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Research Paper

WEAK AMENABILITY OF ULTRAPOWERS OF BANACH ALGEBRAS

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Abstract. We study when weak amenability of Banach algebras are stable under the ultrapower constructions. We extend some general results of weak amenability of Banach algebras to their ultrapowers. A Banach algebra $\mathcal{A}$ is ultra-weakly amenable, if every ultrapower of it is weakly amenable. We further investigate the relationship between ultra-weak amenability, (weak)amenability, and ultra-amenability. We provide an example of an ultra-weakly amenable Banach algebra that is not amenable and vice versa.

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1. Introduction and Background

1.1. Introduction.

The notion of an amenable Banach algebra was defined by B. E. Johnson in 1972 in [10]. One of the main results was that for a locally compact group G, the group algebra $L^1(G)$ is amenable if and only if the group G is amenable.

The concept of weak amenability was introduced by Bade, Curtis, and Dalse in [1]. Johnson introduced an equivalent definition for weak amenability. According to his definition, a Banach algebra $\mathcal{A}$ is weakly amenable if every continuous derivation from $\mathcal{A}$ to $\mathcal{A}'$, the dual space of $\mathcal{A}$, is inner.

The general ultrapower construction was studied with Los's theorem in model theory, in 1955, and was applied to analysis by A. Robinson, in 1966. Heinrich in [Heinrich, 1980], used ultrapowers as a convenient device for the localization of the infinite dimensional properties (P): Banach space E has local (P) if and only if every ultrapower of E has (P).

M. Daws in [5], studied this notion for localization of amenability property. Specifically, a Banach algebra $\mathcal{A}$ is considered ultra-amenable if, for every ultrafilter $\mathcal{U}$, the ultrapower $(\mathcal{A})_{\mathcal{U}}$ exhibits amenability. Furthermore, Daws established that ultra-amenability significantly surpasses regular amenability in strength. Notably, in [5], he illustrated that for numerous locally compact groups G, the ultra-amenability of the associated group algebra $L^1(G)$ is equivalent to the finite nature of the group G. In our investigation, we will draw upon the valuable insights offered in the survey articles referenced as [5] and [6].

In this paper, we define ultra-weak amenability of Banach algebras and study the localization of weak amenability property. We show that every C^*-algebra is ultra-weakly amenable. Hence, ultra-amenability is strictly stronger than ultra-weak amenability. We also study some hereditary properties of ultra-weakly amenable Banach algebras. We show that ultra-weak

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amenability is strictly stronger than weak amenability. It follows that there is no relationship between ultra-weak amenability and amenability in general.
Finally, we show that for many abelian locally compact groups G, ultra-weak amenability of $L^1(G)$ is equivalent to G being finite.

1.2. **Background.**

A filter on a set X is a subset F of $P(X)$ such that: (i) empty set is not in F; (ii) F closed under finite intersection; (iii) F is an upper set. We can partially order the collection of filters on a set by inclusion. Maximal filters exist and are called ultrafilters. If $x \in X$ then the collection $U_x = \{A \subseteq X : x \in A\}$ is an ultrafilter, this ultrafilter is the principal ultrafilter at x. Ultrafilters which are not principal are called non-principal.

We want to study the concept of a limit along a filter. If F is a filter on a set $\mathbb{I}$ and $(x_i)_{i\in\mathbb{I}}$ is a family in a topological space, we write $x = \lim_{i\in F} x_i$ if for each open neighborhood U of x, we have $\{i \in \mathbb{I} : x_i \in U\} \in F$.

Now, we refer to the following proposition of [6] regarding the limit under an ultrafilter.

Proposition 1.1. *Let X be a compact topological space, let $\mathcal{U}$ be an ultrafilter on a set $\mathbb{I}$, and let $(x_i)_{i\in\mathbb{I}}$ be a family in X. Then there exists $x \in X$ with $x = \lim_{i\in\mathcal{U}} x_i$. Furthermore, if X is Hausdorff, then x is unique.*

Let $\mathcal{U}$ be a non-principal ultrafilter on a set $\mathbb{I}$, and E be a Banach space; we form the Banach space

$$l^\infty(\mathrm{E}, \mathbb{I}) = \{(x_i)_{i\in\mathbb{I}} \subseteq \mathrm{E} \quad : \quad \|(x_i)\| := sup_{i\in\mathbb{I}}\|x_i\| < \infty\},$$

and define the closed subspace

$$N_{\mathcal{U}} = \{(x_i)_{i\in\mathbb{I}} \in l^\infty(\mathrm{E}, \mathbb{I}) \quad : \quad \lim_{i\to\mathcal{U}} \|x_i\| = 0\}.$$

Definition 1.2. Let $\mathcal{U}$ be a non-principal ultrafilter on a set $\mathbb{I}$ and E be a Banach space; the quotient space $\frac{l^\infty(\mathrm{E},\mathbb{I})}{N_{\mathcal{U}}}$ is called ultrapower of E with respect to $\mathcal{U}$, and denoted by $(\mathrm{E})_{\mathcal{U}}$:

$$(E)_{\mathcal{U}} := \frac{l^\infty(\mathrm{E}, \mathbb{I})}{N_{\mathcal{U}}}.$$

In this paper, we write (x_i) for the equivalence class it represents.
We can verify that, if $(x_i) \in (\mathrm{E})_{\mathcal{U}}$, then

$$\|(x_i)\| = \lim_{i\to\mathcal{U}} \|x_i\|.$$

Since the map $x \to (x)$ from E into $(E)_{\mathcal{U}}$ is a canonical isometry, where (x) is a constant family in $(E)_{\mathcal{U}}$, so we can consider E, as a closed subspace of $(E)_{\mathcal{U}}$.

Example 1.3. Let E be a finite dimensional Banach space and $\mathcal{U}$ be an ultrafilter, then by [6, Proposition 3.1.11], $(\mathrm{E})_{\mathcal{U}} = \mathrm{E}$.

Let $\mathcal{A}$ be a Banach algebra, then $(\mathcal{A})_{\mathcal{U}}$ becomes a Banach algebra under the pointwise product, and $N_{\mathcal{U}}$ is a closed ideal of it. Thus the quotient space $\frac{l^\infty(\mathrm{E},\mathbb{I})}{N_{\mathcal{U}}}$, that is $(\mathcal{A})_{\mathcal{U}}$, is Banach algebra.

For a Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, E' the dual space of E, is a Banach $\mathcal{A}$-bimodule under the following action:

$$< a \cdot m, x > = < m, x \cdot a >, \quad < m \cdot a, x > = < m, a \cdot x > \qquad (a \in \mathcal{A}, x \in E, m \in E').$$

A derivation $D : \mathcal{A} \to E$, is a (bounded) linear map such that

$$D(ab) = D(a) \cdot b + a \cdot D(b), \qquad (a, b \in \mathcal{A}).$$

The derivation D is inner if it is of the form $a \to a \cdot x - x \cdot a$ for some $x \in E$.

A Banach algebra $\mathcal{A}$ is said to be amenable if every derivation from $\mathcal{A}$ to a dual Banach $\mathcal{A}$-bimodule is inner. For example Haagroup and Lautsten in [], show that a C^*-algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is nuclear.

Moreover, $\mathcal{A}$ is weakly amenable if every derivation from $\mathcal{A}$ to $\mathcal{A}'$ is inner.

The notion of an ultra-amenability was introduced by Daws, that is, a Banach algebra $\mathcal{A}$ is ultra-amenable if every ultrapowers of $\mathcal{A}$ is amenable. For example, let $\mathcal{A}$ be an unital and commutative C*-algebra, $\mathcal{A}$ is ultra-amenable.

Remark 1.4. Ultra-amenability is strictly stronger than amenability, for example let G be an infinite compact group, then $L^1(G)$ is amenable, but by [5, Theorem 5.9], is not ultra-amenable.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{U}$ an ultrafilter, ultrapower $(\mathcal{A})_{\mathcal{U}}$ is unital if and only if $\mathcal{A}$ is unital, by [5, Proposition 2.1]. Also, it is easy to see that $\mathcal{A}$ is commutative if and only if $(\mathcal{A})_{\mathcal{U}}$ is commutative.

2. **Ultra-weak amenability**

Now, We introduce a weak notion of ultra-amenability of Banach algebras.

Definition 2.1. Banach algebra $\mathcal{A}$ is called to be ultra-weakly amenable if provided that every ultrapowers of $\mathcal{A}$ is weakly amenable.

Example 2.2. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{U}$ be an ultrafilter, then $(\mathcal{A})_{\mathcal{U}}$ is a C^*-algebra again, [9, Proposition 3.1]. Since every C^*-algebra is weakly amenable [3, Theorem 2.1], every C^*-algebra is ultra-weakly amenable.

It is easy to check that every ultra-amenable Banach algebra is ultra-weakly amenable. Hence, ultra-amenability is stronger than ultra-weak amenability.

Let $\mathcal{A}$ be a non nuclear C^*-algebra, then $\mathcal{A}$ is ultra-weakly amenable, but it is not amenable. For example $B(H)$, the space of all bounded linear operator on Hilbert space H (when H is infinite dimensional), is not a nuclear C^*-algebra, so it is not amenable. But it is ultra-weakly amenable.

As a consequences of the above descriptions, ultra-amenability is strictly stronger than ultra-weak amenability; and also ultra-weak amenability is not stronger than amenability.

Remark 2.3. Every ultra-weakly amenable Banach algebra is weakly amenable. So ultra-weak amenability is stronger than weak amenability.

In section 2.2, we give an example of ultra-weakly amenable Banach algebra which is not amenable.

2.1. **Ultra-weak amenability and hereditary properties.**

Theorem 2.4. *Let $\mathcal{A}$ be a commutative Banach algebra which is ultra-weakly amenable, and I be a closed ideal in $\mathcal{A}$. Then $\frac{\mathcal{A}}{I}$ is ultra-weakly amenable.*

Proof. Let $\mathcal{U}$ be an ultrafilter.
$\mathcal{A}$ is ultra-weakly amenable, so $(\mathcal{A})_{\mathcal{U}}$ is weakly amenable. Since $(\pi)_{\mathcal{U}} : (\mathcal{A})_{\mathcal{U}} \to (\frac{\mathcal{A}}{I})_{\mathcal{U}}$ is a quotient map, $(\pi)_{\mathcal{U}}(\mathcal{A})_{\mathcal{U}} = (\frac{\mathcal{A}}{I})_{\mathcal{U}}$. By [7, Proposition 2.1], $(\frac{\mathcal{A}}{I})_{\mathcal{U}}$ is weakly amenable, thus $\frac{\mathcal{A}}{I}$ is ultra-weakly amenable. □

Theorem 2.5. *If I and $\frac{\mathcal{A}}{I}$ are ultra-weakly amenable Banach algebras, then so is $\mathcal{A}$.*

Proof. Let $\mathcal{U}$ be an ultrafilter.
I and $\frac{\mathcal{A}}{I}$ are ultra-weakly amenable, so $(I)_{\mathcal{U}}$ and $(\frac{\mathcal{A}}{I})_{\mathcal{U}}$ are weakly amenable. Moreover, $(I)_{\mathcal{U}}$ is a closed ideal in $(\mathcal{A})_{\mathcal{U}}$, and $(\frac{\mathcal{A}}{I})_{\mathcal{U}} \cong \frac{(\mathcal{A})_{\mathcal{U}}}{(I)_{\mathcal{U}}}$, [9, Section 5.3]. Since $\frac{(\mathcal{A})_{\mathcal{U}}}{(I)_{\mathcal{U}}}$ is weakly amenable [2, Proposition 2.8.66], $(\mathcal{A})_{\mathcal{U}}$ is weakly amenable, and hence $\mathcal{A}$ is ultra-weakly amenable. □

Proposition 2.6. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A} = B \oplus I$, when B is a closed subalgebra and I is a closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is ultra-weakly amenable, then so is B.*

Proof. Let $\mathcal{U}$ be an ultrafilter.
$(B)_{\mathcal{U}}$ is a closed subalgebra, and $(I)_{\mathcal{U}}$ is an closed ideal of $(\mathcal{A})_{\mathcal{U}}$. $\mathcal{A}$ is ultra-weakly amenable, so $(\mathcal{A})_{\mathcal{U}} = (B \oplus I)_{\mathcal{U}}$ is weak amenable. It is easy to see that $(B \oplus I)_{\mathcal{U}} = (B)_{\mathcal{U}} \oplus (I)_{\mathcal{U}}$. Therefor, $(B)_{\mathcal{U}}$ is weakly amenable [12, Lemma 2.3]. □

Theorem 2.7. *Let $\mathcal{A}$ be a ultra-weakly amenable and commutative Banach algebra. Suppose that I is a closed ideal in $\mathcal{A}$, and $\mathcal{U}$ be an ultrafilter. Then:*

(i) *$(I)_{\mathcal{U}}$ is ultra-weakly amenable if and only if $\bar{I}^2 = I$;*
(ii) *If I has finite codimension in A, then $(I)_{\mathcal{U}}$ is weakly amenable.*

Proof. (i) It is easy to see that if $\bar{I}^2 = I$, then $\bar{(I)_{\mathcal{U}}}^2 = (I)_{\mathcal{U}}$. Hence, the proof concludes by[2, Theorem 2.8.69 (i)].
(ii) Since I has finite codimension in A, $(I)_{\mathcal{U}}$ has finite codimension in $(A)_{\mathcal{U}}$. Therefor, weak amenability of $(I)_{\mathcal{U}}$ follows from [2, Theorem 2.8.69(ii)].

□

2.2. **Ultra-weak amenability of group algebra $L^1(G)$.**

Let G be a locally compact group, and form the Banach algebra $L^1(G)$, for example and details of this Banach algebra, see [2, Section 3.3]. We want to utilize the concepts of the almost periodic functions, Bohr compactification for a group G, and maximally almost periodic groups.

Definition 2.8. Let $\mathcal{A}$ be a Banach algebra, and for $\mu \in \mathcal{A}'$,

$$L_\mu : \mathcal{A} \to \mathcal{A}'; \quad a \to a.\mu \quad (a \in \mathcal{A})$$

be a compact operator, then μ is almost periodic functional, and denoted by
$\mu \in AP(\mathcal{A}')$. $AP(\mathcal{A}')$ is a closed submodule of $\mathcal{A}$.

Let $\mathcal{A}$ be a Banach algebra, and $\mathcal{U}$ be a non-principal ultrafilter. Define:

$$\sigma_{AP} : (\mathcal{A})_{\mathcal{U}} \to AP(\mathcal{A}')', \quad < \sigma_{AP}((a_i)), \mu >= \lim_{i \to \mathcal{U}} < \mu, a_i >$$

for all $\mu \in AP(\mathcal{A}')$ and $(a_i) \in (\mathcal{A})_{\mathcal{U}}$. Then σ_{AP} is norm-decreasing, algebra homomorphism, and for suitable $\mathcal{U}$, it is surjection [5, Section 5.3].

Proposition 2.9. *Let $\mathcal{A}$ be a commutative ultra-weakly amenable Banach algebra, then $AP(\mathcal{A}')'$ is weakly amenable.*

Proof. For a suitable $\mathcal{U}$, σ_{AP} is surjection, and $\frac{(\mathcal{A})_{\mathcal{U}}}{\ker \sigma_{AP}} \cong AP(\mathcal{A}')'$. Banach algebra $(\mathcal{A})_{\mathcal{U}}$ is weakly amenable and $\ker \sigma_{AP}$ is a closed ideal in $(\mathcal{A})_{\mathcal{U}}$. Since $\frac{(\mathcal{A})_{\mathcal{U}}}{\ker \sigma_{AP}}$ is weak amenable, $AP(\mathcal{A}')'$ is weak amenable. □

Let $\mathcal{A} = L^1(G)$, we write $AP(G)$ for $AP(\mathcal{A}')$. By [11, Lemma 2.10.2], $AP(G)$ is a unital closed $*$-subalgebra of $L^\infty(G)$; so by Gelfand representation theorem, $AP(G) \cong C(G^{AP})$, where G^{AP} is characters space of $AP(G)$.

By the Riesz representation theorem, it can be stated that $AP(G)' = M(G^{AP})$, signifying the space of radon bounded measures on G^{AP}. It is worth noting that the Bohr compactification map from G to G^{AP} might not always be injective. However, in cases where it is injective, or equivalently, when $AP(G)$ distinguishes the points of G, the term "maximally almost periodic" is employed. Notably, all compact groups and abelian locally compact groups are inherently maximally almost periodic due to the fact that the characters of G effectively differentiate the points of G.

Theorem 2.10. *Let G be an infinite abelian locally compact group, then $L^1(G)$ is amenable (in particular is weak amenable), but not ultra-weak amenable.*

Proof. It is clearly that for such group G, $L^1(G)$ is amenable. Suppose that $L^1(G)$ is ultra-weak amenable, then by the above proposition, $\mathcal{M}(G^{AP})$ is weak amenable. Therefor, G^{AP} is discrete [1]. Since G^{AP} is compact, it is finite. However, by G is maximally almost periodic, and so we see that G is also finite, a contradiction. □

Remark 2.11. For every infinite abelian maximally almost periodic group G, $L^1(G)$ is not ultra-weak amenable. Because the Bohr compactification map $G \to G^{AP}$ is injective.

Corollary 2.12. *Ultra-weak amenability is strictly stronger than weak amenability. And amenability is not stronger than ultra-weak amenability.*

Corollary 2.13. *Let G be an abelian discrete group, then $l^1(G)$ is ultra-weak amenable if and only if G is finite.*

Let G be a discrete group, and consider the Banach algebra $l^1(G)$. Let $\mathcal{U}$ be an ultrafilter on a set $\mathbb{I}$, so we can form the ultrapower $(l^1(G))_{\mathcal{U}}$.We form the ultrapower of G, denoted by $(G)_{\mathcal{U}}$. This is the set of all families $(g_i)_{i \in I}$ of elements of G, quotiented by the equivalence relation

$$(g_i) \sim (h_i) \quad \Leftrightarrow \quad \{i \in \mathbb{I} : g_i = h_i\} \in \mathcal{U}.$$

Then $(G)_{\mathcal{U}}$ becomes a (discrete) group for the pointwise product, and we have a canonical map $G \to (G)_{\mathcal{U}}$ formed by sending $g \in G$ to the constant family (g). We can hence identify $l^1((G)_{\mathcal{U}})$ with a 1-complemented subspace of $(l^1(G))_{\mathcal{U}}$.
For study more detail of the above concept, see [6, Section 5.4].

Proposition 2.14. *Let G be an infinite abelian discrete group, and $\mathcal{U}$ be an ultrafilter. Then $l^1((G)_{\mathcal{U}})$ is weakly amenable, but $(l^1(G))_{\mathcal{U}}$ is not necessary weakly amenable in general, (in particular $l^1(G)$ is not ultra-weakly amenable).*

Proof. By the above concept, $(G)_{\mathcal{U}}$ is a discrete group, so $l^1((G)_{\mathcal{U}})$ is weakly amenable (see [1]). But, $l^1(G)$ is not ultra-weak amenable, by Remark 2.11. □

Let G be a locally compact group, recall the notion of measure algebra $\mathcal{M}(G)$. The subset of $\mathcal{M}(G)$ consisting of the continuous measures is denoted by $\mathcal{M}_C(G)$, so that, for $\mu \in \mathcal{M}(G)$, we have $\mu \in \mathcal{M}_C(G)$ if and only if $\mu(\{s\}) = 0 \quad (s \in G)$. Also we have $\mathcal{M}(G) = \mathcal{M}_C(G) \oplus l^1(G)$, and $\mathcal{M}_C(G)$ is an closed ideal, and $l^1(G)$ is a unital closed subalgebra of $\mathcal{M}(G)$(for more details of this concept see [4, Section 1]).

Theorem 2.15. *Let G be an infinite abelian locally compact group, then $\mathcal{M}(G)$ is not ultra-weak amenable.*

Proof. By corollary 2.13, $l^1(G)$ is not ultra-weak amenable. Since $\mathcal{M}(G) = \mathcal{M}_C(G) \oplus l^1(G)$, $\mathcal{M}(G)$ is not ultra-weak amenable, by proposition 2.6. □

Remark 2.16. Notice that in theorems 2.10 and 2.15, an ultrafilter $\mathcal{U}$ (such that $(L^1(G))_{\mathcal{U}}$ and $(\mathcal{M}(G))_{\mathcal{U}}$ is not weakly amenable) have not been specified. By proposition 2.9, $(L^1(G))_{\mathcal{U}}$ is not weakly amenable when there is a surjection $\sigma_{AP} : (L^1(G))_{\mathcal{U}} \to AP(G)'$, unless G be finite. So for infinite group G, if $\mathcal{U}$ is a suitable ultrafilter that $\sigma_{AP} : (L^1(G))_{\mathcal{U}} \to AP(G)'$ be a surjection map, then $(L^1(G))_{\mathcal{U}}$ and so $(\mathcal{M}(G))_{\mathcal{U}}$ is not weakly amenable.

Theorem 2.17. *Let G be an abelian locally compact group such that $L^1(G)$ is ultra-weakly amenable, and let H be a closed normal subgroup of G. Then $\frac{G}{H}$ is finite.*

Proof. Let $\mathcal{U}$ be an ultrafilter. As detailed in [13, 1.9.12], we have a surjective algebra homomorphism $\phi : L^1(G) \to L^1(\frac{G}{H})$. So $(\phi)_{\mathcal{U}} : (L^1(G))_{\mathcal{U}} \to (L^1(\frac{G}{H}))_{\mathcal{U}}$ is surjective homomorphism. Since $\frac{(L^1(G))_{\mathcal{U}}}{\ker(\phi)_{\mathcal{U}}} \cong (L^1(\frac{G}{H}))_{\mathcal{U}}$, $L^1(\frac{G}{H})$ is ultra-weakly amenable, by theorem 2.4. The result now follows from theorem 2.10.

□

Theorem 2.18. *Let G be a locally compact abelian group such that $L^1(G)$ is ultra-weakly amenable. Then G is finite; or G satisfies the following statement:*
(1) *G is not discrete;*
(2) *G is not maximally almost periodic group (so we see that G is not compact);*
(3) *$AP(G)$ is finite-dimensional;*
(4) *if H is a closed normal subgroup of G, then $\frac{G}{H}$ is finite.*

Proof. The parts 1, 2 and 4 are present in this section, we prove part 3. G is a locally compact abelian group such that $L^1(G)$ is ultra-weakly amenable, so by proof of the theorem 2.8, we have that G^{AP} is finite. Thus by $AP(G) \cong C(G^{AP})$, we see $C(G^{AP})$ and so $AP(G)$ is finite-dimensional.

□

Daws in [5], give the following open question:
Is there an infinite locally compact group G such that $L^1(G)$ is ultra-amenable?

By the last theorem, we suspect that, the above open question can be replaced by ultra-weak amenability of $L^1(G)$, where G is infinite abelian locally compact group.
open question:
Is there an infinite abelian locally compact group G such that $L^1(G)$ is ultra-weakly amenable?

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