \in \mathfrak{A}$.

The map D is bounded if there exists a constant $M > 0$ such that

$$\|D(a)\| \leq M\|a\| \qquad (a \in A). \tag{1.7}$$

Although D need not be $\mathbb{C}$-linear, its boundedness implies norm continuity since D preserves subtraction.

Definition 1.2. Let X be a Banach A–$\mathfrak{A}$-module. The *center* of X on $\mathfrak{A}$ is defined by

$$\mathrm{Cen}_{\mathfrak{A}}X = \{x \in X : x \cdot \alpha = \alpha \cdot x,\ \forall \alpha \in \mathfrak{A}\}. \tag{1.8}$$

Each $x \in \mathrm{Cen}_{\mathfrak{A}}X$ naturally defines an inner $\mathfrak{A}$-module derivation given by

$$D(a) = \mathrm{ad}_x(a) = a \cdot x - x \cdot a \qquad (a \in A). \tag{1.9}$$

These derivations are called *inner* $\mathfrak{A}$*-module derivations.*

Moreover:

- If X is a commutative Banach A–$\mathfrak{A}$-module, then $\mathrm{Cen}_{\mathfrak{A}}X = X$.
- If X is bi-commutative, then all inner derivations are identically zero.

Definition 1.3. A Banach algebra A is called $\mathfrak{A}$*-module amenable* if, for every Banach A–$\mathfrak{A}$-module X, every $\mathfrak{A}$-module derivation $D : A \to X^*$ is inner.

Definition 1.4. A Banach algebra A is called *weakly* $\mathfrak{A}$*-module amenable* if every $\mathfrak{A}$-module derivation $D : A \to A^*$ is inner.

We denote

$$\mathcal{Z}^1_{\mathfrak{A}}(A, X) = \{D : A \to X\,;\, D \text{ is an } \mathfrak{A}\text{-module derivation}\},$$
$$\mathcal{B}^1_{\mathfrak{A}}(A, X) = \{D : A \to X\,;\, D \text{ is an inner } \mathfrak{A}\text{-module derivation}\},$$

and define the first $\mathfrak{A}$-module cohomology group by

$$\mathcal{H}^1_{\mathfrak{A}}(A, X) = \mathcal{Z}^1_{\mathfrak{A}}(A, X)/\mathcal{B}^1_{\mathfrak{A}}(A, X). \tag{1.10}$$

Hence, A is $\mathfrak{A}$-module amenable if and only if $\mathcal{H}^1_{\mathfrak{A}}(A, X^*) = 0$ for every Banach A–$\mathfrak{A}$-module X, and weakly $\mathfrak{A}$-module amenable if $\mathcal{H}^1_{\mathfrak{A}}(A, A^*) = 0$.

Definition 1.5. Let A and B be Banach algebras, and let M be a Banach A–B-module, that is, a left Banach A-module and a right Banach B-module. The associated *triangular Banach algebra* is defined by

$$\mathcal{T} = \mathrm{Tri}(A, B, M) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in A,\, b \in B,\, m \in M \right\}, \tag{1.11}$$

equipped with the usual matrix addition, multiplication

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}, \tag{1.12}$$

and the norm

$$\left\| \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right\| = \|a\|_A + \|b\|_B + \|m\|_M. \tag{1.13}$$

It is easy to verify that $\mathcal{T}$ is a Banach algebra under these operations. Since $\mathcal{T}$ is isomorphic, as a Banach space, to the ℓ^1-direct sum $A \oplus B \oplus M$, its dual space satisfies

$$\mathcal{T}^* \simeq A^* \oplus_{\ell^\infty} B^* \oplus_{\ell^\infty} M^* \simeq \begin{pmatrix} A^* & M^* \\ 0 & B^* \end{pmatrix}. \tag{1.14}$$

For $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \mathcal{T}$ and $\begin{pmatrix} \phi & \varphi \\ 0 & \psi \end{pmatrix} \in \mathcal{T}^*$, the canonical action is given by

$$\begin{pmatrix} \phi & \varphi \\ 0 & \psi \end{pmatrix} \left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right) = \phi(a) + \varphi(m) + \psi(b). \tag{1.15}$$

Forrest and Marcoux [3] studied derivations on triangular Banach algebras and, in [4], proved that $\mathcal{T}$ is weakly amenable if and only if both A and B are weakly amenable. Pourabbas and Nasrabadi [12] introduced the notion of *weak module amenability* for triangular Banach algebras and showed that $\mathcal{T}$ is weakly $\mathfrak{T}$-module amenable if and only if the corner algebras A and B are weakly $\mathfrak{A}$-module amenable, where $\mathfrak{T} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in \mathfrak{A} \right\}$.

2. **Preliminary**

Let S be a discrete semigroup. A map $T : S \to S$ is called a *left* (resp. *right*) multiplier on S if it satisfies the condition

$$T(st) = T(s)t \quad (\text{resp. } T(st) = sT(t)), \qquad \forall s, t \in S. \tag{2.1}$$

The set of all left (resp. right) multiplier maps on S is denoted by $\mathrm{Mul}_l(S)$ (resp. $\mathrm{Mul}_r(S)$). An operator T is referred to as a *multiplier map* if $T \in \mathrm{Mul}_l(S) \cap \mathrm{Mul}_r(S)$. The class of all multiplier maps on S is denoted by $\mathrm{Mul}(S)$.

We define a new operation "$\circ$" on S by

$$s \circ t := sT(t), \qquad \forall s, t \in S. \tag{2.2}$$

The semigroup S, equipped with the operation "$\circ$'', forms a new semigroup, which is called the **induced semigroup** and is denoted by S_T. Throughout this paper, we assume that S is a discrete semigroup, $T \in \mathrm{Mul}(S)$, and T is bijective.

When S is a discrete semigroup, the Banach space $\ell^1(S)$ is defined by

$$\ell^1(S) = \{f : S \to \mathbb{C} : \|f\|_1 = \sum_{x \in S} |f(x)| < \infty\}. \tag{2.3}$$

Thus, $\ell^1(S)$ consists of all complex-valued functions f on S that vanish outside at most a countable subset of S. It is clear that both $\ell^1(S)$ and $\ell^1(S_T)$ are Banach algebras with their respective convolution products defined by

$$\delta_s * \delta_t = \delta_{st}, \qquad \delta_s \circledast \delta_t = \delta_{s \circ t} = \delta_{sT(t)}, \qquad \forall s, t \in S. \tag{2.4}$$

The functions in $(\ell^1(S), *)$ and $(\ell^1(S_T), \circledast)$ are equipped with the usual ℓ^1-norm. Since T is a bijective multiplier map on S, it naturally extends to a mapping on point masses in $\ell^1(S)$, given by

$$T(\delta_s) = \delta_{T(s)}, \qquad \forall s \in S. \tag{2.5}$$

Remark 2.1. The set of all finite linear combinations of point masses $\{\delta_s : s \in S\}$ is dense in $\ell^1(S)$, i.e.

$$\overline{\operatorname{span}\{\delta_s : s \in S\}} = \ell^1(S).$$

Since module actions and derivations are continuous, we may identify point masses with elements of the semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$.

Definition 2.2. Let S be a semigroup and let $\ell^1(S)$ denote the Banach algebra of absolutely summable functions on S with convolution. The *canonical left module* associated with $\ell^1(S)$ is defined by

$$M_{\delta_S} = \operatorname{span}\{\delta_m : m \in S\}.$$

It is a left $\ell^1(S)$-module, where the module multiplication is given by

$$f.m = \sum_{t \in S} f(t) m_t, \qquad (f \in \ell^1(S),\, m \in M_{\delta_S}). \tag{2.6}$$

In particular, for a point mass function δ_s, we have

$$\delta_s * m = \sum_{t \in S} \delta_s(t) m_t = m_s, \qquad (\delta_s \in \ell^1(S),\, m \in M_{\delta_S}). \tag{2.7}$$

Thus, the dual module $M^*_{\delta_S}$ becomes a right $\ell^1(S)$-module, where the right action is defined by

$$(\varphi.\delta_s)(\delta_m) = \varphi(\delta_{sm}), \qquad (\varphi \in M^*_{\delta_S},\, \delta_m \in M_{\delta_S},\, \delta_s \in \ell^1(S)). \tag{2.8}$$

Similarly, a compatible right module action on M_{δ_S} can be defined in an analogous manner.

Remark 2.3. In general, $\ell^1(S)$ and M_{δ_S} are not necessarily equal. For example, consider $S = (\mathbb{N}, +)$ and the function $f(n) = 2^{-n}$ for $n \in \mathbb{N}$. Then $f \in \ell^1(\mathbb{N})$ since

$$\|f\|_1 = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty,$$

but $f \notin M_{\delta_{\mathbb{N}}}$, because f cannot be expressed as a finite linear combination of point mass functions.

Lemma 2.4. *Let S be a discrete semigroup and let T be a bijective multiplier map on S. Then the canonical left modules M_{δ_S} and $M_{\delta_{S_T}}$ over $\ell^1(S)$ and $\ell^1(S_T)$, respectively, satisfy*

$$T(M_{\delta_S}) = M_{\delta_{S_T}}.$$

Proof. By definition $M_{\delta_S} = \operatorname{span}\{\delta_m : m \in S\}$. Since T acts on point masses by

$$T(\delta_m) = \delta_{T(m)} \qquad (m \in S),$$

it follows that for every finite linear combination $x = \sum_{i=1}^{n} \alpha_i \delta_{m_i} \in M_{\delta_S}$ we have

$$T(x) = \sum_{i=1}^{n} \alpha_i \, \delta_{T(m_i)} \in \operatorname{span}\{\delta_n : n \in S_T\} = M_{\delta_{S_T}}.$$

Hence $T(M_{\delta_S}) \subseteq M_{\delta_{S_T}}$.

Conversely, since $T : S \to S$ is bijective, for each $n \in S_T$ there exists $m \in S$ with $T(m) = n$, and therefore $\delta_n = T(\delta_m) \in T(M_{\delta_S})$. Thus every basis element of $M_{\delta_{S_T}}$ lies in $T(M_{\delta_S})$, so $M_{\delta_{S_T}} \subseteq T(M_{\delta_S})$.

Combining the two inclusions yields $T(M_{\delta_S}) = M_{\delta_{S_T}}$. □

Lemma 2.5 ([7, Lemma 2.1])**.** *Let S be a semigroup and let $T : S \to S$ be bijective. Then:*

(i) *$T \in \mathrm{Mul}_l(S)$ if and only if $T^{-1} \in \mathrm{Mul}_l(S)$.*
(ii) *If $T \in \mathrm{Mul}_l(S)$, then $T(E_T) = E$ and $T^{-1}(E) = E_T$.*
(iii) *If $T \in \mathrm{Mul}(S)$, then for every $s, t \in S$,*

$$s \circ T(t) = T(s) \circ t, \qquad s \circ T^{-1}(t) = T^{-1}(s) \circ t.$$

Lemma 2.6 ([7, Lemma 3.1])**.** *Let S and S_T be as above and suppose $T \in \mathrm{Mul}_l(S)$. Then*

$$\mathrm{Cent}_{\ell^1(E)}\, \ell^\infty(S) = \mathrm{Cent}_{\ell^1(E_T)}\, \ell^\infty(S_T).$$

3. Module Cohomology of Triangular Banach Algebras from Induced Semigroups

In this section, we investigate the first module cohomology group of triangular Banach algebras arising from induced semigroup algebras. Let S be a discrete semigroup, and let $T \in \mathrm{Mul}(S)$ be a bijective multiplier. Considering the respective module structures, suppose that

$$\mathcal{T} = \begin{bmatrix} \ell^1(S) & M_{\delta_S} \\ & \ell^1(S) \end{bmatrix} \qquad \text{and} \qquad \mathcal{T}_T = \begin{bmatrix} \ell^1(S_T) & M_{\delta_{S_T}} \\ & \ell^1(S_T) \end{bmatrix}.$$

We prove that

$$\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^*) \simeq \mathcal{H}^1_{\mathfrak{T}_T}(\mathcal{T}_T, \mathcal{T}_T^*).$$

This result reveals a structural connection between the cohomological properties of triangular Banach algebras and those of their corresponding induced semigroup algebras. Moreover, it provides deeper insights into the behavior of module derivations and the weak module amenability of these algebras.

Lemma 3.1. *Let S, S_T, M_{δ_S}, and T be as above. Then a map $D : \mathcal{T} \to \mathcal{T}^*$ is a derivation if and only if the map $\widetilde{D} : \mathcal{T}_T \to \mathcal{T}_T^*$ defined by*

$$\widetilde{D}\left(\begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix}\right) := D\left(\begin{bmatrix} \delta_{T(x)} & \delta_{T(m)} \\ & \delta_{T(y)} \end{bmatrix}\right)$$

is also a derivation. Furthermore, D is inner if and only if $\widetilde{D}$ is inner.

Proof. First, let D be a derivation. It is clear that $\widetilde{D}$ is additive. Let $\mathbf{t}_i = \begin{bmatrix} \delta_{x_i} & \delta_{m_i} \\ & \delta_{y_i} \end{bmatrix} \in \mathcal{T}_T$ for $i \in \{1,2,3\}$. By applying Lemmas 2.4 and 2.5, along with relations (2.4) and (2.7), we compute

$$\widetilde{D}(\mathbf{t}_1) \cdot \mathbf{t}_2 + \mathbf{t}_1 \cdot \widetilde{D}(\mathbf{t}_2) = \widetilde{D}(\mathbf{t}_1 \cdot \mathbf{t}_2).$$

For this purpose, we define

$$\Delta = \widetilde{D}(\mathbf{t}_1) \cdot \mathbf{t}_2 + \mathbf{t}_1 \cdot \widetilde{D}(\mathbf{t}_2).$$

Since D is a derivation, we have

$$\begin{aligned} \Delta(\mathbf{t}_3) &= [\widetilde{D}(\mathbf{t}_1)](\mathbf{t}_2 \cdot \mathbf{t}_3) + [\widetilde{D}(\mathbf{t}_2)](\mathbf{t}_3 \cdot \mathbf{t}_1) \\ &= D\left(\begin{bmatrix} \delta_{T(x_1)} & \delta_{T(m_1)} \\ & \delta_{T(y_1)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_2 \circ x_3} & \delta_{x_2 \circ m_3} + \delta_{m_2 \circ y_3} \\ & \delta_{y_2 \circ y_3} \end{bmatrix}\right) \\ &\quad + D\left(\begin{bmatrix} \delta_{T(x_2)} & \delta_{T(m_2)} \\ & \delta_{T(y_2)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_3 \circ x_1} & \delta_{x_3 \circ m_1} + \delta_{m_3 \circ y_1} \\ & \delta_{y_3 \circ y_1} \end{bmatrix}\right) \\ &= \widetilde{D}(\mathbf{t}_1 \cdot \mathbf{t}_2)(\mathbf{t}_3). \end{aligned}$$

Thus, $\widetilde{D}$ satisfies the derivation property. The converse implication follows similarly.

Now, we show that $\widetilde{D}$ is a $\mathfrak{T}_T$-module derivation whenever D is a $\mathfrak{T}$-module derivation. Let $w = \begin{bmatrix} \delta_p & \\ & \delta_p \end{bmatrix} \in \mathfrak{T}_T$ and $t_i = \begin{bmatrix} \delta_{x_i} & \delta_{m_i} \\ & \delta_{y_i} \end{bmatrix} \in \mathcal{T}_T$. Since $T \in \mathrm{Mul}_l(S)$ and $T(p) \in E$, by Lemma 2.5(ii) we obtain

$$\begin{aligned} [w \cdot \widetilde{D}(t_1)](t_2) &= \widetilde{D}\left(\begin{bmatrix} \delta_{x_1} & \delta_{m_1} \\ & \delta_{y_1} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_2} & \delta_{m_2} \\ & \delta_{y_2} \end{bmatrix} \cdot \begin{bmatrix} \delta_p & \\ & \delta_p \end{bmatrix}\right) \\ &= \widetilde{D}\left(\begin{bmatrix} \delta_{x_1} & \delta_{m_1} \\ & \delta_{y_1} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_2 \circ p} & \delta_{m_2 \circ p} \\ & \delta_{y_2 \circ p} \end{bmatrix}\right) \\ &= \widetilde{D}(wt_1)(t_2). \end{aligned}$$

A similar argument using $T \in \mathrm{Mul}_r(S)$ shows that $\widetilde{D}(t_1) \cdot w = \widetilde{D}(t_1 \cdot w)$. Hence $\widetilde{D}$ is a $\mathfrak{T}_T$-module derivation. The converse follows by applying the same reasoning to $T^{-1} \in \mathrm{Mul}(S)$.

Finally, if D is inner, say $D = \mathbf{ad}_{\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix}}$ with $\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \in \mathrm{Cent}_{\mathfrak{T}}\, \mathcal{T}$, then

$$D\left(\begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix}\right) = \begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix} \cdot \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} - \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \cdot \begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix}.$$

By Lemmas 2.4, 2.5, and 2.6, together with (2.4) and (2.7), one obtains

$$\widetilde{D}\left(\begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix}\right) = \begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix} \cdot \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} - \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \cdot \begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix},$$

so $\widetilde{D}$ is inner as well. The converse follows similarly. □

Theorem 3.2. *Let S be a discrete semigroup, and let T be a bijective left multiplier operator on S. Then*

$$\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^*) \simeq \mathcal{H}^1_{\mathfrak{T}_T}(\mathcal{T}_T, \mathcal{T}_T^*).$$

Proof. Consider the map

$$\Gamma : \mathcal{Z}^1_{\mathfrak{X}}(\mathcal{T}, \mathcal{T}^*) \longrightarrow \mathcal{H}^1_{\mathfrak{X}_T}(\mathcal{T}_T, \mathcal{T}^*_T), \qquad \Gamma(D) = \widetilde{D} + \mathcal{B}^1_{\mathfrak{X}_T}(\mathcal{T}_T, \mathcal{T}^*_T),$$

where

$$\widetilde{D}\left(\begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix}\right) := D\left(\begin{bmatrix} \delta_{T(x)} & \delta_{T(m)} \\ & \delta_{T(y)} \end{bmatrix}\right).$$

Clearly, Γ is linear, and Lemma 3.1 shows that Γ is well-defined. For surjectivity, let $P \in \mathcal{Z}^1_{\mathfrak{X}_T}(\mathcal{T}_T, \mathcal{T}^*_T)$ and define $D : \mathcal{T} \to \mathcal{T}^*$ by

$$D\left(\begin{bmatrix} \delta_x & \delta_m \\ & \delta_y \end{bmatrix}\right) := P\left(\begin{bmatrix} \delta_{T^{-1}(x)} & \delta_{T^{-1}(m)} \\ & \delta_{T^{-1}(y)} \end{bmatrix}\right).$$

Then $\Gamma(D) = \widetilde{D} = P$. Lemma 3.1 again shows that $D \in \mathcal{Z}^1_{\mathfrak{X}}(\mathcal{T}, \mathcal{T}^*)$, so Γ is surjective. Moreover, the same lemma yields $\ker \Gamma = \mathcal{B}^1_{\mathfrak{X}}(\mathcal{T}, \mathcal{T}^*)$. Hence,

$$\mathcal{H}^1_{\mathfrak{X}}(\mathcal{T}, \mathcal{T}^*) = \frac{\mathcal{Z}^1_{\mathfrak{X}}(\mathcal{T}, \mathcal{T}^*)}{\mathcal{B}^1_{\mathfrak{X}}(\mathcal{T}, \mathcal{T}^*)} = \frac{\mathcal{Z}^1_{\mathfrak{X}}(\mathcal{T}, \mathcal{T}^*)}{\ker \Gamma} \simeq \operatorname{Im} \Gamma = \mathcal{H}^1_{\mathfrak{X}_T}(\mathcal{T}_T, \mathcal{T}^*_T).$$

□

Corollary 3.3. *A direct consequence of Theorem 3.2 is that the vanishing of the first module cohomology group is preserved under the induced semigroup transformation. In particular, the triangular Banach algebra $\mathcal{T}$ is weakly $\mathfrak{X}$-module amenable if and only if $\mathcal{T}_T$ is weakly $\mathfrak{X}_T$-module amenable.*

4. **Conclusions**

The concluding section begins with a clarifying example, effectively showcasing the concepts detailed in this article. We then emphasize the key points and inherent advantages of utilizing the proposed method.

Example 4.1. Consider the semigroup

$$S = \{i, -i, 2i, -2i, \dots, 0, \pm 1, \pm 2, \pm 3, \dots\},$$

where multiplication is defined as scalar multiplication by complex numbers. Let

$$E = \{0, 1\}$$

be the set of idempotent elements. Define the left multiplier map $T = L_i$ by

$$L_i(x) = ix, \quad \forall x \in S.$$

The induced operation on S is given by

$$a \circ b = aT(b).$$

Thus, the induced semigroup $S_T = (S, \circ)$ has the following multiplication table:

$\circ$	0	1	$-i$	i	2	3	4	$\dots$
0	0	0	0	0	0	0	0	$\dots$
1	0	i	1	-1	$2i$	$3i$	$4i$	$\dots$
$-i$	0	1	$-i$	i	2	3	4	$\dots$
i	0	-1	i	$-i$	-2	-3	-4	$\dots$
2	0	$2i$	2	-2	$4i$	$6i$	$8i$	$\dots$
3	0	$3i$	3	-3	$6i$	$9i$	$12i$	$\dots$
4	0	$4i$	4	-4	$8i$	$12i$	$16i$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$

The set of idempotent elements of S_T is $E_T = \{0, -i\}$. We now define the following triangular Banach algebras:

$$\mathcal{T} = \begin{bmatrix} \ell^1(S) & M_{\delta_S} \\ & \ell^1(S) \end{bmatrix}, \qquad \mathcal{T}_T = \begin{bmatrix} \ell^1(S_T) & M_{\delta_{S_T}} \\ & \ell^1(S_T) \end{bmatrix}.$$

To analyze the module structure on $\ell^1(S_T)$, we consider the action of $M_{\delta_{S_T}}$. This leads to a well-defined module action on $\ell^1(S_T)$, consistent with the properties of $M_{\delta_{S_T}}$.

In the triangular Banach algebra $\mathcal{T}_T$, both corner algebras are convolution algebras, given by $A = B = (\ell^1(S_T), \circledast)$, while $M_{\delta_{S_T}}$ serves as a canonical $\ell^1(S_T)$-module with compatible module actions. This example illustrates the preservation of module cohomology under the induced structure.

Remark 4.2. Example 4.1 demonstrates that in $\mathcal{T}$, the idempotent elements induce a real-linear structure, whereas in $\mathcal{T}_T$, they induce a complex-linear structure. As a result, module derivations on $\ell^1(S)$ remain real-linear due to the induced real structure, whereas those on $\ell^1(S_T)$ inherit a complex-linear nature from the induced multiplication.

This distinction highlights that, in the study of weak module amenability of certain semigroup algebras, real-linear structures often simplify the analysis by reducing computational complexity and improving conceptual clarity. Furthermore, this paper demonstrates how multiplier operators play a crucial role in shaping the algebraic behavior of idempotent elements, facilitating a more structured approach to related mathematical concepts.

It is worth emphasizing that $\ell^1(S)$ and the canonical module M_{δ_S} exhibit inherently different structural properties and functions. While $\ell^1(S)$ consists of functions from S to $\mathbb{C}$, equipped with convolution, the module M_{δ_S} consists of all multipliers that act on $\ell^1(S)$ via pointwise multiplication. This pointwise multiplication differs from the convolution structure of $\ell^1(S)$.

For example, the constant function $g(s) = 1$ for all $s \in S$ serves as a multiplier in M_{δ_S}. However, it does not correspond to any convolution element in $\ell^1(S)$, confirming that $\ell^1(S)$ and M_{δ_S} are structurally distinct.

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