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Research Paper

ADVANCED APPROACHES TO APPROXIMATING CUBIC AND RADICAL CUBIC FUNCTIONAL EQUATIONS IN $G\beta$-NORMED SPACES

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ABSTRACT. This article investigates the approximation of cubic and radical cubic functional equations in G-normed and $G\beta$-normed vector spaces. We define these spaces and employ the Hyers-Ulam-Rassias stability methods to establish the stability of these functional equations. This study illuminates the stability properties of these equations in G-normed and $G\beta$-normed spaces, providing useful insights into their behavior and mathematical properties.

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1. Introduction and Background

During a noteworthy speech at the Mathematical Club of the University of Wisconsin in the autumn of 1940, Ulam [25] addressed a set of unanswered questions. This lecture marked the beginning of the development of functional equation stability theory. "If the suppositions of the theorem holds approximately, can we claim that the corresponding theorem will also hold approximately?" Ulam asked when introducing the stability problem. The essence of the stability problem for functional equations lies in its fundamental question: "If an approximate solution exists for a given functional equation, can this approximation effectively approach an exact solution for the same equation?" In cases where the response is positive, we designate the specific equation as possessing stability.

In 1941, Hyers presented the initial response to Ulam's theorem[7]. This crucial contribution marked the inception of a new mathematical research domain and captivated the interest of many mathematicians. Following this groundbreaking answer, many endeavors have been undertaken to delve into the realm of stability in functional equations, leading to the emergence of diverse theories within this discipline.

Hyers established that for any chosen value of $\epsilon > 0$, and provided that an mapping $f : E_1 \to E_2$ operates on Banach spaces E_1 and E_2 in a approximatively additive manner, meaning that the relation

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

holds, then it becomes feasible to identify a positive constant k and a unique additive mapping $A : E_1 \to E_2$ such that the condition

$$\|A(x) - f(x)\| \leq \epsilon$$

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is satisfied. Approaches employed to validate stability utilizing additive properties are denoted as “direct methods.” If a functional equation’s stability is analogous to Hyers’ theorem and is established using the control parameter ϵ, it is referred to as Hyers-Ulam stability.

In 1950, Aoki made a significant stride in enhancing the Hyers theorem [4]. Aoki’s breakthrough revealed that when $\theta_0 \geq 0$ and $0 \leq p < 1$ are available for a mapping $f : E_1 \to E_2$ between Banach spaces E_1 and E_2, and the relationship

$$\|f(x+y) - f(x) - f(y)\| \leq \theta_0(\|x\|^p + \|y\|^p)$$

holds true, it implies the existence of an additive mapping $A : E_1 \to E_2$ along with $\theta \geq 0$ in such a manner that the relationship

$$\|A(x) - f(x)\| \leq \theta\|x\|^p$$

is established. Notably, in Aoki’s theorem, the control function adheres to the rule $\|x\|^p + \|y\|^p$.

Subsequently, the exploration of functional equation stability encountered a period of dormancy until 1978 when Rassias extended Aoki’s theorem [23]. When the stability of a functional equation is established through the application of a control function similar to Aoki’s theorem, it is termed “Hyers-Ulam-Rassias stability.” The name shift from Aoki to Rassias may stem from the latter’s significant impact on upholding the issue of the stability of functional equations.

The evolution of Hyers-Ulam stability theory owes much to the numerous mathematicians who have enriched it through the introduction and substantiation of novel theorems. By altering the nature of functional equations, control functions, and spaces in the Hyers-Ulam stability theorem, they have embarked on a quest to explore and validate fresh conditions [5, 6, 19, 9, 11, 13, 20, 24, 15, 1, 2, 22] . Beyond its original confines, the Hyers-Ulam stability theorem has found extensive utility in substantiating various results across diverse realms of mathematics.

Definition 1.1. [10] Let F be a field, either $\mathbb{R}$ or $\mathbb{C}$, and X be a vector space over field F. A function $\|.,.\| : X \times X \to [0, \infty)$ is termed a G-norm if it fulfills the following set of conditions.

$(G1)$ $\|u, v\| = 0$ if and only if $u = v = 0$.
$(G2)$ $\|u - v, 0\| \leq \|u - w, 0\| + \|w - v, 0\|$ for all $u, v, w \in X$.
$(G3)$ $\|u - v, u - z\| \leq \|u - w, u - w\| + \|w - v, w - z\|$ for all $u, v, w, z \in X$.
$(G4)$ $\|\lambda u, \lambda v\| = |\lambda|\|u, v\|$ for all $u, v \in X$ and $\lambda \in F$.

The pair $(X, \|.,.\|)$ is termed a G-normed space.

In the above definition, if we replace the property $(G4)$ with the following property, we refer to the pair $(X, \|.,.\|)$ as a $G\beta$-normed vector space.

$$(G4') \ \|\lambda u, \lambda v\| = |\lambda|^\beta \|u, v\| \text{ for all } u, v \in X \text{ and } \lambda \in F,\ 0 < \beta \leq .1$$

Definition 1.2. [10] Let $(X, \|\cdot,\cdot\|)$ represent a G-normed space.
(i) We say that a sequence $\{u_n\}$ in X is a G-Cauchy sequence if, for any $\epsilon > 0$, there exists an integer N such that, for all $n, m > N$,

$$\|u_n - u_m, u_n - u_m\| < \epsilon \text{ and } \|u_n - u_m, 0\| < \epsilon.$$

(ii) We say that a sequence $\{u_n\}$ in X is a G-convergent sequence to $u \in X$ if, for any $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$,

$$\|u_n - u, u_n - u\| < \epsilon \text{ and } \|u_n - u, 0\| < \epsilon.$$

We will symbolize u by $G - \lim_{n\to\infty} u_n$.

Example 1.3. [10] Let $(X, \|\cdot\|)$ be a normed space. It is straightforward to demonstrate that $\|\cdot,\cdot\| : X \times X \to [0, \infty)$, defined by

$$\|u, v\| = \max\{\|u\|, \|v\|\},$$

is a G-norm.

Example 1.4. [12] Consider the linear space $X = \mathrm{C}[0, 1]$ of real-valued continuous functions on the interval $[0, 1]$. We define a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ as follows:

$$\|f, g\| = \max_{0\leq t\leq 1}\{|f(t)| + |g(t)|\} \quad (f, g \in C[0, 1]).$$

Then, the space $(X, \|\cdot,\cdot\|)$ forms a G-normed space.

A G-normed space $(X, \|\cdot,\cdot\|)$ is termed complete if every G-Cauchy sequence is G-convergent.

2. Main results

In this section, we investigate Hyers-Ulam-Rassias stability of the following cubic and radical cubic functional equations in $G-$normed spaces and $G\beta-$ normed spaces:

$$f(2u + v) + f(2u - v) = 2f(u + 2v) - 4f(u + v) + 18f(u) - 12f(v). \tag{2.1}$$

$$f(\sqrt[3]{u^3 + v^3}) = f(u) + f(v). \tag{2.2}$$

The solutions and properties to these two functional equations have been investigated in [8] and [3], respectively. For any function $f : X \to Y$, we define the operator $Cf(u, v) : X \times X \to Y$ by

$$Cf(u, v) = f(2u + v) + f(2u - v) - 2f(u + 2v) + 4f(u + v) - 18f(u) + 12f(v)$$

for all $u, v \in U$. Similarly, for each function $f : \mathbb{R} \to X$, we define the operator $CRf(u, v) : \mathbb{R} \times \mathbb{R} \to X$ as follows

$$CRf(u, v) = f(\sqrt[3]{u^3 + v^3}) - f(u) - f(v)$$

for all $u, v \in \mathbb{R}$. Readers have the option to refer to [3] in order to acquaint themselves with the technique of solving cubic radical function equations.

Theorem 2.1. *Let $\epsilon \geq 0$, p and q be positive real numbers with $p + q < 3$. Assume that $f : X \to Y$ is a mapping such that for all $u, v, z, w \in X$ the following inequality holds:*

$$\begin{aligned} \|Cf(u, v), Cf(z, w)\| \leq \epsilon\big[&\|u, u\|^p\|v, v\|^q + \|z, z\|^p\|w, w\|^q \\ &+ \|u, u\|^p + \|v, v\|^q + \|z, z\|^p + \|w, w\|^q\big]. \end{aligned} \tag{2.3}$$

Under these conditions, then there exists a unique cubic mapping $A : X \to Y$ that satisfies the following properties:

(1) A satisfies (2.1).
(2) For all $u \in X$, we have

$$\big\|A(u) - f(u), A(u) - f(u)\big\| \leq \frac{\epsilon}{8}\left[\frac{1}{1 - 2^{(p-3)}}\|u, u\|^p\right].$$

(3) For all $u \in X$, we have

$$\big\|A(u) - f(u), 0\big\| \leq \frac{\epsilon}{16}\left[\frac{1}{1 - 2^{(p-3)}}\|u, u\|^p\right].$$

Proof. Letting $u = v = z = w = 0$ in (2.3), we get

$$\| - 2f(0), -2f(0)\| \leq 0,$$

which means that $f(0) = 0$. Setting $z = u$ and $v = w = 0$ in (2.3), we have

$$\big\|2f(2u) - 16f(u), 2f(2u) - 16f(u)\big\| \leq 2\epsilon\big\|u, u\big\|^p.$$

Using property $(G4)$ and dividing both sides of the above inequality by 16, we obtain

$$\left\|\frac{f(2u)}{8} - f(u), \frac{f(2u)}{8} - f(u))\right\| \leq \frac{\epsilon}{8}\big\|u, u\big\|^p, \tag{2.4}$$

for all $u \in X$. Replacing u by $2u$ in (2.4), we obtain

$$\begin{aligned}\left\|\frac{f(2^2u)}{8} - f(2u), \frac{f(2^2u)}{8} - f(2u))\right\| &\leq \frac{\epsilon}{8}\big\|2u, 2u\big\|^p \\ &= \frac{\epsilon 2^p}{8}\big\|u, u\big\|^p.\end{aligned}$$

By dividing both sides of the aforementioned inequality by 8, we have

$$\left\|\frac{f(2^2u)}{8^2} - \frac{f(2u)}{8}, \frac{f(2^2u)}{8^2} - \frac{f(2u)}{8}\right\| \leq \frac{\epsilon 2^p}{8^2}\big\|u, u\big\|^p. \tag{2.5}$$

By comparing the two equations (2.4) and (2.5) and utilizing the following inequality

$$\|u - v, u - z\| \leq \|u - w, u - w\| + \|w - v, w - z\|,$$

for all $u, v, z, w \in X$, we get

$$\begin{aligned}\left\|\frac{f(2^2u)}{8^2} - f(u), \frac{f(2^2u)}{8^2} - f(u)\right\| &\leq \left\|\frac{f(2^2u)}{8^2} - \frac{f(2u)}{8}, \frac{f(2^2u)}{8^2} - \frac{f(2u)}{8}\right\| \\ &\quad + \left\|\frac{f(2u)}{8} - f(u), \frac{f(2u)}{8} - f(u))\right\| \\ &\leq \frac{\epsilon}{8}\big\|u, u\big\|^p + \frac{\epsilon 2^p}{8^2}\big\|u, u\big\|^p \\ &= \frac{\epsilon}{8}\left[1 + 2^{(p-3)}\right]\big\|u, u\big\|^p.\end{aligned}$$

In above inequality, if we replace u with $2u$, and subsequently, divide both sides by 8, then we obtain

$$\left\|\frac{f(2^3u)}{8^3}-\frac{f(2u)}{8},\frac{f(2^3u)}{8^3}-\frac{f(2u)}{8}\right\| \leq \frac{\epsilon}{8^2}\left[1+2^{(p-3)}\right]\|2u,2u\|^p$$
$$=\frac{\epsilon}{8}\left[2^{(p-3)}+2^{2(p-3)}\right]\|u,u\|^p.$$

By utilizing the property of $(G3)$, we have

$$\left\|\frac{f(2^3u)}{8^3}-f(u),\frac{f(2^3u)}{8^3}-f(u)\right\| \leq \left\|\frac{f(2^3u)}{8^3}-\frac{f(2u)}{8},\frac{f(2^3u)}{8^3}-\frac{f(2u)}{8}\right\|$$
$$+\left\|\frac{f(2u)}{8}-f(u),\frac{f(2u)}{8}-f(u))\right\|$$
$$\leq \frac{\epsilon}{8}\left[1+2^{(p-3)}+2^{2(p-3)}\right]\|u,u\|^p.$$

Continuing this process, we have,

$$\left\|\frac{f(2^nu)}{8^n}-f(u),\frac{f(2^nu)}{8^n}-f(u)\right\| \leq \frac{\epsilon}{8}\sum_{k=0}^{n-1}2^{k(p-3)}\|u,u\|^p.$$

For all nonnegative integers m,n with $m>n$ and for all $u\in X$, we have

$$\left\|\frac{f(2^mu)}{8^m}-\frac{f(2^nu)}{8^n},\frac{f(2^mu)}{8^m}-\frac{f(2^nu)}{8^n}\right\|$$
$$=\frac{1}{8^n}\left\|\frac{f(2^{m-n}.2^nu)}{8^{m-n}}-f(2^nu),\frac{f(2^{m-n}.2^nu)}{8^{m-n}}-f(2^nu)\right\|$$
$$\leq \frac{\epsilon}{8}\sum_{k=n}^{m-1}2^{k(p-3)}\|u,u\|^p. \tag{2.6}$$

Putting $v=z=w=0$ in (2.3), we get

$$\|f(2u)-16f(u),0\| \leq \epsilon\|u,u\|^p,$$

for all $u\in X$. Therefore

$$\left\|\frac{f(2u)}{16}-f(u),0\right\| \leq \frac{\epsilon}{16}\|u,u\|^p.$$

In a similar manner, and using

$$(G2)\ \|u-v,0\| \leq \|u-w,0\|+\|w-v,0\|,$$

we can demonstrate that

$$\left\|\frac{f(2^nu)}{8^n}-f(u),0\right\| \leq \frac{\epsilon}{16}\sum_{k=0}^{n-1}2^{k(p-3)}\|u,u\|^p,$$

for all $u\in X$. Therefore, we have

$$\left\|\frac{f(2^mu)}{8^m}-\frac{f(2^nu)}{8^n},0\right\| \leq \frac{\epsilon}{16}\sum_{k=n}^{m-1}2^{k(p-3)}\|u,u\|^p, \tag{2.7}$$

for all nonnegative integers m, n with $m > n$ and for $u \in X$. From inequalities (2.6) and (2.7), we can deduce that the sequence $\left\{\frac{1}{8^n} f(2^n u)\right\}$ is a $G-$Cauchy sequence every $u \in X$. Since Y is complete, the sequence $\left\{\frac{1}{8^n} f(2^n u)\right\}$ is $G-$convergent. Consequently, we can establish the mapping $A : X \to Y$ as follows

$$A(u) := G - \lim_{n\to\infty} \frac{1}{8^n} f(2^n u)$$

for all $u \in X$. Setting $z = w = 0$ and substituting $2^n u$ and $2^n v$ for u and v in (2.3), respectively, we obtain

$$\begin{aligned}
\|CA(u,v), 0\| &= \lim_{n\to\infty} \frac{1}{8^n} \big\|Cf(2^n u, 2^n v), 0\big\| \\
&\leq \lim_{n\to\infty} \frac{\epsilon}{8^n} \left[\|2^n u, 2^n u\|^p \|2^n v, 2^n v\|^q + \|2^n u, 2^n u\|^p + \|2^n v, 2^n v\|^q\right] \\
&= \lim_{n\to\infty} \frac{\epsilon}{8^n} \left[2^{n(p+q)} \|u,u\|^p \|v,v\|^q + 2^{np} \|u,u\|^p + 2^{nq} \|v,v\|^q\right] \\
&= \lim_{n\to\infty} \epsilon \left[2^{n(p+q-3)} \|u,u\|^p \|v,v\|^q + 2^{n(p-3)} \|u,u\|^p + 2^{n(q-3)} \|v,v\|^q\right].
\end{aligned}$$

As n tends to infinity, the right-hand side of the above inequality becomes equivalent to zero, since p and q are two positive real numbers and $p+q < 3$. Thus $CA(u,v) = 0$ for all $u, v \in X$. Hence the mapping $A : X \to Y$ is a cubic mapping. By substituting $n = 0$ and taking the limit $m \to \infty$ in inequalities (2.6) and (2.7), we get

$$\|A(u) - f(u), A(u) - f(u)\| \leq \frac{\epsilon}{8} \left[\frac{1}{1 - 2^{(p-3)}}\right] \|u,u\|^p$$

and

$$\|A(u) - f(u), 0\| \leq \frac{\epsilon}{16} \left[\frac{1}{1 - 2^{(p-3)}}\right] \|u,u\|^p.$$

In the following, we establish that $A : X \to Y$ is a unique cubic mapping. Let $A' : X \to Y$ be another cubic mapping satisfying aforementioned inequalities. Then

$$\begin{aligned}
\|A(u) - A'(u), 0\| &= \frac{1}{8^n} \big\|A(2^n u) - f(2^n u), 0\big\| + \big\|A'(2^n u) - f(2^n u), 0\big\| \\
&\leq \frac{1}{8^n} \left[\frac{\epsilon}{16} . \frac{1}{1 - 2^{(p-3)}} \|2^n u, 2^n u\|^p \right. \\
&\left. + \frac{\epsilon}{16} . \frac{1}{1 - 2^{(p-3)}} \|2^n u, 2^n u\|^p \right] \\
&= \frac{1}{8^n} \left[\frac{\epsilon}{8} . \frac{2^{np}}{1 - 2^{(p-3)}} \|u,u\|^p\right] \\
&= \frac{\epsilon}{8} \left[\frac{2^{n(p-3)}}{1 - 2^{(p-3)}} \|u,u\|^p\right] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore $A(u) = A'(u)$ for all $u \in X$. □

Theorem 2.2. *Let $\epsilon \geq 0$, p and q be positive real numbers with $p + q > 3$. Assume that $f : X \to Y$ is a mapping such that for all $u, v, z, w \in X$ the inequality (2.3) holds. Under* *these conditions, then there exists a unique cubic mapping $A : X \to Y$ that satisfies (2.1) and*

$$\big\|A(u) - f(u), A(u) - f(u)\big\| \leq \frac{\epsilon}{8}\Big[\frac{1}{2^{(3-p)} - 1}\|u, u\|^p\Big],$$

and

$$\big\|A(u) - f(u), 0\big\| \leq \frac{\epsilon}{16}\Big[\frac{1}{2^{(3-p)} - 1}\|u, u\|^p\Big],$$

for all $u \in X$.

Proof. Putting $z = u$ and $v = w = 0$ in (2.3) and replacing u by $\frac{u}{2}$, we get

$$\Big\|2f(u) - 16f\big(\frac{u}{2}\big), 2f(u) - 16f\big(\frac{u}{2}\big)\Big\| \leq 2\epsilon\big\|\frac{u}{2}, \frac{u}{2}\big\|^p$$

for all $u \in X$. Dividing by 2, we obtain

$$\Big\|f(u) - 8f\big(\frac{u}{2}\big), f(u) - 8f\big(\frac{u}{2}\big)\Big\| \leq \epsilon\big\|\frac{u}{2}, \frac{u}{2}\big\|^p = \frac{\epsilon}{2^p}\|u, u\|^p. \tag{2.8}$$

Again, replacing u by $\frac{u}{2}$ in the above inequality, multiplying both sides by 8 and utilizing the property $(G3)$, we have

$$\begin{aligned}\Big\|8f\big(\frac{u}{2}\big) - 8^2 f\big(\frac{u}{2^2}\big), 8f\big(\frac{u}{2}\big) - 8^2 f\big(\frac{u}{2^2}\big)\Big\| &\leq \frac{8\epsilon}{2^p}\big\|\frac{u}{2}, \frac{u}{2}\big\|^p \\ &= \epsilon 2^{-2p+3}\|u, u\|^p.\end{aligned} \tag{2.9}$$

Utilizing $(G3)$ and comparing (2.8) and (2.9), we get

$$\Big\|f(u) - 8^2 f\big(\frac{u}{2^2}\big), f(u) - 8^2 f\big(\frac{u}{2^2}\big)\Big\| \leq \epsilon 2^{-p}\big[1 + 2^{3-p}\big]\|u, u\|^p.$$

Continuing this process, i.e., replacing $\frac{u}{2}$ with u, multiplying both sides by 8, and utilizing property $G3$, we have the following inequality.

$$\Big\|f(u) - 8^n f\big(\frac{u}{2^n}\big), f(u) - 8^n f\big(\frac{u}{2^n}\big)\Big\| \leq \epsilon 2^{-p}\Big[\sum_{k=0}^{n-1} 2^{k(3-p)}\Big]\|u, u\|^p.$$

For all nonnegative integers m, n with $m > n$ and for all $u \in X$, we have

$$\Big\|8^m f\big(\frac{u}{2^m}\big) - 8^n f\big(\frac{u}{2^n}\big), 8^m f\big(\frac{u}{2^m}\big) - 8^n f\big(\frac{u}{2^n}\big)\Big\| \leq \epsilon 2^{-p}\Big[\sum_{k=n}^{m-1} 2^{k(3-p)}\Big]\|u, u\|^p. \tag{2.10}$$

By following an entirely similar approach, we arrive at the following inequality

$$\Big\|8^m f\big(\frac{u}{2^m}\big) - 8^n f\big(\frac{u}{2^n}\big), 0\Big\| \leq \epsilon 2^{1-p}\Big[\sum_{k=n}^{m-1} 2^{k(3-p)}\Big]\|u, u\|^p. \tag{2.11}$$

From inequalities (2.10) and (2.11), we can deduce that the sequence $\big\{8^n f(\frac{u}{2^n})\big\}$ is a $G-$Cauchy sequence for every $u \in X$. Since Y is complete, the sequence $\big\{8^n f(\frac{u}{2^n})\big\}$ is $G-$convergent. Consequently, we can establish the mapping $A : X \to Y$ as follows

$$A(u) := G - \lim_{n\to\infty} 8^n f\big(\frac{u}{2^n}\big).$$

Since the continuation of the proof is similar to the previous theorem, we omit it. □

Example 2.3. Assume that U and V are the vector spaces of continuous real-valued functions on the interval $[0,1]$. Let $g, f, h, i \in U$ such that $g = f$ and $h = i$. Consider $p = 1$ and $q = 1$ being positive real numbers. We define the G-norm as follows

$$\|g, h\| = \max\{\|g\|, \|h\|\}.$$

We also define the function $H : U \to V$ as follows

$$H(g)(u) = g^3(u)$$

for all $u \in [0,1]$ and $g \in U$. It is clear that

$$CH(g,h) = H(2g+h) + H(2g-h) - 2H(g+2h) + 4H(g+h) - 18H(g) + 12H(h) = 0.$$

The right-hand side of inequality (2.3) is equal to zero. Consequently, the inequality (2.3) always holds. Therefore, all the conditions of Theorem 2.1 hold, and there exists a unique cubic mapping A that satisfies conditions (1), (2), and (3).

Theorem 2.4. *Let $\Pi : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be a mapping such that*

$$\Upsilon(u,v) = \sum_{k=1}^{\infty} \left(\frac{1}{2^{\beta}}\right)^{k} \Pi\left(2^{\frac{k}{3}}u, 2^{\frac{k}{3}}v\right) < \infty$$

for all $u, v \in \mathbb{R}$. Suppose that $f : \mathbb{R} \to U$ is a mapping such that for all $u, v, z, w \in \mathbb{R}$ the following inequality holds.

$$\|CRf(u,v), CRf(z,w)\| \le \Pi(u,v) + \Pi(z,w). \tag{2.12}$$

Then there exists a unique radical cubic mapping $F : \mathbb{R} \to U$ satisfying the functional equation (2.2) and the inequality

$$\|f(u) - F(u), f(u) - F(u)\| \le 2^{1-\beta}\Upsilon(u,v) \text{ and } \|f(u) - F(u), 0\| \le 2^{-\beta}\Upsilon(u,v).$$

Proof. Putting $z = w = v = u$ in (2.12), we obtain

$$\left\|f(2^{\frac{1}{3}}u) - 2f(u), f(2^{\frac{1}{3}}u) - 2f(u)\right\| \le 2\Pi(u,u).$$

Dividing both sides for 2 in above inequality, and utilizing $(G4')$, we get

$$\left\|\frac{1}{2}f(2^{\frac{1}{3}}u) - f(u), \frac{1}{2}f(2^{\frac{1}{3}}u) - f(u)\right\| \le \frac{2}{2^{\beta}}\Pi(u,u). \tag{2.13}$$

Replacing u by $2^{\frac{1}{3}}u$ in (2.13), we obtain

$$\left\|\frac{1}{2}f(2^{\frac{2}{3}}u) - f(2^{\frac{1}{3}}u), \frac{1}{2}f(2^{\frac{2}{3}}u) - f(2^{\frac{1}{3}}u)\right\| \le \frac{2}{2^{\beta}}\Pi(2^{\frac{1}{3}}u, 2^{\frac{1}{3}}u).$$

Therefore,

$$\left\|\frac{1}{2^2}f(2^{\frac{2}{3}}u) - \frac{1}{2}f(2^{\frac{1}{3}}u), \frac{1}{2^2}f(2^{\frac{2}{3}}u) - \frac{1}{2}f(2^{\frac{1}{3}}u)\right\| \le \frac{2}{2^{2\beta}}\Pi(2^{\frac{1}{3}}u, 2^{\frac{1}{3}}u). \tag{2.14}$$

Comparing (2.13), (2.14) and utilizing

$$\|u - v, u - z\| \le \|u - w, u - w\| + \|w - v, w - z\|,$$

we obtain

$$\left\|\frac{1}{2^2}f(2^{\frac{2}{3}}u) - f(u), \frac{1}{2^2}f(2^{\frac{2}{3}}u) - f(u)\right\| \le \frac{2}{2^{\beta}}\left[\Pi(u,u) + \frac{1}{\beta}\Pi(2^{\frac{1}{3}}u, 2^{\frac{1}{3}}u)\right].$$

With continuing this process, i.e., substituting $2u$ for u and utilizing the property $(G4)$, we have

$$\left\| \frac{1}{2^n} f(2^{\frac{n}{3}}u) - f(u), \frac{1}{2^n} f(2^{\frac{n}{3}}u) - f(u) \right\| \leq \frac{2}{2^\beta} \left[\sum_{k=0}^{n} \left(\frac{1}{2^\beta} \right)^k \Pi\big(2^{\frac{k}{3}}u, 2^{\frac{k}{3}}u\big) \right],$$

for all $u \in \mathbb{R}$. For all integers m, n with $m > k \geq 0$, we get

$$\left\| \frac{1}{2^n} f(2^{\frac{n}{3}}u) - \frac{1}{2^m} f(2^{\frac{m}{3}}u), \frac{1}{2^n} f(2^{\frac{n}{3}}u) - \frac{1}{2^n} f(2^{\frac{n}{3}}u) \right\| \leq \frac{2}{2^\beta} \left[\sum_{k=n}^{m-1} \left(\frac{1}{2^\beta} \right)^k \Pi\big(2^{\frac{k}{3}}u, 2^{\frac{k}{3}}u\big) \right], \tag{2.15}$$

for all $u \in \mathbb{R}$ with $m > n \geq 0$. Using a completely similar approach, by setting $u = v$ and $z = w = 0$ in (2.12) and utilizing $(G3)$, we get

$$\left\| \frac{1}{2^n} f(2^{\frac{n}{3}}u) - \frac{1}{2^m} f(2^{\frac{m}{3}}u), 0 \right\| \leq \frac{1}{2^\beta} \left[\sum_{k=n}^{m-1} \left(\frac{1}{2^\beta} \right)^k \Pi\big(2^{\frac{k}{3}}u, 2^{\frac{k}{3}}u\big) \right]. \tag{2.16}$$

From inequalities (2.15) and (2.16). We can deduce that the sequence $\left\{\frac{1}{2^n} f(2^{\frac{n}{3}}u)\right\}$ is a $G-$Cauchy sequence for all $u \in \mathbb{R}$. Since X is complete, the sequence $\left\{\frac{1}{2^n} f(2^{\frac{n}{3}}u)\right\}$ is $G-$convergent. Consequently, we can establish the mapping $F : \mathbb{R} \to X$ as follows

$$F(u) := G - \lim_{n\to\infty} \frac{1}{2^n} f(2^{\frac{n}{3}}u)$$

for all $u \in \mathbb{R}$. Setting $z = w = 0$ and substituting $2^{\frac{n}{3}}u$ and $2^{\frac{n}{3}}v$ for u and v in (2.12), respectively, we obtain

$$\begin{aligned} \|CRF(u, v), 0\| &= \lim_{n\to\infty} \frac{1}{2^n} \big\|CRf(2^{\frac{n}{3}}u, 2^{\frac{n}{3}}v), 0\big\| \\ &\leq \frac{1}{2^{\beta n}} \Pi \left(2^{\frac{n}{3}}u, 2^{\frac{n}{3}}v\right) = 0. \end{aligned}$$

Thus

$$F\sqrt[3]{u^3 + v^3} = F(u) + F(v).$$

So, F is a radical cubic mapping using by Theorem 3.1 from [3]. The proof of the uniqueness of the mapping F is similar to the Theorem 2.3.

□

Example 2.5. Given the assumptions of Example 2.3, it suffices to define

$$H(g)(u) = \sqrt[3]{g(u)} \quad \text{for all } g \in C[0, 1].$$

In this case, the right-hand side of inequality (2.12) evaluates to zero, thus the inequality (2.12) always holds. Consequently, there exists a unique radical cubic function F that satisfies Theorem (2.3).

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