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Research Paper

# WEAKLY STRETCH 4-DIMENSIONAL FINSLER MANIFOLDS

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Abstract. There are some interesting non-Riemannian curvatures in Riemann-Finsler geometry. Recently, the author introduced a new non-Riemannian quantity named mean stretch curvature. Taking trace with respect to fundamental tensor in first and second variables of stretch curvature gives rise the mean stretch curvature. A Finsler metric is said to be weakly stretch metric if has vanishing mean stretch curvature. In this paper, we are going to study the mean stretch curvature of 4-dimensional Finsler manifolds. First, we find the necessary and sufficient condition under which a 4-dimensional Finsler manifold is weakly stretch. Then, we show that the main scalars of a weakly stretch metric satisfies some certain PDEs.

**MSC(2010):** 53B40, 53C60.
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## 1. Introduction

In 1926, Berwald introduced a new non-Riemannian quantity for the class of Finsler metrics called by the stretch curvature and denoted it by $\mathbf{\Sigma}_y$ which can be considered as an meaningful extension of Landsberg curvature [3]. Let us recall some Finslerian notions and curvatures. For this aim, let $(M, F)$ be a Finsler manifold. The third order derivatives of $1/2F_x^2$ at non-zero vector $y \in T_xM_0$ is called the Cartan torsion $\mathbf{C}_y$ of $F$. The rate of change of $\mathbf{C}$ along Finslerian geodesics is called the Landsberg curvature of $F$ and denoted by $\mathbf{L}$. Then, $F$ is called a Landsberg metric if it has vanishing Landsberg curvature, namely $\mathbf{L} = 0$. For a non-zero vector $y \in T_xM_0$, one can define the stretch curvature $\mathbf{\Sigma}_y : T_xM \times T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(q, u, v, w) := \Sigma_{\ ijkl}(y)q^iu^jv^kw^l$, where

$$\Sigma_{ijkl} := L_{ijk|l} - L_{ijl|k}. \tag{1.1}$$

Here " | " is the horizontal derivation with respect to the Berwald connection of Finsler metric $F$. The family $\mathbf{\Sigma} := \{\mathbf{\Sigma}_y\}_{y\in TM_0}$ is called the stretch curvature. $F$ is called a stretch metric if $\mathbf{\Sigma} = 0$. As a geometric meaning, Berwald showed that the stretch curvature of $F$ satisfies $\mathbf{\Sigma} = 0$ if and only if the length of an arbitrary vector is unchanged under the parallel displacement along an infinitesimal parallelogram. $F$ is said to be stretch metric whenever $\mathbf{\Sigma} = 0$. Then, this curvature was studied by researchers such as Shibata [7] and Matsumoto [4]. In [10], Tayebi-Tabatabaeifar showed that every Douglas-Randers metric with vanishing stretch curvature is a Berwald metric. It results that, a Douglas-Randers metric is R-quadratic if and only if it is a Berwald metric. In [8], Tayebi-Najafi proved that every homogeneous

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$(\alpha,\beta)$-metric is a stretch metric if and only if it is a Berwald metric. In [9], Tayebi-Sadeghi showed that a regular $(\alpha,\beta)$-metric of non-Randers type satisfying $\mathbf{S}=0$ is a stretch metric if and only if it is a Berwald metric. Let $F$ be an almost regular non-Randers type $(\alpha,\beta)$-metric. Suppose that $F$ is not Berwaldian. They found a family of stretch $(\alpha,\beta)$-metrics which are not Landsberg metrics [9].

In [5], Najafi-Tayebi introduced a new non-Riemannian quantity named mean stretch curvature. Taking trace with respect to $\mathbf{g}_y$ in first and second variables of $\mathbf{\Sigma}_y$ gives rise the mean stretch curvature $\bar{\mathbf{\Sigma}}_y$, namely,

$$\bar{\mathbf{\Sigma}} := trace(\mathbf{\Sigma}).$$

A Finsler metric is said to be *weakly stretch metric* if $\bar{\mathbf{\Sigma}}=0$. $F$ satisfying $\bar{\mathbf{\Sigma}}=0$ is called weakly stretch metric. The class of weakly stretch metric metrics contains the class of stretch metrics [1][2]. Najafi-Tayebi proved that every compact weakly stretch manifold with negative flag curvature reduces to a Riemannian manifold [5].

In order to study of mean stretch curvature, we focus on the class of 4-dimensional Finsler manifolds. For studying of 4-dimensional Finsler spaces, the special and useful Miron frame was founded and developed method by Radu Miron. It works under the assumption that the Finsler function is positive-definite. Then one can define a local field of orthonormal frame $(\ell,m,n,p)$ called the Miron frame. Let $(M,F)$ be a 4-dimensional Finsler manifold. We study 4-dimensional Finsler manifolds and define a local field of orthonormal frame $(\ell^i,m^i,n^i,p^i)$ called the Miron frame. This frame is locally defined in a neighborhood of each point of $M$. It is interesting to find the necessary and sufficient condition under which a 4-dimensional Finsler manifold is weakly stretch (see Lemma 3.1). Then, we prove the following.

**Theorem 1.1.** *Let $(M,F)$ be 4-dimensional Finsler manifold. Suppose that $F$ is a weakly stretch metric. Then the main scalars of $F$ satisfies following*

$$(\mathcal{A}''+\mathcal{B}''+\mathcal{C}'')=(h_0^2+j_0^2)(\mathcal{A}+\mathcal{B}+\mathcal{C}), \tag{1.2}$$

$$2(\mathcal{A}'+\mathcal{B}'+\mathcal{C}')h_0+(\mathcal{A}+\mathcal{B}+\mathcal{C})h_{0|0}-j_0k_0(\mathcal{A}+\mathcal{B}+\mathcal{C})=0, \tag{1.3}$$

$$2(\mathcal{A}'+\mathcal{B}'+\mathcal{C}')j_0+(\mathcal{A}+\mathcal{B}+\mathcal{C})j_{0|0}+h_0k_0(\mathcal{A}+\mathcal{B}+\mathcal{C})=0, \tag{1.4}$$

*where $\mathcal{A}=\mathcal{A}(x,y)$, ..., $\mathcal{H}=\mathcal{H}(x,y)$ are scalar functions on $TM$ and called the main scalars of $F$, and $\mathcal{A}':=\mathcal{A}_{|s}y^s$, ..., $\mathcal{H}':=\mathcal{H}_{|s}y^s$ denote the horizontal derivation of main scalars along Finslerian geodesics, , and $\mathcal{A}'':=\mathcal{A}'_{|s}y^s$, ..., $\mathcal{H}'':=\mathcal{H}'_{|s}y^s$, and $h_s=h_s(x,y)$, $j_s=j_s(x,y)$ and $k_s=k_s(x,y)$ are called the h-connection vectors, and $h_0=h_sy^s$, $j_0=j_sy^s$ and $k_0=k_sy^s$, $h_{0|0}:=h_{0|s}y^s$ and $j_{0|0}:=j_{0|s}y^s$.*

## 2. PRELIMINARY

Let $M$ be an $n$-dimensional $C^\infty$ manifold, $TM=\bigcup_{x\in M}T_xM$ the tangent bundle and $TM_0:=TM-\{0\}$ the slit tangent bundle. A Finsler structure on $M$ is a function $F:TM\to[0,\infty)$ with the following properties:
(i) $F$ is $C^\infty$ on $TM_0$;
(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, i.e.,

$$F(x,\lambda y)=\lambda F(x,y), \qquad \forall\lambda>0,$$

(iii) The following quadratic form $\mathbf{g}_y:T_xM\times T_xM\to\mathbb{R}$ is positively defined on $TM_0$

$$\mathbf{g}_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y+su+tv)\Big]_{s=t=0}, \quad u,v\in T_xM.$$

Then the pair $(M,F)$ is called a Finsler manifold.

Let $x\in M$ and $F_x:=F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$, one can define $\mathbf{C}_y:T_xM\times T_xM\times T_xM\to\mathbb{R}$ by

$$\mathbf{C}_y(u,v,w):=\frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0},\quad u,v,w\in T_xM.$$

The family $\mathbf{C}:=\{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}=0$ if and only if $F$ is Riemannian.

For $y\in T_xM_0$, define $\mathbf{I}_y:T_xM\to\mathbb{R}$ by

$$\mathbf{I}_y(u):=\sum_{i=1}^{n}g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_j),$$

where $\{\partial_i\}$ is a basis for $T_xM$ at $x\in M$. The family $\mathbf{I}:=\{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y)=0$ and $\mathbf{I}_{\lambda y}=\lambda^{-1}\mathbf{I}_y$, $\lambda>0$. Therefore, $\mathbf{I}_y(u):=I_i(y)u^i$, where $I_i:=g^{jk}C_{ijk}$.

The Landsberg tensor $\mathbf{L}_y:T_xM\times T_xM\times T_xM\to\mathbb{R}$ defined by $\mathbf{L}_y(u,v,w):=L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk}:=C_{ijk|s}y^s$$

$u=u^i\partial/\partial x^i|_x$, $v=v^i\partial/\partial x^i|_x$ and $w=w^i\partial/\partial x^i|_x$. The family $\mathbf{L}:=\{\mathbf{L}_y\}_{y\in TM_0}$ is said the Landsberg curvature of $F$. $F$ is called a Landsberg metric if $\mathbf{L}=0$.

The mean Landsberg curvature $\mathbf{J}_y:T_xM\to\mathbb{R}$ defined by $\mathbf{J}_y(u):=J_i(y)u^i$, where

$$J_i:=g^{jk}L_{ijk}.$$

A Finsler metric is said to be weakly Landsbergian if $\mathbf{J}=0$.

For a vector $y\in T_xM$, the Landsberg and mean Landsberg curvature of $F$ can be defined by following

$$\mathbf{L}_y(u,v,w):=\frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\Big(U(t),V(t),W(t)\Big)\Big]|_{t=0},\quad \mathbf{J}_y(u):=\frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\Big(U(t)\Big)\Big]|_{t=0},$$

where $\sigma=\sigma(t)$ is the geodesic with $\sigma(0)=x$, $\dot\sigma(0)=y$ and $U(t),V(t),W(t)$ are three linearly parallel vector fields along $\sigma$ with $U(0)=u,V(0)=v,W(0)=w$. Then the Landsberg (resp. mean Landsberg) curvature measures the rate of change of the Cartan (resp. mean Cartan) torsion along Finslerian geodesics.

For $y\in T_xM_0$, define $\bar{\mathbf{\Sigma}}_y:T_xM\times T_xM\to\mathbb{R}$ by $\bar{\mathbf{\Sigma}}_y(u,v):=\bar\Sigma_{ij}(y)u^iv^j$, where $\bar\Sigma_{ij}:=g^{kl}\Sigma_{klij}$. In local coordinate, it is defined by

$$\bar\Sigma_{ij}=2(J_{i|j}-J_{j|i}).$$

$F$ is called a weakly stretch metric if it satisfies $\bar{\mathbf{\Sigma}}=0$.

Throughout this paper, we use the Berwald connection on Finsler manifolds. The $h$- and $v$-covariant derivatives of a Finsler tensor field are denoted by “ | ” and “, ” respectively.

## 3. Proof of Theorems

In this section, we are going to prove Theorem 1.1. For this aim, let $(M, F)$ be a 4-dimensional Finsler manifold. Suppose that $\ell_i := F_{y^i}$ is the unit vector along the element of support, $m_i$ is the unit vector along mean Cartan torsion $I_i$, i.e.,

$$m_i := \frac{1}{||\mathbf{I}||} I_i,$$

where $||\mathbf{I}|| := \sqrt{g^{ij} I_i I_j}$, and $n_i$ and $p_i$ are unit vectors orthogonal to the vectors $\ell_i$ and $m_i$. Then the quadruple $(\ell_i, m_i, n_i, p_i)$ is called the Miron frame. In this frame, we have

$$g_{ij} = \ell_i \ell_j + m_i m_j + n_i n_j + p_i p_j. \tag{3.1}$$

In order to prove Theorem 1.1, we need the following.

**Lemma 3.1.** *A 4-dimensional Finsler manifold $(M, F)$ is a weakly stretch manifold if and only if its main scalars satisfy*

$$\begin{aligned} &(\mathcal{A}'_{|k} + \mathcal{B}'_{|k} + \mathcal{C}'_{|k}) m_i - (\mathcal{A}'_{|i} + \mathcal{B}'_{|i} + \mathcal{C}'_{|i}) m_k = (\mathcal{A}' + \mathcal{B}' + \mathcal{C}')(h_i n_k + j_i p_k) - (\mathcal{A}' + \mathcal{B}' \\ &+ \mathcal{C}')(h_k n_i + j_k p_i) - (\mathcal{A}_{|k} + \mathcal{B}_{|k} + \mathcal{C}_{|k})(h_0 n_i + j_0 p_i) + (\mathcal{A}_{|i} + \mathcal{B}_{|i} + \mathcal{C}_{|i})(h_0 n_k + j_0 p_k) \\ &-(\mathcal{A} + \mathcal{B} + \mathcal{C})\Big[h_{0|k} n_i - h_{0|i} n_k + j_{0|k} p_i - j_{0|i} p_k\Big] - (\mathcal{A} + \mathcal{B} + \mathcal{C})\Big[(k_k p_i - k_i p_k - h_k m_i \\ &+ h_i m_k) h_0 + (j_i m_k - j_k m_i + k_i n_k - k_k n_i) j_0\Big]. \end{aligned} \tag{3.2}$$

*Proof.* Taking a vertical derivative of (3.1) yields the Cartan torsion as follows

$$\begin{aligned} FC_{ijk} = {} & \mathcal{A} m_i m_j m_k + \mathcal{B}(m_i n_j n_k + n_i m_j n_k + n_i n_j m_k) + \mathcal{C}(m_i p_j p_k + p_i m_j p_k + p_i p_j m_k) \\ & + \mathcal{D}(m_i m_j n_k + m_i n_j m_k + n_i m_j m_k) + \mathcal{E} n_i n_j n_k + \mathcal{F}(m_i m_j p_k + m_i p_j m_k + p_i m_j m_k) \\ & + \mathcal{G}(n_i n_j p_k + n_i p_j n_k + p_i n_j n_k) + \mathcal{H}(m_i n_j p_k + m_i p_j n_k + n_i m_j p_k + n_i p_j m_k \\ & + p_i m_j n_k + p_i n_j m_k) - (\mathcal{D} + \mathcal{E})(n_i p_j p_k + p_i n_j p_k + p_i p_j n_k) - (\mathcal{F} + \mathcal{G}) p_i p_j p_k, \end{aligned} \tag{3.3}$$

where $\mathcal{A} = \mathcal{A}(x, y)$, $\mathcal{B} = \mathcal{B}(x, y)$, $\mathcal{C} = \mathcal{C}(x, y)$, $\mathcal{D} = \mathcal{D}(x, y)$, $\mathcal{E} = \mathcal{E}(x, y)$, $\mathcal{F} = \mathcal{F}(x, y)$, $\mathcal{G} = \mathcal{G}(x, y)$ and $\mathcal{H} = \mathcal{H}(x, y)$ are scalar functions on $TM$ and called the main scalars of $F$. By (3.1), we have

$$g^{ij} = \ell^i \ell^j + m^i m^j + n^i n^j + p^i p^j. \tag{3.4}$$

Contracting (3.3) with (3.4) gives us the following

$$F I_k = (\mathcal{A} + \mathcal{B} + \mathcal{C}) m_k. \tag{3.5}$$

(3.5) shows that a positive-definite 4-dimensional Finsler metric $F$ is Riemannian if and only if $\mathcal{A} + \mathcal{B} + \mathcal{C} = 0$.

The horizontal derivation of Miron frame are given by following

$$\ell_{i|j} = 0, \tag{3.6}$$

$$m_{i|s} = h_s n_i + j_s p_i, \tag{3.7}$$

$$n_{i|s} = k_s p_i - h_s m_i, \tag{3.8}$$

$$p_{i|s} = -j_s m_i - k_s n_i, \tag{3.9}$$

where $h_s = h_s(x,y)$, $j_s = j_s(x,y)$ and $k_s = k_s(x,y)$ are called the h-connection vectors (for more details, see [6]). Multiplying (3.7)-(3.9) with $y^s$ yield

$$m_{i|0} := m_{i|s}y^s = h_0 n_i + j_0 p_i, \tag{3.10}$$

$$n_{i|0} := n_{i|s}y^s = k_0 p_i - h_0 m_i, \tag{3.11}$$

$$p_{i|0} := p_{i|s}y^s = -j_0 m_i - k_0 n_i, \tag{3.12}$$

where $h_0 = h_s y^s$, $j_0 = j_s y^s$ and $k_0 = k_s y^s$. Taking a horizontal derivation of (3.5) along Finslerian geodesic and using (3.10)-(3.12) imply the following

$$FJ_i = (\mathcal{A}' + \mathcal{B}' + \mathcal{C}')m_i + (\mathcal{A} + \mathcal{B} + \mathcal{C})(h_0 n_i + j_0 p_i). \tag{3.13}$$

Taking a horizontal derivation of (3.13) yields

$$\begin{aligned} FJ_{i|k} = {} & (\mathcal{A}'_{|k} + \mathcal{B}'_{|k} + \mathcal{C}'_{|k})m_i + (\mathcal{A}' + \mathcal{B}' + \mathcal{C}')(h_k n_i + j_k p_i) + (\mathcal{A}_{|k} + \mathcal{B}_{|k} + \mathcal{C}_{|k})(h_0 n_i + j_0 p_i) \\ & +(\mathcal{A} + \mathcal{B} + \mathcal{C})(h_{0|k} n_i + j_{0|k} p_i) + (\mathcal{A} + \mathcal{B} + \mathcal{C})\Big[(k_k p_i - h_k m_i)h_0 - (j_k m_i + k_k n_i)j_0\Big]. \end{aligned} \tag{3.14}$$

By replacing $i \leftrightarrow k$ in (3.14) it follows that

$$\begin{aligned} FJ_{k|i} = {} & (\mathcal{A}'_{|i} + \mathcal{B}'_{|i} + \mathcal{C}'_{|i})m_k + (\mathcal{A}' + \mathcal{B}' + \mathcal{C}')(h_i n_k + j_i p_k) + (\mathcal{A}_{|i} + \mathcal{B}_{|i} + \mathcal{C}_{|i})(h_0 n_k + j_0 p_k) \\ & +(\mathcal{A} + \mathcal{B} + \mathcal{C})(h_{0|i} n_k + j_{0|i} p_k) + (\mathcal{A} + \mathcal{B} + \mathcal{C})\Big[(k_i p_k - h_i m_k)h_0 - (j_i m_k + k_i n_k)j_0\Big]. \end{aligned} \tag{3.15}$$

(3.14) − (3.15) implies

$$\begin{aligned} F\Sigma_{ik} = {} & (\mathcal{A}'_{|k} + \mathcal{B}'_{|k} + \mathcal{C}'_{|k})m_i - (\mathcal{A}'_{|i} + \mathcal{B}'_{|i} + \mathcal{C}'_{|i})m_k + (\mathcal{A}' + \mathcal{B}' + \mathcal{C}')(h_k n_i + j_k p_i) \\ & -(\mathcal{A}' + \mathcal{B}' + \mathcal{C}')(h_i n_k + j_i p_k) + (\mathcal{A}_{|k} + \mathcal{B}_{|k} + \mathcal{C}_{|k})(h_0 n_i + j_0 p_i) \\ & -(\mathcal{A}_{|i} + \mathcal{B}_{|i} + \mathcal{C}_{|i})(h_0 n_k + j_0 p_k) + (\mathcal{A} + \mathcal{B} + \mathcal{C})\Big[h_{0|k} n_i - h_{0|i} n_k + j_{0|k} p_i - j_{0|i} p_k\Big] \\ & +(\mathcal{A} + \mathcal{B} + \mathcal{C})\Big[(k_k p_i - k_i p_k - h_k m_i + h_i m_k)h_0 + (j_i m_k - j_k m_i + k_i n_k - k_k n_i)j_0\Big]. \end{aligned} \tag{3.16}$$

By (3.16), we get (3.2). □

**Proof of Theorem 1.1:** Contracting (3.16) in $y^k$ gives us

$$\begin{aligned} F\Sigma_{i0} = {} & (\mathcal{A}'_{|0} + \mathcal{B}'_{|0} + \mathcal{C}'_{|0})m_i + (\mathcal{A}' + \mathcal{B}' + \mathcal{C}')(h_0 n_i + j_0 p_i) + (\mathcal{A}' + \mathcal{B}' + \mathcal{C}')(h_0 n_i + j_0 p_i) \\ & +(\mathcal{A} + \mathcal{B} + \mathcal{C})(h_{0|0} n_i + j_{0|0} p_i) + (\mathcal{A} + \mathcal{B} + \mathcal{C})\Big[(k_0 p_i - h_0 m_i)h_0 - (j_0 m_i + k_0 n_i)j_0\Big]. \end{aligned} \tag{3.17}$$

where

$$\mathcal{A}' := \mathcal{A}_{|s}y^s, \quad \mathcal{B}' := \mathcal{B}_{|s}y^s, \quad \mathcal{C}' := \mathcal{C}_{|s}y^s.$$

and

$$\mathcal{A}'' := \mathcal{A}'_{|s}y^s, \quad \mathcal{B}'' := \mathcal{B}'_{|s}y^s, \quad \mathcal{C}'' := \mathcal{C}'_{|s}y^s.$$

By assumption, (3.17) reduces to

$$\begin{aligned} & (\mathcal{A}'' + \mathcal{B}'' + \mathcal{C}'')m_i + 2(\mathcal{A}' + \mathcal{B}' + \mathcal{C}')(h_0 n_i + j_0 p_i) + (\mathcal{A} + \mathcal{B} + \mathcal{C})(h_{0|0} n_i + j_{0|0} p_i) \\ & +(\mathcal{A} + \mathcal{B} + \mathcal{C})\Big[(k_0 p_i - h_0 m_i)h_0 - (j_0 m_i + k_0 n_i)j_0\Big] = 0, \end{aligned} \tag{3.18}$$

which can be written as follows

$$\begin{aligned}&\Big[(\mathcal{A}''+\mathcal{B}''+\mathcal{C}'')-(h_0^2+j_0^2)(\mathcal{A}+\mathcal{B}+\mathcal{C})\Big]m_i\\ &+\Big[2(\mathcal{A}'+\mathcal{B}'+\mathcal{C}')h_0+(\mathcal{A}+\mathcal{B}+\mathcal{C})h_{0|0}-j_0k_0(\mathcal{A}+\mathcal{B}+\mathcal{C})\Big]n_i\\ &+\Big[2(\mathcal{A}'+\mathcal{B}'+\mathcal{C}')j_0+(\mathcal{A}+\mathcal{B}+\mathcal{C})j_{0|0}+k_0h_0(\mathcal{A}+\mathcal{B}+\mathcal{C})\Big]p_i=0.\end{aligned}\tag{3.19}$$

Contracting (3.19) with $m^i$, $n^i$ and $p^i$ imply (1.2), (1.3) and (1.4), respectively. □

**Corollary 3.2.** *Let $(M,F)$ be a 4-dimensional Finsler manifold with constant main scalars. Suppose that $F$ is a weakly stretch metric. Then $F$ is a Riemannian metric or its main scalars satisfy*

$$h_0^2+j_0^2=0,\tag{3.20}$$

$$h_{0|0}-j_0k_0=0,\tag{3.21}$$

$$j_{0|0}+h_0k_0=0.\tag{3.22}$$

*Proof.* For a Finsler metric $F$ with constant main scalars, the relation (3.2) reduces to following

$$\begin{aligned}(\mathcal{A}+\mathcal{B}+\mathcal{C})\Big\{&h_{0|k}n_i-h_{0|i}n_k+j_{0|k}p_i-j_{0|i}p_k+(k_kp_i-k_ip_k-h_km_i+h_im_k)h_0\\ &+(j_im_k-j_km_i+k_in_k-k_kn_i)j_0\Big\}=0.\end{aligned}\tag{3.23}$$

If $\mathcal{A}+\mathcal{B}+\mathcal{C}=0$, then by (3.5) one can see that $F$ is Riemannian. Suppose that $F$ is not a Riemannian metric. Then, (3.23) implies that

$$\begin{aligned}&h_{0|k}n_i-h_{0|i}n_k+j_{0|k}p_i-j_{0|i}p_k+(k_kp_i-k_ip_k-h_km_i+h_im_k)h_0\\ &+(j_im_k-j_km_i+k_in_k-k_kn_i)j_0=0.\end{aligned}\tag{3.24}$$

Multiplying (3.24) with $y^k$ yields

$$(h_0^2+j_0^2)m_i-(h_{0|0}-j_0k_0)n_i-(j_{0|0}+h_0k_0)p_i=0.\tag{3.25}$$

Contracting (3.25) with $m^i$, $n^i$ and $p^i$, implies (3.20), (3.21) and (3.22), respectively. □

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