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Research Paper

PSEUDO-AMENABILITY AND BIFLATNESS OF WEIGHTED ALGEBRAS AND THEIR SECOND DUAL ON INVERSE SEMIGROUPS

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Abstract. This paper proves that when S is an inverse semigroup, pseudo-amenability, amenability and approximately amenability of $\ell^1(S,\omega)$ are equivalent, $E(S)$ is finite and every maximal subgroup of S is amenable. In addition, for a discrete inverse semigroup of S, we prove when $(E(S),\leq)$ is uniformly locally finite, then $\ell^1(S,\omega)^{**}$ is pseudo-amenable if and only if $\ell^1(S)$, it is pseudo-amenable and S is finite. Also, we prove that for discrete inverse semigroup S, if $\ell^1(S,\omega)$ has a bounded approximate identity, then $\ell^1(S,\omega)^{**}$ is amenable if and only if $\ell^1(S)$ is biprojective and S is finite. Moreover, we show that for an Archimedean semigroup S, if Ω be bounded on every maximal subgroup G of S, pseudo-amenability, amenability and approximate amenability of $\ell^1(S,\omega)$ are equivalent. Finally, we investigate the amenability and pseudo-amenability of $\ell^1(S,\omega)$, where S is a left (right) zero semigroup or it is a rectangular band semigroup.

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1. Introduction and Preliminaries

For a Banach algebra $\mathfrak{A}$ the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule in a natural manner and the multiplication map $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}$ defined by $\pi(a \otimes b) = ab$ for $a, b \in \mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule homomorphism.

Amenability for Banach algebras introduced by B. E. Johnson [14]. Let $\mathfrak{A}$ be a Banach algebra and E be a Banach $\mathfrak{A}$-bimodule. A continuous linear operator $D : \mathfrak{A} \longrightarrow E$ is a *derivation* if it satisfies $D(ab) = D(a) . b + a . D(b)$ for all $a, b \in \mathfrak{A}$. Given $x \in E$, the *inner* derivation $ad_x : \mathfrak{A} \longrightarrow E$ is defined by $ad_x(a) = a . x - x . a$. A Banach algebra $\mathfrak{A}$ is *amenable* if for every Banach $\mathfrak{A}$-bimodule E, every derivation from $\mathfrak{A}$ into E^*, the dual of E, is inner.

An *approximate diagonal* for a Banach algebra $\mathfrak{A}$ is a net $(m_i)_i$ in $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $a . m_i - m_i . a \longrightarrow 0$ and $a\pi(m_i) \longrightarrow a$, for each $a \in \mathfrak{A}$. The concept of pseudo-amenability introduced by F. Ghahramani and Y. Zhang in [8]. A Banach algebra $\mathfrak{A}$ is *pseudo-amenable* if it has an approximate diagonal. It is well-known that amenability of $\mathfrak{A}$ is equivalent to the existence of a *bounded* approximate diagonal. One may see [15, 16, 17] for more details and related notions.

The notions of biprojectivity and biflatness of Banach algebras introduced by Helemskiĭ in [11]. A Banach algebra $\mathfrak{A}$ is *biprojective* if there is a bounded $\mathfrak{A}$-bimodule homomorphism $\rho : \mathfrak{A} \longrightarrow \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $\pi o \rho = I_{\mathfrak{A}}$, where $I_{\mathfrak{A}}$ is the identity map on $\mathfrak{A}$. We say that $\mathfrak{A}$ is *biflat*

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if there is a bounded $\mathfrak{A}$-bimodule homomorphism $\rho : \mathfrak{A} \longrightarrow (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ such that $\pi^{**} o \rho = k_{\mathfrak{A}}$, where $k_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{A}^{**}$ is the natural embedding of $\mathfrak{A}$ into its second dual.

Definition 1.1. Let S be a semigroup and

$$\ell^1(S) = \{f : S \to \mathbb{C}, \|f\|_1 = \sum_{s\in S} \mid f(s) \mid < \infty\}.$$

We define the convolution of two elements $f, g \in \ell^1(S)$ by $(f * g)(s) = \sum_{uv=s} f(u)g(v)$, where $\sum_{uv=s} f(u)g(v) = 0$, when there are no elements $u, v \in S$ with $uv = s$. Then $(\ell^1(S), *, \|.\|_1)$ becomes a Banach algebra, that is called the *semigroup algebra of S*.
Let S be a semigroup. A continuous function $\omega : S \longrightarrow (0, \infty)$ is a *weight* on S if $\omega(st) \leq \omega(s)\omega(t)$, for all $s, t \in S$. Then it is standard that

$$\ell^1(S,\omega) = \left\{ f = \sum_{s\in S} f(s)\delta_s : \|f\|_\omega = \sum_{s\in S} |f(s)|\omega(s) < \infty \right\}$$

is a Banach algebra with the convolution product $\delta_s * \delta_t = \delta_{st}$. These algebras are called *weighted semigroup algebras.*

Definition 1.2. Let $(P, \leq)$ is a partially ordered set. Then $(P, \leq)$ is *locally finite* if $(x] = \{y \in S : y \leq x\}$ is finite for every $x \in S$ and it is *uniformly locally finite* if $\sup\{|(x]| : s \in S\} < \infty$.
We recall that a semigroup S is an *inverse semigroup* if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. The maximal subgroup of S at $p \in E(S)$ is denoted by G_p. It is known that $G_p = \{s \in S : ss^* = s^*s = p\}$.

In this note, we study the earlier mentioned properties of Banach algebras for weighted semigroup algebras. Firstly in section 2, we prove that for discrete inverse semigroup S, if $\ell^1(S,\omega)$ has a bounded approximate identity then $\ell^1(S,\omega)^{**}$ is amenable if and only if $\ell^1(S)$ is biprojective and S is finite.

Then, in section 3, we show that for an Archimedean semigroup S, if Ω be bounded on every maximal subgroup G of S, pseudo-amenability, amenability and approximate amenability of $\ell^1(S,\omega)$ are equivalent, and also if S be a weakly cancellative commutative semigroup, we obtain some results on pseudo-amenability of the second dual of $\ell^1(S,\omega)$. Finally, in section 4, we characterize amenability and pseudo-amenability of $\ell^1(S,\omega)$, for some certain class of semigroups.

2. Pseudo-amenability and Biflatness of $\ell^1(S,\omega)^{**}$

In the section, we prove that for a discrete inverse semigroup S, if $\ell^1(S,\omega)$ has a bounded approximate identity, then $\ell^1(S,\omega)^{**}$ is amenable if and only if $\ell^1(S)$ is biprojective and S is finite. Also, for the semigroup $S = M^0(G, I)$, we also prove that if I is non-empty and finite and $\ell^1(S,\omega)$ has a bounded approximate identity, then the biflatness of $\ell^1(S,\omega)^{**}$ and the finiteness of G are equivalent.

Theorem 2.1. *Let S be a finite inverse semigroup and ω be a weight on S. If $\ell^1(S,\omega)^{**}$ is pseudo-amenable, then $\ell^1(S,\omega)$ is biflat.*

Proof. Let $\ell^1(S,\omega)^{**}$ is pseudo-amenable, by [8, proposition 2.3], $\ell^1(S,\omega)$ is pseudo-amenable. It is well-known that every finite inverse semigroup is amenable. So S is amenable semigroup and by [2, theorem 8], $\ell^1(S)$ is amenable. Since S is finite, ω is bounded on the whole of S, so by [25, theorem 3.6], $\ell^1(S,\omega)$ is amenable, therefore $\ell^1(S,\omega)$ is biflat. □

Theorem 2.2. *Let S be a discrete inverse semigroup, ω be a weight on S, and $\ell^1(S,\omega)$ has a bounded approximate identity.Then the following are equivalent:*
(i) $\ell^1(S,\omega)^{**}$ is amenable.
(ii) $\ell^1(S)$ is biprojective and S is finite.
(iii) $\ell^1(S,\omega)^{**}$ is biprojective.

Proof. $(i) \to (ii)$ Let $\ell^1(S,\omega)^{**}$ is amenable, by [25, theorem 3.7], $\ell^1(S)$ is amenable and S is finite, so $\ell^1(S)$ is biflat and by [21, thorem 3.7 i], S is uniformly locally finite and for each $p \in E(S)$, G_p is amenable group. Since S is finite, G_p is finite and [21, thorem 3.7ii] shows that $\ell^1(S)$ is biprojective, as required.
$(ii) \to (iii)$ Finiteness of S implies that Ω is finite and $\ell^1(S) \cong \ell^1(S,\omega)$. Moreover, $\ell^1(S)$ is finite-dimensional and $\ell^1(S,\omega)^{**} \cong \ell^1(S)$, so $\ell^1(S,\omega)^{**}$ is biprojective.
$(iii) \to (i)$ Since $\ell^1(S,\omega)^{**}$ is biprojective,it is biflat and by our hypothesis, $\ell^1(S,\omega)^{**}$ has a bounded approximate identity[23], therefore $\ell^1(S,\omega)^{**}$ is amenable, as required. □

Corollary 2.3. *Let G be a group, I be a non-empty set and let $S = M^0(G,I)$ be the Brandt semigroup over G with index set I, ω be a weight on S and $\ell^1(S)$ has a bounded approximate identity. Then $\ell^1(S,\omega)^{**}$ is biflat if and only if G be finite.*

Proof. Let $\ell^1(S,\omega)^{**}$ is biflat, since I is finite and $E(S) = \{(e_G)_{ii} \quad , \quad i \in I\} \cup \{0\}$ is finite. $\ell^1(S,\omega)^{**}$ has a bounded approximate identity, now by theorem 2.2, $\ell^1(S)$ is biprojective and by [21, thorem 3.7i], each maximal subgroup of S is finite and so G is finite. For consequently, if G be finite, by our hypothesis, the finiteness of I implies that S is finite and $\ell^1(S) \cong \ell^1(S,\omega) \cong \ell^1(S,\omega)^{**}$. So by [5, corollary 3.12], $\ell^1(S,\omega)^{**}$ is biflat. □

Theorem 2.4. *Let S be a discrete inverse semigroup, ω be a weight on S and $(E(S),\leq)$ be a uniformly locally finite. Then the following are equivalent:*
(i) $\ell^1(S,\omega)^{**}$ is pseudo-amenable.
(ii) $\ell^1(S)$ is pseudo-amenable and S is finite.
(iii) $\ell^1(S)^{**}$ is pseudo-amenable.

Proof. We first prove that S is finite. Since S is discrete, then S has a principal series

$$S_1 \trianglelefteq S_2 \trianglelefteq S_3 \trianglelefteq ..... \trianglelefteq S_{n-1} \trianglelefteq S_n = S.$$

such that each quotient S_{i+1}/S_i is a regular Rees matrix semigroup of the form $M^0(G_i,P_i,n_i)$ for each i , where $n_i \in \mathbb{N}$ and $S_1 \cup \{G_i \ : \ 2 \leq i \leq n\}$ is the set of all maximal subgroups of S. Furthermore, S_1 is an ideal subgroup of S. $\ell^1_0(S_1,\omega)$ is an ideal of $\ell^1(S,\omega)$ and $\ell^1(S/S_1,\tilde{\omega})$ are pseudo-amenable. Since S_1 is a group, $\ell^1(S_1,\omega)$ has a bounded approximate identity and $\ell^1(S_1,\omega)^{**}$ is pseudo-amenable and S_1 is amenable group. Since S_1 is a group, it is finite by [8]. Hence S_2/S_1 is a completely semigroup and by [8, proposition 2.2], $\ell^1(S_{i+1}/S_i,\tilde{\omega})^{**}$ is pseudo-amenability and by [25, Theorem 2.1], for $2 \leq i \leq n$, we have

$$\frac{\ell^1(S_2/S_1,\tilde{\omega})^{**}}{\mathbb{C}\delta_0} \cong M_n(\ell^1(G_i,\omega))^{**} \cong M_n(\ell^1(G_i,\omega)^{**})$$

$\ell^1(S_2/S_1,\tilde{\omega})^{**}$ is pseudo-amenable, thus $M(\ell^1(G_i,\omega)^{**})$ is pseudo-amenable and so $\ell^1(G_i,\omega)$ is pseudo-amenable. Now, by[8, theorem 4.2], G_i is finite, so S_2/S_1 is finite. Then S_2 is finite. We will continue to conclude that S is finite.
$\ell^1(S,\omega)^{**}$ is pseudo-amenable, then by [8, proposition 2.3], $\ell^1(S,\omega)$ is pseudo- amenable. Since S is an inverse semigroup, we may define the relation ρ on S by $s\rho t \leftrightarrow \exists e \in E(S)$ such that $es = et$. It is easy to see that ρ is an equivalence relation on S and the quotient space

$G_S = S/\rho$ is a group,[18]. Let $\theta : S \to G_S$ be the quotient map, it follows that $\ell^1(G_S, \omega_S)$ is a homomorphic image of $\ell^1(S,\omega)$ where as $\omega_S = \omega|_S$ and so by [8, proposition 2.2], it is pseudo-amenable. Since G_S is a group then G_S has an identity kind e, so δ_e is identity for $\ell^1(G_S, \omega_S)$ and $\ell^1(G_S, \omega_S)$ is approximately amenable [8, theorem 2.3], so by [22, theorem 2.4], G_S is an amenable group. It follows that S is an amenable inverse semigroup and by [2, theorem 1], each maximal subgroup of S is amenable moreover by [6, theorem 1], $\ell^1(S)$ is pseudo-amenable.
$(ii) \leftrightarrow (iii)$Since S is finite so $\ell^1(S)^{**} \cong \ell^1(S)$, as required.
$(ii) \to (i)$Let $\ell^1(S)$ is pseudo-amenable and S be finite, by [8, theorem 3.7] and [2, theorem 1], S is amenable inverse semigroup. Since S is finite, ω is bounded on the whole of S and so $\ell^1(S) \cong \ell^1(S,\omega)$. Moreover, $\ell^1(S)^{**} \cong \ell^1(S)$, so $\ell^1(S,\omega)^{**}$ is pseudo-amenable. □

Theorem 2.5. *Let S be a uniformly locally finite inverse semigroup, ω be a weight on S and Ω is bounded on every maximal subgroup G of S and $\ell^1(S,\omega)$ has a bounded approximate identity.Then the following are equivalent:*
(i) $\ell^1(S,\omega)$ is pseudo-amenable.
(ii) $E(S)$ is finite and every maximal subgroup of S, is amenable.
(iii) $\ell^1(S,\omega)$ is amenable.
(iv) $\ell^1(S,\omega)$ is approximate amenable.

Proof. $(i) \to (ii)$ Let $\ell^1(S,\omega)$ is pseudo-amenable, by proof of theorem 2.4, S is an amenable semigroup and therefoe each maximal subgroup of S is amenable by [2, theorem 1]. Now, we prove that $E(S)$ is finite. Since $(S, \leq)$ is uniformly locally finite, then by [21, proposition 2.14], $(E(S), \leq)$ is uniformly locally finite, so $\ell^1(S)$ is pseudo-amenable [6, theorem 3.7]. Let $\omega = 1$, $\ell^1(S,\omega) = \ell^1(S)$, and by our hypothesis, $\ell^1(S)$ has a bounded approximate identity. By [8, proposition 3.2], $\ell^1(S)$ is approximate amenable and so by [25, theorem 4.1], $E(S)$ is finite, as required.
$(ii) \to (iii)$ This follows by applying [22, theorem 4.3] and [25, theorem 3.6].
$(iii) \to (iv)$This implication is trivial.
$(iv) \to (i)$ By our hypothesis and [8, proposition 3.2], $\ell^1(S,\omega)$ is pseudo-amenable. □

Corollary 2.6. *Let S be an inverse semigroup with $E(S)$ finite, ω be a weight on S and Ω is bounded on every maximal subgroup G of S. $\ell^1(S,\omega)$ is pseudo-amenable if and only if $\ell^1(S,\omega)$ be amenable.*

Proof. Let $\ell^1(S,\omega)$ is pseudo-amenable, by theorem 2.5, S is amenable inverse semigroup and by [2, theorem 1], each maximal subgroup of S is amenable. So by [2, theorem 8], $\ell^1(S)$ is amenable. Now, By our hypothesis and [25, theorem 3.6], $\ell^1(S,\omega)$ is amenable. Consequently is clear. □

3. The pseudo-amenability weighted algebras and their second dual on commutative semigroups

In the section, we show that for an Archimedean semigroup S, if Ω be bounded on every maximal subgroup G of S, pseudo-amenability, amenability and approximate amenability of $\ell^1(S,\omega)$ are equivalent and also if S be a weakly cancellative commutative semigroup, we obtain some results on pseudo-amenability of the second dual of $\ell^1(S,\omega)$. Recall that a semigroup S is *archimedean* if S is commutative and for each $s,t \in S$ there exists $n \in \mathbb{N}$ such that

$$s^n \in tS = \{tu \ : \ u \in S\}.$$

Theorem 3.1. *Let S be an archimedean semigroup, ω be a weight on S and Ω is bounded on every maximal subgroup G of S. Then the following are equivalent:*
(i) *$\ell^1(S,\omega)$ is pseudo-amenable.*
(ii) *S is an amenable group.*
(iii) *$\ell^1(S)$ is pseudo-amenable.*
(iv) *$\ell^1(S,\omega)$ is amenable.*
(v) *$\ell^1(S,\omega)$ is approximately amenable.*

Proof. $(i) \rightarrow (ii)$ Suppose that $\ell^1(S,\omega)$ is pseudo-amenable. Thus $\ell^1(S,\omega)$ has an approximate identity and $\overline{\ell^1(S,\omega)^2} = \ell^1(S,\omega)$. We conclude that $S^2 = S$, and $S = S^2 = \cup_{s\in S} sS$. We deduce that $S = sS$ for all $s \in S$. Fix an element $s \in S$. There exist $u, v \in S$ such that $s = su$ and $u = sv$. Thus,

$$u^2 = svsv = suv = sv = u.$$

So $u \in E(S)$, on the other hand $S = uS$. It follows that u is identity element of S. Now, if $st = sb$ then there exists $v \in S$ such that $u = sv$. So,

$$bv = svb = ub = b \quad , \quad t = ut = svt = stv = s$$

Thus S is cancellative and by proof of the [5, theorem 3.6ii], S is a group. Now by proof of the theorem 2.4, S ia an amenable group.
$(ii) \longleftrightarrow (iii)$ Follows from [20, theorem 4.2].
$(ii) \rightarrow (iv)$ Suppse S is an amenable group, by [4], $\ell^1(S)$ is amenable, So by [25, theorem 3.6], $\ell^1(S,\omega)$ is amenable.
$(iv) \rightarrow (v)$ It is clear.
$(v) \rightarrow (i)$ If S is commutative, then $\ell^1(S,\omega)$ is commutative, so by [8, corollary 3.4], $\ell^1(S,\omega)$ is pseudo-amenable. □

Now, we obtain some results on pseudo-amenability of the second dual of commutative weighted semigroup algebras.
The commutative semigroup S is called *Weakly cancellative* if for each $s, t \in S$, $s^{-1}t = \{u \in S \ : \ su = t\}$ is finite.

Proposition 3.2. *Let S be a weakly cancellative commutative semigroup, ω be a weight on S and $\omega \geq 1$. If $\ell^1(S)^{**}$ is unital, then the following statements are equivalent:*
(i) *$\ell^1(S,\omega)$ is amenable and S is finite.*
(ii) *$\ell^1(S,\omega)^{**}$ is amenable.*
(iii) *$\ell^1(S,\omega)^{**}$ is approximately amenable.*
(iv) *$\ell^1(S,\omega)^{**}$ is pseudo- amenable.*

Proof. $(i) \rightarrow (ii)$ Since S is finite, ω is bounded on the whole of S and so, $\ell^1(S,\omega) \cong \ell^1(S)$, moreover, $\ell^1(S)$ is finite-dimensional and so $\ell^1(S) \cong \ell^1(S)^{**}$. Therefore, $\ell^1(S,\omega)^{**}$ is amenable.
$(ii) \rightarrow (iii)$ It is clear.
$(iii) \rightarrow (iv)$ S is commutative, so $\ell^1(S,\omega)$ is commutative, then $\ell^1(S,\omega)^{**}$ is commutative. Now by [8, corollary 3.4], the proof is complete.
$(iv) \rightarrow (i)$ If $\omega = 1$, $\ell^1(S,\omega)^{**} \cong \ell^1(S)^{**}$, now by [24, corollare 3.6], implies that exists $t \in S$ such that tS is finite. So, for $s \in S$, by our hypothesis $t^{-1}S$ has to be finite. On the other hand, since $S = \cup\{t^{(-1)}s : s \in tS\}$, We conclude that S is finite. $\ell^1(S,\omega)$ is pseudo-amenable, then by theorem 3.1, $\ell^1(S,\omega)$ has to be amenable. □

4. **Pseudo-amenability of the weighted semigroup algebras**

In this section we give a necessary condition for biflatness of a band weighted semigroup algebra.
Recall that S is a right(left) zero semigroup if for each $s, t \in S$, $st = t(st = s)$.

Proposition 4.1. *Let S be a right(left) zero semigroup and ω be a weight on S. Then $\ell^1(S,\omega)$ is biprojective, So it is biflat.*

Proof. Suppose that S is a right zero semigroup. For each $f, g \in \ell^1(S,\omega)$,

$$f * g = \sum_{r \in s} \left(\sum_{uv=r} f(u)g(v) \right) \delta_r = \left(\sum_{s \in S} f(s) \right) g = \varphi_s(f)g.$$

Where φ_s is the augementation character on $\ell^1(S,\omega)$. Take an arbitary element $t_\circ \in S$ and define:

$$\rho : \ell^1(S,\omega) \longrightarrow \ell^1(S,\omega) \otimes \ell^1(S,\omega) = \ell^1(S \times S, \omega \otimes \omega) // \ \rho(f) = \delta_{t_\circ} \otimes f, \ f \in \ell^1(S,\omega)$$

Then for each $f, g \in \ell^1(S,\omega)$ we have

$$\begin{aligned} \rho(f * g) &= \delta_{t_\circ} \otimes (f * g) = f_s(f)\delta_{t_\circ} \otimes g \\ &= f.(\delta_{t_\circ} \otimes g) = f.\rho(g), \end{aligned}$$

and

$$\begin{aligned} f(f * g) &= \delta_{t_\circ} \otimes (f * g) = (\delta_{t_\circ} \otimes \rho).g \\ &= \rho(f).g, \end{aligned}$$

Moreover,

$$\pi o \rho(f) = \pi(\delta_{t_\circ} \otimes f) = \delta_{t_\circ} * f = f_s(\delta_{t_\circ})f = f.$$

Then $\ell^1(S,\omega)$ is biprojective. The proof for a left zero semigroup is similar. □

Definition 4.2. Given two semigroups S_1 and S_2, we say that a weight ω on $S := S_1 \times S_2$ is *separable* if there exist two weights ω_1 and ω_2 on S_1 and S_2, respectively such that $\omega = \omega_1 \otimes \omega_2$. It is easy to verify that $\ell^1(S,\omega) \cong \ell^1(S_1,\omega_1)\widehat{\otimes}\ell^1(S_2,\omega_2)$.

Definition 4.3. Let S be a semigroup and let $E(S) = \{p \in S : p^2 = p\}$. We say that S is a *band semigroup* if $S = E(S)$. A band semigroup S satisfying $sts = s$, for each $s, t \in S$ is called a *rectangular band semigroup*. For a rectangular band semigroup S, it is known that $S \simeq L \times R$, where L and R are left and right zero semigroups, respectively [12, theorem 1.1.3].

Proposition 4.4. *Let S be a rectangular band semigroup and ω be a separable weight. Then $\ell^1(S,\omega)$ is biprojective, So it is biflat.*

Proof. In view of earlier argument, it follows from proposition 4.1, it follows that $\ell^1(L,\omega_1)$ and $\ell^1(R,\omega_2)$ are biprojective. The result now follows from [21, proposition 2.4] . □

Theorem 4.5. *Let S be a rectangular band semigroup and ω be a weight on S. Then $\ell^1(S,\omega)$ is amenable if and only if S singleton.*

Proof. From [25, theorem 3.6], $\ell^1(S)$ is amenable, So by [4, theorem 3.3], the result now follows. □

Definition 4.6. Let A be Banach algebra, Λ be a semilattice and $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of closed subalgebras of A. Suppose that A is ℓ^1-directsum of A_α is as Banach space such that;

$$A_\alpha A_\beta \subseteq A_{\alpha\beta},\ (\alpha, \beta \in \Lambda);$$

Then A is called ℓ^1-graded of A_α is over the semilattice Λ and is denoted by $A = \bigoplus_{\alpha\in\Lambda}^{\ell^1} A_\alpha$. Let S be a semigroup and S^1 denote the unitization of S. We define the equivalence relation τ on S by

$$a\tau b \Longleftrightarrow S^1aS^1 = S^1bS^1,\ (a, b \in S).$$

If S is a regular semigroup, then by the argument in $[5, section 2.4]$,

$$a\tau b \Longleftrightarrow SaS = SbS,\ (a, b \in S).$$

Now let S be a band semigroup, then by [5, theorem4.4.1], S is a semilattice of rectangular band semigroups. Indeed, $S = \bigcup_{\alpha\in Y} S_\alpha$ where $Y = \frac{S}{\tau}$ and for each $\alpha = [S] \in Y$, $S_\alpha = [S]$.

Theorem 4.7. *Let S be a band semigroup and ω be a weight on S. If $\ell^1(S, \omega)$ is amenable, then S is finite and each $\tau-$class is singleton.*

Proof. By the above argument, let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice of rectangular band semigroup and $\ell^1(S, \omega)$ is amenable. Then $E(S) = S$ is finite and so $Y = \frac{S}{\tau}$ is a finite semilattice. Hence $\ell^1(S, \omega)$ is ℓ^1-graded of Banach algebras on finite semilattice of Y. Indeed, We have

$$\ell^1(S, \omega) \cong \bigoplus_{\alpha\in Y}^{\ell^1} \ell^1(S_\alpha, \omega_\alpha)$$

Where $\omega_\alpha|_{S_\alpha}$. By [3, proposition 3.1], $\ell^1(S_\alpha, \omega_\alpha)$ is amenable for $\alpha \in Y$. Since S_α is are rectangular band semigroups, by theorem 4.5, S_α is singleton for each $\alpha \in Y$ and the proof is complete. □

Now, we study pseudo-amenability of the weighted semigroup algebras $\ell^1(S, \omega)$ in terms of the amenability of S. Precisely, we show that pseudo-amenability of $\ell^1(S, \omega)$ implies that S is amenable for left cancellative semigroups and band semigroups.

Theorem 4.8. *Let S be a band semigroup and ω be a weight on S. If $\ell^1(S, \omega)$ is pseudo-amenable, then S is a semilattice and so is amenable.*

Proof. By the after argument of definition 4.6, Suppose $S = \bigcup_{\alpha\in Y} S_\alpha$ is a semilattice of rectangular band semigroup. Indeed, We have $S_\alpha S_\beta \subseteq S_{\alpha\beta}$, $(\alpha, \beta \in Y)$. It follows that $\ell^1(S, \omega)$ is ℓ^1-graded of $\ell^1(S_\alpha, S_\beta)$ is over the semilattice Y.(i.e. $\ell^1(S, \omega) = \bigoplus_{\alpha\in Y} \ell^1(S_\alpha, \omega_\alpha)$ and $\omega_\alpha = \omega|_{S_\alpha}, \forall\alpha$).
Take $\alpha_\circ \in Y$. It is easy to see that $\bigoplus_{\alpha\leq\alpha_\circ}^{\ell^1} \ell^1(S_\alpha, \omega_\alpha)$ is a closed complemented ideal of $\ell^1(S, \omega)$. By [8, proposition 2.5] , it follows that $\bigoplus_{\alpha\leq\alpha_\circ}^{\ell^1} \ell^1(S_\alpha, \omega_\alpha)$ has a left and right approximate identity. On the other hand, Since $\ell^1(S_{\alpha_\circ}, \omega_{\alpha_\circ})$ is a homomorphic image of $\bigoplus_{\alpha\leq\alpha_\circ}^{\ell^1} \ell^1(S_\alpha, \omega_\alpha)$, We conclude that $\ell^1(S_{\alpha_\circ}, \omega_{\alpha_\circ})$ has a left and right approximate identity. By [12, theorem 1.1.3], it follows that $S_{\alpha_\circ}$ is isomorphic to $L \times R$ where L and R are left and right zero-semigroups, respectively. Indeed we have

$$\ell^1(S_{\alpha_\circ}, \omega_{\alpha_\circ}) \cong \ell^1(L \times R, \omega_{\alpha_\circ}) \cong \ell^1(L, \omega_{\alpha_1})\hat{\otimes}\ell^1(R, \omega_{\alpha_2})$$

That $\omega_{\alpha_1} = \omega|_L,\ \omega_{\alpha_2} = \omega|_R$ as Banach algebras. Thus the map

$\psi : \ell^1(L, \omega_{\alpha_1})\hat{\otimes}\ell^1(R, \omega_{\alpha_2}) \longrightarrow \ell^1(L, \omega_{\alpha_1})//\ \psi(f \otimes g) = \phi_R(g)f//\ f \in \ell^1(L, \omega_{\alpha_1}),\ g \in \ell^1(R, \omega_{\alpha_2})$

is an epimorphism of Banach algebras, where ϕ_R is the augmentation character on $\ell^1(R, \omega_{\alpha_2})$. It follows that $\ell^1(L, \omega_{\alpha_1})$ has a left and right approximate identity. Since L is a left zero semigroup. We conclude that L is singleton. Similary, We can show that R is singleton and so S is isomorphic to Y. The completes the proof. □

Corollary 4.9. *Let S be a uniformly locally finite band semigroup and ω be a weight on S, If $\ell^1(S, \omega)$ is pseudo-amenable, then S is semilattice.*

Theorem 4.10. *Let S be a left cancellative semigroup and ω be a weight on S. Then the following are equivalent:*
(i) $\ell^1(S, \omega)$ is pseudo-amenable.
(ii) S is an amenable group.
(iii) $\ell^1(S, \omega)$ is amenable.

Proof. $(i) \rightarrow (ii)$ Suppose that $(m_\alpha) \subseteq \ell^1(S, \omega) \otimes \ell^1(S, \omega)$ is an approximate diagonal for $\ell^1(S, \omega)$. Then $(\pi(m_\alpha))$ is an approximate identity for $\ell^1(S, \omega)$ and for each $s \in S$ we have

$$\delta_s * \pi(m_\alpha) \longrightarrow \delta_s$$

It follows that there exists $t \in S$ such that $st = s$. Thus for any $z \in S$ we have $stz = sz$. Since S is left cancellative, it follows that $tz = z$ and so t is a left identity for S. It follows that

$$\delta_t * \pi(m_\alpha) = \pi(m_\alpha) \longrightarrow \delta_t,$$

Hence for each $s \in S$ we have

$$\delta_s * \pi(m_\alpha) \longrightarrow \delta_s \text{ and } \delta_s * \pi(m_\alpha) \longrightarrow \delta_{st},$$

It follows that $s = st$ and infact t is the identity of S. On the other hand, by the argument of [3, theorem 1 and corollary 2], it is easy to see that S is a regular semigroup. Thus for each $s \in S$ there exists s^* such that $ss^*s = s = st$ and $s^*ss^*s = s^* = s^*t$.
Now since S is left cancellative we have $s^*s = ss^* = t$. Hence we conclude that S is an amenable group.
$(ii) \rightarrow (iii)$ If S be an amenable group, [10, Section 5], $\ell^1(S, \omega)$ is amenable.
$(iii) \rightarrow (i)$ It is clear. □

Corollary 4.11. *Let S be a left cancellative semigroup with an approximate identity and ω be a weight on S. $\ell^1(S, \omega)$ is pseudo-amenable if and only if $\ell^1(S, \omega)$ be biflat.*

Proof. If $\ell^1(S, \omega)$ is pseudo-amenable, from theorem 4.10, $\ell^1(S, \omega)$ is amenable, then $\ell^1(S, \omega)$ is biflat, conversely by [8, proposition 3.5], is clear. □

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