



ON THE EXISTENCE OF FIXED POINTS OF A NONLINEAR OPERATORS FOR A SYSTEM OF BOUNDARY VALUE PROBLEMS

BEHNAZ FARNAM, ROBABEH SAHANDI TOROGH\*, AND SAKINEH SAHANDI TOROGH

ABSTRACT. Recently, a lot of attention has been given to the study of multi points boundary value problems. This paper is mainly concerned with the existence of positive solutions for a class of nonlinear boundary value problems. We obtain at least one positive solution to this problem. We introduce a completely continuous operator such that, the fixed points of this operator are positive solutions of the problem. We establish some theorems to prove the existence of solutions for this system. A completely continuous operator is defined and by using the fixed point theorem in cones, the existence of solutions is proved..

**MSC(2010):** 34B15; 34B20; 47H11; 47H07.

**Keywords:** Positive solution, fixed point index, boundary value problem, Jensen's inequality.

1. Introduction

In this paper, we study the existence of positive solutions for a class of multi point boundary value problems:

$$(1.1) \quad \begin{cases} \Delta_p u = F_1(u, v) + \xi_1(t) \\ \Delta_p v = F_2(u, v) + \xi_2(t) \end{cases} \quad \begin{cases} u'(0) = 0, u(1) = \sum_{i=1}^n a_i u(\eta_i) \\ v'(0) = 0, v(1) = \sum_{i=1}^n b_i v(\eta_i) \end{cases}$$

where

$$\Delta_p x = \phi_p(x'), \phi_p(x) = |s|^{p-2} x, p > 1, \phi_q = (\phi_p)^{-1}, \frac{1}{p} + \frac{1}{q} = 1, a_i \geq 0, b_i \geq 0,$$

$$0 \leq \sum_{i=1}^n a_i < 1, 0 \leq \sum_{i=1}^n b_i < 1, 0 < \eta_1 < \eta_2 < \dots < \eta_n < \frac{1}{2}, i = 1, 2, \dots, n.$$

and  $F_i(u, v) \in C([0, +\infty) \times [0, +\infty), [0, +\infty))$ ,  $\xi_i \in C([0, 1], [0, +\infty))$ .

Recently, a lot of attention has been given to the study of multi points boundary value problems.

See, [1-9,11,12] and the references therein.

*Date:* Received: November 20, 2023, Accepted: May 10, 2024.

\*Corresponding author.

The main purpose of this paper is to establish the existence of positive solution for problem (1.1) using the fixed point theory.

In this paper, we use and extend the result of the papers [5,12].

In [10] authors, studied the existence of solution to the problem

$$\begin{aligned} x''(t) + a(t)x'(t) + b(t)x(t) + f(t, x(t), x'(t)) &= 0, \quad t \in (0, 1) \\ x(0) = 0, x(1) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \\ x'(0) = 0, x(1) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i). \end{aligned}$$

A completely continuous operator is defined and by using the fixed point theorem in cones, the existence of solutions is proved.

In [3] the authors define a cone and completely continuous operator and then by using the fixed point theorems in cone, they prove the existence of solutions to the problem :

$$\begin{aligned} (\varphi_p(x'(t)))' + q(t)f(t, x(t), x'(t)) &= 0, \quad t \in (0, 1) \\ x(0) = 0 = x(1), \\ x'(0) = 0 = x'(1). \end{aligned}$$

## 2. The preliminary lemmas

**Definitions.**[1]. Let  $E$  be a real Banach space. A nonempty convex closed set  $K \subset E$  is said to be a cone provided that

i)  $au \in K$  for all  $u \in K$  and  $a \geq 0$  and

ii)  $u, -u \in K$  implies that  $u = 0$ .

The main tool of this paper is the following theorem.

**Theorem 2.1.** ([10]). Let  $E$  be a real Banach space and  $K \subset E$  a cone. Suppose that  $\Omega \subset E$  is a bounded open set and  $T : \bar{\Omega} \cap K \rightarrow K$  is a completely continuous operator.

Let  $x_0 \in K \setminus \{0\}$

(I) If  $x - Tx \neq \eta x_0$  for  $\eta \geq 0, x \in \delta\Omega \cap K$ , then  $i(T, \Omega \cap k, k) = 0$ , where  $i$  indicates the fixed point index on  $K$ .

(II) Let  $E$  be a real Banach space and  $K$  a cone in  $E$ . Suppose that  $\Omega \subset E$  is a bounded open set with  $0 \in \Omega$  and  $T : \bar{\Omega} \cap K \rightarrow K$  is a completely continuous operator. If  $x - \eta Tx \neq \eta x_0$  for  $\eta \in [0, 1], x \in \delta\Omega \cap K$ , then  $i(T, \Omega \cap k, k) = 1$ .

Let  $P := C([0, 1], \mathbf{R})$  and

$$K := \{u \in P : u(t) \geq 0, t \in [0, 1]\}, \|u\| := \max\{|u(t)| : t \in [0, 1]\}$$

and

$$\begin{aligned} \|(u, v)\| &:= \max\{\|u\|, \|u'\|, \|v\|, \|v'\|\}, (u, v) \in P \times P, \\ B_r &= \{(u, v) \in P^2 : \|(u, v)\| < r\} \end{aligned}$$

for  $r > 0$ .

Then  $(P, \|\cdot\|)$  is a real Banach space and  $K, K^2$  are cones.

let, we define the following operators:

$$L(u, v) = (L_1(u, v)(t), L_2(u, v)(t))$$

such that

$$L_1(u, v)(t) := \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (F_1(u(r), v(r)) + \xi_1(r)) dr \right) ds$$

$$L_2(u, v)(t) := \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (F_2(u(r), v(r)) + \xi_2(r)) dr \right) ds$$

So,  $L : K^2 \rightarrow K^2$  is an completely continuous operator.

**Remark 2.2.** Suppose that  $x \in K$  is concave on  $[0,1]$ ,  $\|x\| = x(1)$ . then  $\|x\| \leq \frac{\pi^2}{4} \int_0^1 x(t) \sin \frac{\pi}{2} t dt$ .

**Lemma 2.3.** (*Jensen's Integral Inequality for Nonnegative Concave Functions*) Suppose that  $u \in C([a, b], \mathbf{R})$ ,  $\phi \in C(\mathbf{R}^+, \mathbf{R}^+)$ . If  $\phi$  is concave, then

$$\phi\left(\frac{1}{b-a} \int_a^b u(t) dt\right) \geq \frac{1}{b-a} \int_a^b \phi(u(t)) dt.$$

In particular, if  $b - a \leq 1$ , then we have

$$\left(\int_a^b u(t) dt\right)^\alpha \geq (b-a)^{\alpha-1} \int_a^b u^\alpha(t) dt \geq \int_a^b u^\alpha(t) dt$$

for  $0 < \alpha \leq 1$ .

Let  $C$  is a cone in a Banach space  $(K, \|\cdot\|)$ , the Bonsall cone spectral radius of  $T$  is defined by

$$R_C(T) := \lim_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} = \inf_{m \geq 1} \|T^m\|^{\frac{1}{m}}$$

**Lemma 2.4.** ([5]), Let  $C$  is a cone in a Banach space  $(K, \|\cdot\|)$ , and  $T : C \rightarrow C$  is a countinuous homogeneous. If

$$R_C(T) < 1, u, u_0 \in C$$

satisfy  $u \leq Tu + u_0$ , then  $u \leq (I - T)_C^{-1} u_0$ , where the Bonsall cone spectral radius of  $T$  is

$$R_C(T) := \lim_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} = \inf_{m \geq 1} \|T^m\|^{\frac{1}{m}}$$

and  $(I - T)_C^{-1}$  is the inverse operator of  $I - T$  on  $C$ .

### 3. Main Results

Let :

$H_1$ ) There are nonnegative constants  $\alpha, \beta, \gamma, \delta, c$  such that  $R_{K^2}(T_2) < 1$  and

$$(F_1(u, v) + \xi_1(t)) \leq \alpha u^{p-1} + \beta v^{p-1} + c, (F_2(u, v) + \xi_2(t)) \leq \gamma u^{p-1} + \delta v^{p-1} + c$$

for  $u, v \in [0, +\infty), t \in [0, 1]$

we define

$T_2 : K^2 \rightarrow K^2$  by  $T_2(u, v)(t) =$

$$\left( \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (\alpha u^{p-1}(r) + \beta v^{p-1}(r)) dr \right) ds, \right. \\ \left. \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (\gamma u^{p-1}(r) + \delta v^{p-1}(r)) dr \right) ds \right)$$

$H_2$ ) There are two continuous nonnegative functions  $\chi, \Upsilon$  and two constants  $\tau > \frac{\pi^4}{16}, \rho > 0$  satisfying

i)  $\chi^p$  is concave on  $[0, +\infty)$

ii)  $(F_1(u, v) + \xi_1(t)) \geq \chi(v^{p'-1}), (F_2(u, v) + \xi_2(t)) \geq \Upsilon(u^{p'-1})$

for all  $u, v \in [0, +\infty)$ ,

iii)  $\chi^p(\Upsilon^p(w)) \geq \iota'w$  for all  $w \in [0, +\infty)$

**Theorem 3.1.** *Suppose that  $H_1, H_2$  be satisfied. Then the problem (1.1) has at least one positive solution.*

*Proof.* Let

$$\mathfrak{S} := \{(u, v) \in K^2 \mid (u, v) = \eta T(u, v), 0 \leq \eta \leq 1\},$$

If  $(u_0, v_0) \in \mathfrak{S}$  so,  $\exists \eta_0 \in [0, 1]$  such that  $(u_0, v_0) = \eta_0 T(u_0, v_0)$  then,

$$u_0(t) \leq \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (F_1(u_0(r), v_0(r)) + \xi_1(r)) dr \right) ds,$$

$$v_0(t) \leq \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (F_2(u_0(r), v_0(r)) + \xi_2(r)) dr \right) ds$$

Suppose that  $\varepsilon, c_0 > 0, e \geq 0$  such that

$$(1 + \varepsilon)R_{K^2}(T_2) < 1, \varphi_p^{-1}(e + c) \leq (1 + \varepsilon)\varphi_p^{-1}(e) + c_0,$$

$H_1$  implies that,

$$u_0(t) \leq \frac{(1+\varepsilon)}{1-\sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (\alpha(u_0(r))^{p-1} + \beta(v_0(r))^{p-1}) dr \right) ds + c_0,$$

$$v_0(t) \leq \frac{(1+\varepsilon)}{1-\sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (\gamma u^{p-1}(r) + \delta v^{p-1}(r)) dr \right) ds + c_0,$$

lemma 2.4 implies that

$$(u_0, v_0) \leq (1+\varepsilon)T_2(u_0, v_0) + (c_0, c_0), (u_0, v_0) \leq (I - (1+\varepsilon)T_2)_{K^2}^{-1}(c_0, c_0).$$

Then,  $\mathfrak{S}$  is bounded. For

$$R > \sup\{\| (u, v) \| \mid (u, v) \in \mathfrak{S}\}$$

we have

$$(u, v) \neq \eta T(u, v), \forall (u, v) \in \partial B_R \cap K^2,$$

theorem 2.1 implies that

$$(3.1) \quad i(T, B_R \cap K^2, K^2) = 1$$

We define

$$\mathfrak{S}' := \{(u, v) \in \overline{B_r} \cap K^2 \mid (u, v) = T(u, v) + \eta(z_0, z_0), \eta \geq 0\}, \text{ where } z_0 = 2t - t^2,$$

Now, we suppose that  $(u_0, v_0) \in \mathfrak{S}'$  this implies that  $\eta_0 \geq 0$ ,  $(u_0, v_0) = T(u_0, v_0) + \eta(z_0, z_0)$ ,

$$u_0(t) \geq \frac{1}{1-\sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (F_1(u_0(r), v_0(r)) + \xi_1(r)) dr \right) ds,$$

$$v_0(t) \geq \frac{1}{1-\sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_p^{-1} \left( \int_s^1 (F_2(u_0(r), v_0(r)) + \xi_2(r)) dr \right) ds$$

Jensen's inequality for concave functions implies that

$$u_0^{p'-1}(t) \geq \frac{1}{1-\sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^1 (\min\{t, s\}) (F_1(u_0(r), v_0(r)) + \xi_1(r))^{p'} ds,$$

$$v_0^{p'-1}(t) \geq \frac{1}{1-\sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^1 (\min\{t, s\}) ((F_2(u_0(r), v_0(r)) + \xi_2(r))^{p'}) ds$$

$H_2$  implies that

$$(3.2) \quad u_0^{p'-1}(t) \geq \int_0^1 (\min\{t, s\}) m_1^{p'} (v_0^{p'-1}(s)) ds, \quad v_0^{p'-1}(t) \geq \int_0^1 (\min\{t, s\}) n_1^{p'} (u_0^{p'-1}(s)) ds,$$

then, we have

$$\begin{aligned} u_0^{p'-1}(t) &\geq \int_0^1 (\min\{t, s\}) m_1^{p'} \left( \int_0^1 (\min\{r, s\}) n_1^{p'} (u_0^{p'-1}(r)) dr \right) ds \\ &\geq t' \int_0^1 \int_0^1 (\min\{t, s\}) (\min\{r, s\}) u_0^{p'-1}(r) dr ds \end{aligned}$$

so,

$$\int_0^1 u_0^{p'-1}(t) \sin \frac{\pi}{2} t dt \geq \frac{16t'}{\pi^4} \int_0^1 u_0^{p'-1}(t) \sin \frac{\pi}{2} t dt, \text{ so } \int_0^1 u_0^{p'-1}(t) \sin \frac{\pi}{2} t dt = 0, \text{ then } u_0 = 0.$$

From (2) we have  $m_1^{p'} (v_0^{p'-1})(t) = 0$  so,  $v_0 = 0$ , then  $\mathfrak{S}' = 0$ . So

$$(u, v) \neq T(u, v) + \eta(z_0, z_0), \eta \geq 0.$$

By Theorem 2.1, we can obtain

$$(3.3) \quad i(T, B_R \cap K^2, K^2) = 0$$

from

$$i(T, (B_R \setminus \overline{B_r}) \cap K^2, K^2) = 1 - 0.$$

Then,  $T$  has at least one positive solution.  $\square$

**Theorem 3.2.** *Let assumptions  $H_2, H_3, H_4$  be satisfied, where:*

$H_3$ ) *There are two constants  $\alpha > \frac{\pi^4}{16}, d > 0$  and two nonnegative functions  $\chi_1, \Upsilon_1 \in C([0, +\infty), [0, +\infty))$  such that*

*i)  $\chi_1^{p_1}$  is concave on  $[0, +\infty)$*

*ii)  $(F_1(u, v) + \xi_1(t)) \geq \chi_1(v^{p_2-1}) - c, (F_2(u, v) + \xi_2(t)) \geq \Upsilon_1(u^{p_1-1}) - c$  for all  $u, v \in [0, +\infty)$ ,*

*iii)  $\chi_1^{p_1} (\Upsilon_1^{p_2}(w)) \geq \iota w - d$  for all  $w \in [0, +\infty)$*

$H_4$ ) *There is a positive constant  $l > 0$  such that for every  $u, v \in [0, l]$*

$$(F_1(u, v) + \xi_1(t)) \leq (F_1(l, l) + \xi_1(t)), (F_2(u, v) + \xi_2(t)) \leq (F_2(l, l) + \xi_2(t)),$$

$$\frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^1 \varphi_{p_1}^{-1} \left( \int_s^1 (F_1(l, l) + \xi_1(t)) dr \right) ds < l,$$

$$\frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^1 \varphi_{p_2}^{-1} \left( \int_s^1 (F_2(l, l) + \xi_2(t)) dr \right) ds < l$$

*Then the problem (1.1) has at least two positive solutions.*

*Proof.* From  $H_3, H_2$  and Theorem 3.1 we have,

$$i(T, B_R \cap K^2, K^2) = 0$$

, suppose that  $r < y < R$ . By  $H_4$  for all  $(u, v) \in \overline{B}_z \cap K^2$ , so

$$\|T_1(u, v)\| = T_1(u, v)(1) \leq \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} \left( \int_s^1 (F_1(y, y) + \xi_1(t)) dr \right) ds < y,$$

$$\|T_2(u, v)\| = T_2(u, v)(1) \leq \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1} \left( \int_s^1 (F_2(y, y) + \xi_2(t)) dr \right) ds < y,$$

then, for  $(u, v) \in \partial B_z \cap K^2$ , we conclude that

$$\|T(u, v)\| < \|(u, v)\|, (u, v) \neq \eta T(u, v), 0 \leq \eta \leq 1.$$

By Theorem 2.1 we can find,

$$(3.4) \quad i(T, B_z \cap K^2, K^2) = 1.$$

By (3),(4) we have,

$$i(T, (B_R \setminus \overline{B}_z) \cap K^2, K^2) = 0 - 1, i(T, (B_z \setminus \overline{B}_r) \cap K^2, K^2) = 1 - 0 = 1.$$

Therefore, T has at least two fixed points on  $B_R \setminus \overline{B}_z) \cap K^2$ . Then the proof is completed.  $\square$

### Acknowledgement(s) :

We would like to thank the referee(s) for his comments and suggestions on the manuscript.

### REFERENCES

- [1] Y. Wang, W. Zhao, W.Ge, Multiple positive solutions for boundary value problems of second order delay differential equations with one-dimensional p-Laplacian, J. Math. Anal. Appl. 326, 641-654, (2007).
- [2] Robabeh Sahandi Torogh, Existence of positive solutions for  $(p_1, p_2)$ Laplacian system to Dirichlet boundary conditions, International Journal of Fundamental Physical Sciences, Vol 6, No 4, 17-22, (2016).
- [3] Y. Wang, W. Ge, Triple positive solutions for two-point boundary value problems with one-dimensional p-Laplacian, Appl. Anal, 84 , 821-831,(2005).
- [4] L. Yang, X. Liu, C. Shen, Positive solutions for second-order m-point boundary value problems with nonlinearity depending on the first derivative, Electronic Journal of Differential Equations, Vol. 2006 , No. 24, 1-10, (2006).
- [5] Z. Yang, X. Wang,H. Li, Positive solutions for a system of second order quasilinear boundary value problems, Nonlinear Analysis, 195, 11749, 1-13, (2020).
- [6] C. Bai, J. Fang, Existence of multiple positive solutions for nonlinear m-point boundary value problems, Appl. Math. Comput 140 (2003) 297-305C. Bai, J. Fang, Existence of multiple positive solutions for nonlinear m-point boundary value problems, Appl. Math. Comput 140, 297-305, (2003).
- [7] Robabeh Sahandi Torogh, On the existence of solutions for P-Laplacian systems with integral Boundary conditions, International Journal of Fundamental Physical Sciences (IJFPS) 7 (4), 38-41, (2017).
- [8] R. Ma, Positive solutions for a second order three-point boundary value problems, Appl. Math. Lett. 14 1-5, (2001).
- [9] H. Lian, H. Pang, W. Ge, Triple positive solutions for boundary value problems on infinite intervals, Nonlinear Analysis 67 (1), 2199-2207, (2007).
- [10] D. Gue, V. Lakshmikantham, Nonlinear problems in abstract cones, Academic Press, Boston, (1988).

- [11] L. Yang, X. Liu, C. Shen, Positive solutions for second-order m-point boundary value problems with nonlinearity depending on the first derivative, *Electronic Journal of Differential Equations*, Vol. 2006 (2006), No. 24, 1-10.
- [12] Robabeh Sahandi Torogh, B. Farnam, Solvability for a typical system of boundary value problems by fixed point theory, *Mathematics and Computational Sciences*, Vol 1(2), 42-47, (2020).

(Behnaz Farnam) DEPARTMENT OF MATHEMATICS, QOM UNIVERSITY OF TECHNOLOGY, QOM, IRAN.

(Robabeh Sahandi Torogh) DEPARTMENT OF MATHS, VARAMIN-PISHVA BRANCH, ISLAMIC AZAD UNIVERSITY, TEHRAN, IRAN

*Email address:* [sahandi@iauvaramin.ac.ir](mailto:sahandi@iauvaramin.ac.ir)

(Sakineh Sahandi Torogh) DEPARTMENT OF EDUCATIONAL SCIENCES, KARAJ BRANCH, ISLAMIC AZAD UNIVERSITY, KARAJ, IRAN.