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Research Paper

HOMOGENEOUS FINSLER SPACES WITH SPECIAL (α, β)-METRIC

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ABSTRACT. In this paper, we consider the special (α, β)-metric such that it is satisfying $F(\alpha, \beta) = \beta + a\alpha + \frac{\beta^2}{\alpha}$, $a \in \mathbb{R}$. We have investigate the geometric properties of this metric in homogeneous spaces. We investigate the existence of invariant vector fields. Also, we obtain the explicit formula for the S-curvature and mean Berwald curvature of homogeneous Finsler space with this (α, β)-metric. Geodesics and geodesic vectors are other topics that we study for these spaces.

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1. Introduction

Finsler geometry is just Riemannian geometry without the quadratic restriction. In Riemannian geometry, the restriction of the metric to a tangent space is an inner product and hence tangent spaces at different points are linearly isometric to each other. However, in Finsler spaces this is no longer true. Indeed, in a Finsler space, tangent spaces at different points can be very different. Matsumoto was the one who introduced for the first time an important class of Finsler metrics, which were called (α, β)-metrics [8]. These metrics are in fact a generalization of Randers metrics. In recent years, many studies have been conducted on these types of metrics due to their special importance (see [6, 7]). We note that, an (α, β)-metric is a Finsler metric F of the form

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. There are many examples of (α, β)-metrics such as Randers metric $F = \alpha + \beta$, Matsumoto metric $F = \frac{\alpha^2}{\alpha+\beta}$, square root-metric $F = \sqrt{\alpha(\alpha + \beta)}$, 3-power metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and exponential metric $F = \alpha \exp(\frac{\beta}{\alpha})$.

In this paper, we consider the special (α, β)-metric as follows

$$F(\alpha, \beta) = \beta + a\alpha + \frac{\beta^2}{\alpha}, \quad a \in \mathbb{R}. \tag{1.1}$$

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Then we study the S-curvature and E-curvature of metrics of type (1.1) and give formula for these curvatures. Also, we give some result about invariant vector fields on homogeneous Finsler spaces with (α,β)-metric of type (1.1).

2. Preliminaries

Let M be a smooth n-dimensional C^∞ manifold and TM be its tangent bundle. We note that, $TM := \cup_{x\in M} T_xM$. Therefore, a Finsler metric on manifold M is a non-negative function $F : TM \to \mathbb{R}$ such that [1]:

1) F is smooth on the slit tangent bundle $TM^0 := TM\backslash\{0\}$.
2) $F(x,\lambda y) = \lambda F(x,y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.
3) The bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow R$ is positive definite such that

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}F^2(x, y + su + tv)|_{s=t=0}. \tag{2.1}$$

The pair (M,F) is called Finsler space. Now, in the following we give an explicit definition of (α,β)-metric.

Definition 2.1. Assume that $\alpha = \sqrt{\tilde{a}_{ij}(x)y^iy^j}$ be a norm induced by a Riemannian metric $\tilde{a}$ and $\beta(x,y) = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Suppose

$$b := \|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}.$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2.2}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \leq b < b_0.$$

Then F is a Finsler metric if we have

$$\|\beta(x)\|_\alpha < b_0,$$

for any $x \in M$. A Finsler metric in the form (2.2) is called an (α,β)-metric.

Recall that, for the special (α,β)-metric of type (1.1) we have:

$$\phi(s) = s^2 + s + a, \quad a \in \mathbb{R}. \tag{2.3}$$

Here we give some basic concepts about Finsler spaces and Lie groups Which will be needed in the next sections and the continuation of the work.

Definition 2.2. Assume that G be a Lie group. Also, let M be a smooth n-dimensional manifold. Therefore a smooth map $f : G \times M \to M$ satisfying

$$f\big(g_2, f(g_1,x)\big) = f(g_2g_1, x), \quad \forall g_1, g_2 \in G, \quad x \in M,$$
$$f(e,x) = x, \quad \forall x \in M.$$

is called a smooth action of G on M.

Recall that, for an arbitrary n-dimensional smooth manifold M and a Lie group G, If G acts smoothly on M, then G is called a Lie transformation group of M. Let G be a Lie group and H, its closed subgroup. Then there exists a unique differentiable structure on the left coset space G/H with the induced topology that turns G/H into a smooth manifold such that G is a Lie transformation group of G/H. We note that, a Riemannian homogeneous space is a Riemannian manifold (M, g) on which the isometry group $I(M)$ acts transitively. In the following we have the definition of homogeneous Finsler spaces.

Definition 2.3. Let (M, F) be a connected Finsler space and $I(M, F)$ be the group of isometries of (M, F). If the action of $I(M, F)$ is transitive on M, then (M, F) is said to be a homogeneous Finsler space.

Now, assume that G be a Lie group acting transitively on a smooth manifold M. Then for $x \in M$, the isotropy subgroup G_x of G is a closed subgroup and then according to what was said G is a Lie transformation group of G/G_x. Further, G/G_x is diffeomorphic to M.

Theorem 2.4. [3] *Suppose that (M, F) be a Finsler space and $G = I(M, F)$ the group of isometries of M. Then $G = I(M, F)$ is a Lie transformation group of M. Also, let $a \in M$ and $I_a(M, F)$ be the isotropy subgroup of $I(M, F)$ at point a. Then $I_a(M, F)$ is compact.*

Let (M, F) be a homogeneous Finsler space, i.e. $G = I(M, F)$ acts transitively on M. For $a \in M$, let $H = I_a(M, F)$ be a closed isotropy subgroup of G which is compact. Then H is a Lie group itself being a closed subgroup of G. Write M as the quotient space G/H.

Definition 2.5. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. Then the direct sum decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ such that $Ad(h)(\mathfrak{m}) \subset \mathfrak{m}$, $\forall h \in H$, is called a reductive decomposition of $\mathfrak{g}$, and if such decomposition exists, then $(G/H, F)$ is called reductive homogeneous space.

Therefore, we can write any homogeneous Finsler space as a coset space of a connected Lie group with an invariant Finsler metric. Here, the Finsler metric F is viewed as G invariant Finsler metric on M.

Definition 2.6. A one-parameter subgroup of a Lie group G is a homomorphism $\xi : R \to G$, such that $\xi(0) = e$, where e is the identity of G.

Definition 2.7. Let G be a Lie group with identity element e and $\mathfrak{g}$ its Lie algebra. The exponential map $exp : \mathfrak{g} \to G$ is defined by

$$exp(tY) = \xi(t), \quad \forall \quad t \in R,$$

where $\xi : R \to G$ is unique one-parameter subgroup of G with $\dot{\xi}(0) = Y_e$.

In the case of reductive homogeneous manifold, we can identify the tangent space $T_H(G/H)$ of G/H at the origin $eH = H$ with $\mathfrak{n}$ through the map

$$Y \to \frac{d}{dt} exp(tX)H|_{t=0}, \quad Y \in \mathfrak{n},$$

since M is identified with G/H and Lie algebra of any Lie group G is viewed as T_eG.

3. Invariant vector field of homogeneous Finsler spaces with special (α,β)-metrics of type (1.1)

Let M be a n-dimensional smooth manifold. Also, let for any $x \in M$, T_xM be the tangent space at point x. Denote by $TM := \cup_{x\in M} T_xM$ the tangent bundle of M and by $T^*M := \cup_{x\in M} T^*_xM$ the cotangent bundle of M where T^*_xM is the cotangent space at point x and it is the dual space of T_xM. Then the Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T^*_xM such that $\big\langle dx^i(x), dx^j(x)\big\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T^*_xM induces a linear isomorphism between T^*_xM and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that we have $\tilde{a}\big(y, \tilde{X}(x)\big) = \beta(x,y)$. Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$. Therefore we can write (α,β)-metrics as follows:

$$F(x,y) = \alpha(x,y)\phi\Big(\frac{\tilde{a}(\tilde{X}(x),y)}{\alpha(x,y)}\Big), \tag{3.1}$$

where for any $x \in M$, $\sqrt{\tilde{a}\big(\tilde{X}(x), \tilde{X}(x)\big)} = \|\tilde{X}(x)\|_\alpha < b_0$.

Thus for (α,β)-metric $F(\alpha,\beta) = \beta + a\alpha + \frac{\beta^2}{\alpha}$, $\ a \in \mathbb{R}$ we have:

$$F(x,y) = \tilde{a}(X_x, y_x) + a\sqrt{\tilde{a}(y_x,y_x)} + \frac{\tilde{a}(X_x,y_x)^2}{\sqrt{\tilde{a}(y_x,y_x)}}, \quad a \in \mathbb{R}. \tag{3.2}$$

Now we have the following Proposition about $I(M,F)$ and $I(M,\tilde{a})$ and relation between these groups.

Proposition 3.1. *Assume that (M,F) be a Finsler space with metric of type (3.2). Also, let $I(M,F)$ be the group of isometries of (M,F) and $I(M,\tilde{a})$ be the group of isometries of Riemannian space $(M,\tilde{a})$. Then $I(M,F)$ is a closed subgroup of $I(M,\tilde{a})$.*

Proof. Suppose that $x \in M$ and $\delta : (M,F) \to (M,F)$ be an isometry. Thus,

$$F(x,Y) = F\big(\delta(x), d\delta_x(Y)\big), \quad \forall Y \in T_xM.$$

Therefore, for any $a \in \mathbb{R}$ we have:

$$\begin{aligned} &\tilde{a}(X_x,Y) + a\sqrt{\tilde{a}(Y,Y)} + \frac{\tilde{a}(X_x,Y)^2}{\sqrt{\tilde{a}(Y,Y)}} \\ &= \tilde{a}(X_{\delta(x)}, d\delta_x(Y)) + a\sqrt{\tilde{a}(d\delta_x(Y), d\delta_x(Y))} + \frac{\tilde{a}(X_{\delta(x)}, d\delta_x(Y))^2}{\sqrt{\tilde{a}(d\delta_x(Y), d\delta_x(Y))}}. \end{aligned} \tag{3.3}$$

Replacing Y by $-Y$ in (3.3) implies that

$$\begin{aligned} &-\tilde{a}(X_x,Y) + a\sqrt{\tilde{a}(Y,Y)} + \frac{\tilde{a}(X_x,Y)^2}{\sqrt{\tilde{a}(Y,Y)}} \\ &= -\tilde{a}(X_{\delta(x)}, d\delta_x(Y)) + a\sqrt{\tilde{a}(d\delta_x(Y), d\delta_x(Y))} + \frac{\tilde{a}(X_{\delta(x)}, d\delta_x(Y))^2}{\sqrt{\tilde{a}(d\delta_x(Y), d\delta_x(Y))}}. \end{aligned} \tag{3.4}$$

By subtracting relation (3.4) from relation (3.3), we get

$$\tilde{a}(X_x,Y) = \tilde{a}(X_{\delta(x)}, d\delta_x(Y)). \tag{3.5}$$

Now by adding equations (3.3) and (3.4) and use equation (3.5), we get

$$\tilde{a}(Y,Y) = \tilde{a}(d\delta_x(Y), d\delta_x(Y)).$$

Thus δ is an isometry with respect to the Riemannian metric $\tilde{a}$ and $d\delta_x(X_x) = X_{\delta(x)}$. Therefore $I(M, F)$ is a closed subgroup of $I(M, \tilde{a})$. □

By using Proposition 3.1, if (M, F) is a homogeneous Finsler space with metric of type (3.2), then the Riemannian space $(M, \tilde{a})$ is also homogeneous. Also, M can be written as a coset space G/H, where $G = I(M, F)$ is a Lie transformation group of M and H be the compact isotropy subgroup $I_x(M, F)$ of $I(M, F)$ at some point $x \in M$ [4].

Assume that $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H, respectively. If $\mathfrak{g}$ can be written as a direct sum of sub-spaces $\mathfrak{h}$ and $\mathfrak{m}$ of $\mathfrak{g}$ such that $Ad(h)\mathfrak{m} \subset \mathfrak{m}$, $\forall h \in H$, then $(G/H, F)$ is a reductive homogeneous space. Thus, homogeneous Finsler space with metric of type (3.2) can be written as a coset space of a connected Lie group with metric of type (3.2). Here, the metric of type (3.2) is viewed as G invariant Finsler metric on M.

Theorem 3.2. *Suppose that $F(\alpha, \beta) = \beta + a\alpha + \frac{\beta^2}{\alpha}$, $a \in \mathbb{R}$ be a G-invariant special (α, β)-metric on G/H and X be the vector field corresponding to 1-form β. Then α and X are G-invariant Riemannian metric and the vector field respectively.*

Proof. Assume that F be G-invariant metric on G/H, i.e.

$$F(Y) = F(Ad(h)Y), \quad \forall h \in H, \quad Y \in \mathfrak{m}.$$

By relation (3.2) for any $a \in \mathbb{R}$ we have:

$$\begin{aligned} &\tilde{a}(X, Y) + a\sqrt{\tilde{a}(Y, Y)} + \frac{\tilde{a}(X, Y)^2}{\sqrt{\tilde{a}(Y, Y)}} \\ &= \tilde{a}(X, Ad(h)Y) + a\sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)} + \frac{\tilde{a}(X, Ad(h)Y)^2}{\sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)}}. \end{aligned} \tag{3.6}$$

Replacing Y by $-Y$ in (3.6) implies that

$$\begin{aligned} &-\tilde{a}(X, Y) + a\sqrt{\tilde{a}(Y, Y)} + \frac{\tilde{a}(X, Y)^2}{\sqrt{\tilde{a}(Y, Y)}} \\ &= -\tilde{a}(X, Ad(h)Y) + a\sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)} + \frac{\tilde{a}(X, Ad(h)Y)^2}{\sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)}}. \end{aligned} \tag{3.7}$$

By subtracting relation (3.7) from relation (3.6), we have

$$\tilde{a}(X, Y) = \tilde{a}(X, Ad(h)Y). \tag{3.8}$$

By adding relations (3.6) and (3.7) and use relation (3.8), we get

$$\tilde{a}(Y, Y) = \tilde{a}(Ad(h)Y, Ad(h)Y).$$

Thus, α is a G-invariant Riemannian metric and we have

$$Ad(h)X = X,$$

which proves that X is also G-invariant. □

4. Homogeneous geodesics of homogeneous Finsler spaces with special (α, β)-metrics of type (1.1)

In this section we consider special (α, β)-metrics of type (1.1) and study homogeneous geodesic of this spaces. A Finsler space (M, F) is called a homogeneous Finsler space if the group of isometries of (M, F), $I(M, F)$ acts transitively on M. Recall that, any homogeneous Finsler manifold $M = G/H$ is a reductive homogeneous space. Let $(G/H, F)$ be a homogeneous Finsler space and e be the identity of G. A non-zero vector $X \in \mathfrak{g}$ is called a geodesic vector if the curve $exp(tX).eH$ is a geodesic of $(G/H, F)$. In [5], the second author proved the following result that gives a criterion for a non-zero vector to be a geodesic vector in a homogeneous Finsler space. The following lemma is known as the geodesic lemma.

Lemma 4.1. [5] *A non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if*

$$g_{Y_\mathfrak{m}}(Y_\mathfrak{m}, [Y, Z]_\mathfrak{m}) = 0, \quad \forall Z \in \mathfrak{g}.$$

Now in the following Theorem we study the necessary and sufficient condition for a $0 \neq Y$ in a homogeneous Finsler space with special (α, β)-metric of type (3.2) to be a geodesic vector.

Theorem 4.2. *Assume that $(G/H, F)$ be a special (α, β)-metric with*

$$F(x, y) = \tilde{a}(X_x, Y_x) + a\sqrt{\tilde{a}(Y_x, Y_x)} + \frac{\tilde{a}(X_x, Y_x)^2}{\sqrt{\tilde{a}(Y_x, Y_x)}}, \quad a \in \mathbb{R},$$

defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then, X is a geodesic vector of $(G/H, \tilde{a})$ if and only if X is a geodesic vector of $(G/H, F)$.

Proof. By using equation (2.1) and after some calculations for $a \in \mathbb{R}$ we get

$$\begin{aligned} g_Y(U, V) = {}& \tilde{a}(U, V)(q^2 + q + a)^2 + \tilde{a}(Y, U)(q^2 + q + a)(2q + 1)\Big(\frac{\tilde{a}(X, V)}{\sqrt{\tilde{a}(Y, Y)}} \\ &- \frac{\tilde{a}(X, Y)\tilde{a}(Y, V)}{(\tilde{a}(Y, Y))^{\frac{3}{2}}}\Big) + \Big((2q + 1)^2 + 2(q^2 + q + a)\Big)\Big(\frac{\tilde{a}(X, V)}{\sqrt{\tilde{a}(Y, Y)}} \\ &- \frac{\tilde{a}(X, Y)\tilde{a}(Y, V)}{(\tilde{a}(Y, Y))^{\frac{3}{2}}}\Big) \times \Big(\tilde{a}(X, U)\sqrt{\tilde{a}(Y, Y)} - \frac{\tilde{a}(Y, U)\tilde{a}(X, Y)}{\sqrt{\tilde{a}(Y, Y)}}\Big) \\ &+ \frac{(q^2 + q + a)(2q + 1)}{\sqrt{\tilde{a}(Y, Y)}}\Big(\tilde{a}(X, U)\tilde{a}(Y, V) - \tilde{a}(U, V)\tilde{a}(X, Y)\Big), \end{aligned} \tag{4.1}$$

where $q = \frac{\tilde{a}(X,Y)}{\sqrt{\tilde{a}(Y,Y)}}$. So for all $Z \in \mathfrak{g}$ we have

$$g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X, Z]_\mathfrak{m}) = \tilde{a}(X_\mathfrak{m}, [X, Z]_\mathfrak{m})\big(- q^4 - q^3 + aq + a^2 + (2q + 1)F(y)\big).$$

Therefore, $g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X, Z]_\mathfrak{m}) = 0$ if and only if

$$\tilde{a}(X_\mathfrak{m}, [X, Z]_\mathfrak{m}) = 0.$$

□

Theorem 4.3. *Assume that $(G/H, F)$ be a special (α, β)-metric with*

$$F(x, y) = \tilde{a}(X_x, Y_x) + a\sqrt{\tilde{a}(Y_x, Y_x)} + \frac{\tilde{a}(X_x, Y_x)^2}{\sqrt{\tilde{a}(Y_x, Y_x)}}, \quad a \in \mathbb{R},$$

defined by the Riemannian metric $\tilde{a}$ and the vector field X. Suppose that $Y \in \mathfrak{g} - \{0\}$ be a vector which $\tilde{a}(X, [Y, Z]_{\mathfrak{m}}) = 0$, for all $Z \in \mathfrak{g}$. Then, Y is a geodesic vector of $(G/H, F)$ if and only if Y is a geodesic vector of $(G/H, \tilde{a})$.

Proof. By using the equation (4.1) and after some computations for any $Z \in \mathfrak{g}$ we get

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}) = \tilde{a}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}})\big(- q^4 - q^3 + aq + a^2\big) \\ + \tilde{a}(X, [Y, Z]_{\mathfrak{m}})\big((2q + 1)F(Y)\big).$$

where $q = \frac{\tilde{a}(X,Y)}{\sqrt{\tilde{a}(Y,Y)}}$. This completes the proof. □

5. S-curvature of homogeneous Finsler spaces with special (α, β)-metric of type (1.1)

In this section we consider the special (α, β)-metric of type (1.1) and then we give explicit formula for S-curvature of these metrics on the homogeneous Finsler spaces. Assume that V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{b_i\}$ of V set

$$\sigma_F = \frac{Vol(B^n)}{Vol\{(y^i) \in R^n| \quad F(y^i b_i) < 1\}},$$

where Vol means the volume of a subset in the standard Euclidean space R^n and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{b_i\}$. But

$$\tau(y) = \ln \frac{\sqrt{det(g_{ij}(y))}}{\sigma_F}, \quad y \in V - \{0\},$$

is independent of the choice of basis. We call $\tau = \tau(y)$ the distortion of (V, F).

Assume that (M, F) be a Finsler space, $\tau(x, y)$ be the distortion of the Minkowski norm F_x on $T_x(M)$ and σ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Therefore the quantity

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]|_{t=0},$$

is called the S-curvature of the Finsler space (M, F).

Recall that the formula for S-curvature of an (α, β)-metric in the local coordinate system introduced in [2] as follows:

$$S = \Big(2\psi - \frac{f^{'}(b)}{bf(b)}\Big)(r_0 + s_0) - \frac{\Phi}{2\alpha\Delta^2}(r_{00} - 2\alpha Q s_0), \tag{5.1}$$

where

$$
\begin{aligned}
Q &= \frac{\phi'}{\phi - s\phi'},\\
\Delta &= 1 + sQ + (b^2 - s^2)Q',\\
\psi &= \frac{Q'}{2\Delta},\\
\Phi &= (sQ' - Q)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'',\\
r_{ij} &= \frac{1}{2}(b_{i|j} + b_{j|i}), \quad r_j = b^i r_{ij}, \quad r_0 = r_i y^i, \quad r_{00} = r_{ij} y^i y^j,\\
s_{ij} &= \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j = b^i s_{ij}, \quad s_0 = s_i y^i.
\end{aligned}
$$

Now, for special (α, β)-metric of type (1.1) with $\phi(s) = s^2 + s + a$, $a \in \mathbb{R}$, after some calculation we get

$$
\begin{aligned}
Q &= \frac{\phi'}{\phi - s\phi'} = \frac{2s+1}{a - s^2},\\
Q' &= \frac{2(s^2 + s + a)}{(a - s^2)^2},\\
Q'' &= \frac{2(2s^3 + 3s^2 + 6as + a)}{(a - s^2)^3},\\
\Delta &= 1 + sQ + (b^2 - s^2)Q'\\
&= \frac{-3s^4 - 3s^3 + a^2 + as - 2as^2 + 2b^2s^2 + 2sb^2 + 2ab^2}{(a - s^2)^2},\\
\Phi &= (sQ' - Q)(n\Delta + 1 + sQ) + (s^2 - b^2)(1 + sQ)Q''\\
&= \frac{\left\{\begin{aligned} &- s^4 - 3ns^4 - s^3 - 3ns^3 - 2nas^2 + 2ns^2b^2 + as + nas + 2nsb^2 + a^2 + na^2\\ &+ 2nab^2 + 4s^7 + 10s^6 + 6s^5 - 2a^2b^2 - 4b^2s^5 - 6b^2s^4 - 4s^4b^2 - 6s^3b^2 + 16as^5\\ &+ 20as^4 + 12a^2s^3 + 2a^2s^2 + 2as^3 - 16ab^2s^3 - 8ab^2s^2 - 12sa^2b^2 - 12as^2b^2\\ &- 2asb^2 \end{aligned}\right\}}{(a - s^2)^4}.
\end{aligned}
$$

By using below formula

$$
S(H, y) = \frac{\Phi}{2\alpha\Delta^2}\big(\langle [v, y]_{\mathfrak{m}}, y\rangle + \alpha Q \langle [v, y]_{\mathfrak{m}}, v\rangle\big), \tag{5.2}
$$

which is proven in [9] and after substituting the above values in (5.2), we get the following Theorem for S-curvature of homogeneous Finsler space with special (α, β)-metric of type (1.1).

Theorem 5.1. *Assume that $M = G/H$ be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and $F(\alpha, \beta) = \beta + a\alpha + \frac{\beta^2}{\alpha}$, $a \in \mathbb{R}$, be a*

G-invariant special (α,β)-metric on G/H. Then the S-curvature of F is given by

$$S(H,y)=\frac{A}{2B^2}\left(\frac{1}{\alpha}\langle[v,y]_{\mathfrak{m}},y\rangle+\frac{2s+1}{a-s^2}\langle[v,y]_{\mathfrak{m}},v\rangle\right), \tag{5.3}$$

where $v\in\mathfrak{m}$ corresponds to the 1-form β and $\mathfrak{m}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H and

$$\begin{aligned}A=&-s^4-3ns^4-s^3-3ns^3-2nas^2+2ns^2b^2+as+nas+2nsb^2+a^2+na^2\\&+2nab^2+4s^7+10s^6+6s^5-2a^2b^2-4b^2s^5-6b^2s^4-4s^4b^2-6s^3b^2+16as^5\\&+20as^4+12a^2s^3+2a^2s^2+2as^3-16ab^2s^3-8ab^2s^2-12sa^2b^2-12as^2b^2\\&-2asb^2,\\B=&-3s^4-3s^3+a^2+as-2as^2+2b^2s^2+2sb^2+2ab^2.\end{aligned}$$

Suppose that (M,F) be an n-dimensional Finsler space and there exists a smooth function $c(x)$ on M and a closed 1-form ω such that

$$S(x,y)=(n+1)\big(c(x)F(y)+\omega(y)\big),\quad x\in M,\quad y\in T_x(M).$$

Therefore (M,F) is said to have almost isotropic S-curvature. In addition, if ω is zero, then (M,F) is said to have isotropic S-curvature. Also, if ω is zero and $c(x)$ is constant, then we say that (M,F) has constant S-curvature.

Theorem 5.2. *Suppose that $M=G/H$ be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, and $F(\alpha,\beta)=\beta+a\alpha+\frac{\beta^2}{\alpha}$, $a\in\mathbb{R}$, be a G-invariant special (α,β)-metric on G/H. Then $(G/H,F)$ has isotropic S-curvature if and only if it has vanishing S-curvature.*

Proof. First, let G/H has isotropic S-curvature, therefore we have:

$$S(x,y)=(n+1)c(x)F(y),\ x\in G/H,\quad y\in T_x(G/H).$$

Set $x=H$ and $y=v$ in (5.3), hence we get $c(H)=0$. Consequently $S(H,y)=0$, $\forall y\in TH(G/H)$. Since F is a homogeneous metric, we have $S=0$ everywhere. The converse is obvious. □

6. Mean Berwald curvature of homogeneous Finsler spaces with special (α,β)-metric of type (1.1)

In this section we consider the special (α,β)-metric of type (1.1) and then we give explicit formula for E-curvature of these metrics on the homogeneous Finsler spaces. Assume that $E_{ij}=\frac{1}{2}\frac{\partial^2}{\partial y^i\partial y^j}(\frac{\partial G^m}{\partial y^m})(x,y)$, where G^m are spray coefficients. Then $\Xi:=E_{ij}dx^i\otimes dx^j$ is a tensor on $TM\backslash\{0\}$, which called E tensor. E tensor can also be viewed as a family of symmetric forms defined as

$$E_y:T_xM\times T_xM\to\mathbb{R},$$
$$E_y(u,v)=E_{ij}(x,y)u^iv^j,$$

where $u=u^i\frac{\partial}{\partial x^i}|_x, v=v^i\frac{\partial}{\partial x^i}|_x\in T_xM$. Therefore the collection $\{E_y:y\in TM\backslash\{0\}\}$ is called E-curvature or Mean Berwald curvature. We need some symbolization as follow. At the origin, we have $a_{ij}=\delta^i_j$, then

$$y_i = a_{ij}y^j = \delta^i_j y^j = y^i,\ \ \alpha_{y^i} = \frac{y_i}{\alpha}, \beta_{y^i} = b_i,\ \ s_{y^i} = \frac{\partial}{\partial y^i}(\frac{\beta}{\alpha}) = \frac{b_i\alpha - sy_i}{\alpha^2}, \tag{6.1}$$

$$s_{y^iy^j} = \frac{\partial}{\partial y^j}\left(\frac{b_i\alpha - sy_i}{\alpha^2}\right) = \frac{-(b_iy_j + b_jy_i)\alpha + 3sy_iy_j - \alpha^2 s\delta^i_j}{\alpha^4}.$$

In the equation (5.3) set

$$C := \frac{A}{2B^2} = \frac{\left\{\begin{array}{l} -s^4 - 3ns^4 - s^3 - 3ns^3 - 2nas^2 + 2ns^2b^2 + as + nas + 2nsb^2 + a^2 + na^2 \\ + 2nab^2 + 4s^7 + 10s^6 + 6s^5 - 2a^2b^2 - 4b^2s^5 - 6b^2s^4 - 4s^4b^2 - 6s^3b^2 + 16as^5 \\ + 20as^4 + 12a^2s^3 + 2a^2s^2 + 2as^3 - 16ab^2s^3 - 8ab^2s^2 - 12sa^2b^2 - 12as^2b^2 \\ - 2asb^2 \end{array}\right\}}{2(-3s^4 - 3s^3 + a^2 + as - 2as^2 + 2b^2s^2 + 2sb^2 + 2ab^2)^2}.$$

Therefore after some calculation we get

$$\frac{\partial C}{\partial y^j} := \frac{\left\{\begin{array}{l} (s^2+s+a)((3s^2-2b^2-a)(20s^4+24s^3+3(12a-4b^2)s^2+2(-3n-6b^2+2a-1)s \\ -12ab^2) - 12s(4s^5+6s^4+(12a-4b^2)s^3+(-3n-6b^2+2a-1)s^2-12ab^2s \\ +(2b^2+a)n-2ab^2+a)) - (2s+1)(3s^2-2b^2-a)(4s^5+6s^4+(12a-4b^2)s^3 \\ +(-3n-6b^2+2a-1)s^2-12ab^2s+(2b^2+a)n-2ab^2+a) \end{array}\right\}}{2\left(s^2+s+a\right)^2\left(3s^2-2b^2-a\right)^3} s_{y^j},$$

and then for any $a\in\mathbb{R}$ we have

$$
\begin{aligned}
\frac{\partial^2 C}{\partial y^i \partial y^j} :=& \\
&\frac{\left\{\begin{aligned}&(s^2+s+a)((3s^2-2b^2-a)(20s^4+24s^3+3(12a-4b^2)s^2+2(-3n-6b^2+2a-1)s\\ &-12ab^2)-12s(4s^5+6s^4+(12a-4b^2)s^3+(-3n-6b^2+2a-1)s^2-12ab^2s\\ &+(2b^2+a)n-2ab^2+a))-(2s+1)(3s^2-2b^2-a)(4s^5+6s^4+(12a-4b^2)s^3\\ &+(-3n-6b^2+2a-1)s^2-12ab^2s+(2b^2+a)n-2ab^2+a)\end{aligned}\right\}}{2\left(s^2+s+a\right)^2\left(3s^2-2b^2-a\right)^3} s_{y^iy^j}\\
&+\frac{\left\{\begin{aligned}&36s^{11}+162s^{10}+(-24b^2+636a+162)s^9+(-270n-156b^2+1290a-36)s^8\\ &+(-405n-96b^4+(-888a-126)b^2+768a^2+990a-135)s^7+((342b^2-72a-162)n\\ &-304b^4+(-1432a-48)b^2+1628a^2+354a-54)s^6+((450b^2+9a)n+(-96a-384)b^4\\ &+(-216a^2-978a-60)b^2+960a^3+822a^2+213a)s^5+((-144b^4+(18a+162)b^2-36a^2\\ &+81a)n+(-624a-144)b^4+(-696a^2-414a-36)b^2+996a^3+180a^2+117a)s^4+((\\ &-156b^4-12ab^2+33a^2)n+(8-32a)b^6+(-832a^2-312a+4)b^4+(-424a^3-282a^2\\ &-176a)b^2+316a^4+154a^3+127a^2)s^3+((24b^6+(48a-48)b^4+(66a^2-48a)b^2+24a^3\\ &-12a^2)n+(72a-720a^2)b^4+(-312a^3-12a^2-24a)b^2+114a^4+66a^3-12a^2)s^2\\ &+((24b^6+36ab^4+18a^2b^2+3a^3)n+(96a^2-24a)b^6+(-192a^3-24a^2+12a)b^4+(-48a^4\\ &-6a^3+12a^2)b^2+36a^5+3a^3)s+((8-8a)b^6+12ab^4+(6a^3+6a^2)b^2+2a^4+a^3)n\\ &+(32a^2-8a)b^6+(16a^3-16a^2+4a)b^4+(4a^4+2a^3+4a^2)b^2+2a^5+4a^4+a^3\end{aligned}\right\}}{(s^2+s+a)^3(3s^2-2b^2-a)^4} s_{y^i}s_{y^j}.
\end{aligned}
$$

Now by consider this calculation, we calculate the mean Berwald curvature of (α,β)-metric of type (1.1). By using the equation (5.3), we can rewrite S-curvature at the origin as $S(H,y)=\zeta+\varrho$ where

$$\zeta=\frac{C}{\alpha}\langle[v,y]_{\mathfrak{m}},y\rangle \quad \textit{and} \quad \varrho=\frac{C(2s+1)}{a-s^2}\langle[v,y]_{\mathfrak{m}},v\rangle.$$

Then, we can write

$$E_{ij}=\frac{1}{2}\frac{\partial^2 S}{\partial y^i\partial y^j}=\frac{1}{2}\left(\frac{\partial^2\zeta}{\partial y^i\partial y^j}+\frac{\partial^2\varrho}{\partial y^i\partial y^j}\right). \tag{6.2}$$

Therefore by using the equations (6.1), the terms $\frac{\partial^2\zeta}{\partial y^i\partial y^j}$ and $\frac{\partial^2\varrho}{\partial y^i\partial y^j}$ are calculated as follows:

$$
\begin{aligned}
\frac{\partial\zeta}{\partial y^j}&=\frac{\partial}{\partial y^j}\left(\frac{C}{\alpha}\langle[v,y]_{\mathfrak{m}},y\rangle\right)\\
&=\left(\frac{1}{\alpha}\frac{\partial C}{\partial y^j}-\frac{C}{\alpha^2}\frac{y_j}{\alpha}\right)\langle[v,y]_{\mathfrak{m}},y\rangle+\frac{C}{\alpha}\big(\langle[v,v_j]_{\mathfrak{m}},y\rangle+\langle[v,y]_{\mathfrak{m}},v_j\rangle\big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\zeta}{\partial y^i \partial y^j} &= \frac{\partial}{\partial y^i}\left[\left(\frac{1}{\alpha}\frac{\partial C}{\partial y^j} - \frac{C y_j}{\alpha^3}\right)\langle [v,y]_{\mathfrak{m}}, y\rangle + \frac{C}{\alpha}\big(\langle [v,v_j]_{\mathfrak{m}}, y\rangle + \langle [v,y]_{\mathfrak{m}}, v_j\rangle\big)\right] \\
&= \left(\frac{1}{\alpha}\frac{\partial^2 C}{\partial y^i \partial y^j} - \frac{y_i}{\alpha^3}\frac{\partial C}{\partial y^j} - \frac{y_j}{\alpha^3}\frac{\partial C}{\partial y^i} - \frac{C}{\alpha^3}\delta_i^j + \frac{3C}{\alpha^5} y_i y_j\right)\langle [v,y]_{\mathfrak{m}}, y\rangle \\
&\quad + \left(\frac{1}{\alpha}\frac{\partial C}{\partial y^j} - \frac{C y_j}{\alpha^3}\right)\big(\langle [v,v_i]_{\mathfrak{m}}, y\rangle + \langle [v,y]_{\mathfrak{m}}, v_i\rangle\big) \\
&\quad + \left(\frac{1}{\alpha}\frac{\partial C}{\partial y^i} - \frac{C y_i}{\alpha^3}\right)\big(\langle [v,v_j]_{\mathfrak{m}}, y\rangle + \langle [v,y]_{\mathfrak{m}}, v_j\rangle\big) \\
&\quad + \frac{C}{\alpha}\big(\langle [v,v_j]_{\mathfrak{m}}, v_i\rangle + \langle [v,v_i]_{\mathfrak{m}}, v_j\rangle\big).
\end{aligned}
$$

Also we have the following equations

$$
\begin{aligned}
\frac{\partial \varrho}{\partial y^j} &= \frac{\partial}{\partial y^j}\left(\frac{C(2s+1)}{a-s^2}\langle [v,y]_{\mathfrak{m}}, v\rangle\right) \\
&= \left[\frac{2s+1}{a-s^2}\frac{\partial C}{\partial y^j} + \frac{2C\left(s^2+s+a\right)}{\left(s^2-a\right)^2} s_{y^j}\right]\langle [v,y]_{\mathfrak{m}}, v\rangle + \frac{C(2s+1)}{a-s^2}\langle [v,v_j]_{\mathfrak{m}}, v\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \varrho}{\partial y^i \partial y^j} &= \frac{\partial}{\partial y^i}\left[\left(\frac{2s+1}{a-s^2}\frac{\partial C}{\partial y^j} + \frac{2C\left(s^2+s+a\right)}{\left(s^2-a\right)^2} s_{y^j}\right)\langle [v,y]_{\mathfrak{m}}, v\rangle + \frac{C(2s+1)}{a-s^2}\langle [v,v_j]_{\mathfrak{m}}, v\rangle\right] \\
&= \left(\frac{2s+1}{a-s^2}\frac{\partial^2 C}{\partial y^i \partial y^j} + \frac{2\left(s^2+s+a\right)}{\left(s^2-a\right)^2} s_{y^i}\frac{\partial C}{\partial y^j} + \frac{2(s^2+s+a)}{(s^2-a)^2} s_{y^j}\frac{\partial C}{\partial y^i}\right. \\
&\quad \left. - \frac{2C\left(2s^3+3s^2+6as+a\right)}{\left(s^2-a\right)^3} s_{y^i} s_{y^j} + \frac{2C(s^2+s+a)}{(s^2-a)^2} s_{y^i y^j}\right)\langle [v,y]_{\mathfrak{m}}, v\rangle \\
&\quad + \left(\frac{2s+1}{a-s^2}\frac{\partial C}{\partial y^j} + \frac{2C\left(s^2+s+a\right)}{\left(s^2-a\right)^2} s_{y^j}\right)\langle [v,v_i]_{\mathfrak{m}}, v\rangle \\
&\quad + \left(\frac{2s+1}{a-s^2}\frac{\partial C}{\partial y^i} + \frac{2C\left(s^2+s+a\right)}{\left(s^2-a\right)^2} s_{y^i}\right)\langle [v,v_j]_{\mathfrak{m}}, v\rangle.
\end{aligned}
$$

By substituting all above values in equation (6.2), we get the following Theorem.

Theorem 6.1. *Suppose that $M = G/H$ be a reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and $F(\alpha,\beta) = \beta + a\alpha + \frac{\beta^2}{\alpha}$, $\quad a \in \mathbb{R}$, be a G-invariant (α,β)-metric on G/H. Then the mean Berwald curvature of the homogeneous*

Finsler space with metric F is given by

$$
\begin{aligned}
E_{ij}(H,y) = & \left(\frac{1}{\alpha}\frac{\partial^2 C}{\partial y^i \partial y^j} - \frac{y_i}{\alpha^3}\frac{\partial C}{\partial y^j} - \frac{y_j}{\alpha^3}\frac{\partial C}{\partial y^i} - \frac{C}{\alpha^3}\delta_i^j + \frac{3C}{\alpha^5}y_i y_j\right)\frac{\langle [v,y]_{\mathfrak{m}}, y\rangle}{2} \\
& + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial C}{\partial y^j} - \frac{C y_j}{\alpha^3}\right)\left(\langle [v,v_i]_{\mathfrak{m}}, y\rangle + \langle [v,y]_{\mathfrak{m}}, v_i\rangle\right) \\
& + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial C}{\partial y^i} - \frac{C y_i}{\alpha^3}\right)\left(\langle [v,v_j]_{\mathfrak{m}}, y\rangle + \langle [v,y]_{\mathfrak{m}}, v_j\rangle\right) \\
& + \frac{C}{2\alpha}\left(\langle [v,v_j]_{\mathfrak{m}}, v_i\rangle + \langle [v,v_i]_{\mathfrak{m}}, v_j\rangle\right) \\
& + \left(\frac{2s+1}{a-s^2}\frac{\partial^2 C}{\partial y^i \partial y^j} + \frac{2\left(s^2+s+a\right)}{\left(s^2-a\right)^2}s_{y^i}\frac{\partial C}{\partial y^j} + \frac{2(s^2+s+a)}{(s^2-a)^2}s_{y^j}\frac{\partial C}{\partial y^i}\right. \\
& \left. - \frac{2C\left(2s^3+3s^2+6as+a\right)}{\left(s^2-a\right)^3}s_{y^i}s_{y^j} + \frac{2C(s^2+s+a)}{(s^2-a)^2}s_{y^i y^j}\right)\frac{\langle [v,y]_{\mathfrak{m}}, v\rangle}{2} \\
& + \left(\frac{2s+1}{a-s^2}\frac{\partial C}{\partial y^j} + \frac{2C\left(s^2+s+a\right)}{\left(s^2-a\right)^2}s_{y^j}\right)\frac{\langle [v,v_i]_{\mathfrak{m}}, v\rangle}{2} \\
& + \left(\frac{2s+1}{a-s^2}\frac{\partial C}{\partial y^i} + \frac{2C\left(s^2+s+a\right)}{\left(s^2-a\right)^2}s_{y^i}\right)\frac{\langle [v,v_j]_{\mathfrak{m}}, v\rangle}{2}.
\end{aligned} \tag{6.3}
$$

where $v \in \mathfrak{m}$ corresponds to the 1-form β and $\mathfrak{m}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

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