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Research Paper

CRITICAL POINT APPROACHES FOR A CLASS OF SECOND-ORDER BOUNDARY VALUE PROBLEMS WITH VARIABLE EXPONENTS

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Abstract. The first important discovery on electrorheological fluids was contributed by Willis Winslow in 1949. The viscosity of these fluids depends upon the electric field of the fluids. He discovered that the viscosity of such fluids as instance lithium polymetachrylate in an electrical field is an inverse relation to the strength of the field. The field causes string-like formations in the fluid, parallel to the field. They can increase the viscosity five orders of magnitude. This event is called the Winslow effect. Electrorheological fluids also have functions in robotics and space technology. In this paper, using variational methods and critical point theory, we establish the existence of two and infinitely many solutions for the following boundary value problem with variable exponent

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u(x) = f(x,u), \quad x \in]0,1[, \\ |u'(0)|^{p(0)-2}u'(0) = -g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = -h(u(1)). \end{cases}$$

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1. Introduction

In the present paper, we study the following boundary value problem involving an ordinary differential equation with $p(x)$-Laplacian operator, and nonhomogeneous Neumann conditions:

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u(x) = f(x,u), \quad x \in]0,1[, \\ |u'(0)|^{p(0)-2}u'(0) = -g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = -h(u(1)) \end{cases}$$

where $p \in C([0,1],\mathbb{R})$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continue function, $g, h : \mathbb{R} \to \mathbb{R}$ are nonnegative continuous functions. Moreover, $g, h, \alpha \in L^\infty([0,1])$, with $ess \inf_{[0,1]} \alpha > 0$. The operator $-\Delta_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian which becomes p-Laplacian when $p(x) \equiv p$(a constant). Indeed, the $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. In recent years, the investigation of differential equations and variational problems with variable exponent has become a new and interesting topic. It arises from the nonlinear elasticity theory, the theory of electrorheological fluids, see [15]. The first important discovery

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on electrorheological fluids was contributed by Willis Winslow in 1949. The viscosity of these fluids depends upon the electric field of the fluids. He discovered that the viscosity of such fluids as instance lithium polymetachrylate in an electrical field is an inverse relation to the strength of the field. The field causes string-like formations in the fluid, parallel to the field. They can increase the viscosity five orders of magnitude. This event is called the Winslow effect. Electrorheological fluids also have functions in robotics and space technology. Many experimental researches have been done chiefly in the USA, as in NASA laboratories.

We refer to [2, 3, 4, 8, 9, 11, 12, 16, 17] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems. For example, in [17], Yao by using the variational method, under appropriate assumptions on f and g, obtained a number of results on existence and multiplicity of solutions for the nonlinear Neumann boundary value problem of the following

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u)|u|^{p(x)-2}u = \lambda f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial v} = \mu g(x,u), & \text{on } \partial\Omega, \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$, $p \in C(\overline{\Omega})$ and $p(x) > 1$. Deng in [3], based on a local mountain pass theorem without (PS)- condition and Ricceri's variational principle, obtained the existence and multiplicity of non-trivial solutions for the following $p(x)$-Laplacian double perturbed Neumann problem with nonlinear boundary condition

$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2} = f(x,u) + \lambda h_1(x,u), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \gamma} = g(x,u) + \mu h_2(x,u), & \text{on } \partial\Omega. \end{cases}$$

Where Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary, with $p \in C(\overline{\Omega})$, $p(x) > 1$, $\lambda, \mu \in \mathbb{R}$, $a \in L^\infty(\Omega)$ with $ess\inf_{x\in\Omega} a(x) = a^- > 0$, and γ is the outward unit normal to $\partial\Omega$. D'Aguì in [2] by utilizing variational methods, established the existence of an unbounded sequence of weak solutions for a parametric version of the problem (1.1) contains the parameters $\lambda > 0$ and $\mu \geq 0$. In [9] based on the variational methods and critical-point theory the existence of at least three solutions for the following elliptic problems driven by a $p(x)$-Laplacian, was established

$$\begin{cases} -\Delta_{p(x)}u = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \Omega, \end{cases}$$

and the existence of at least one nontrivial solution was also proved. Where $\Omega \subset \mathbb{R}^N, N \geq 1$, is a nonempty bounded open set with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$ satisfies the condition

$$N < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) < +\infty,$$

$\lambda > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. In [16] using critical point theory and variational methods the existence of at least three weak solutions for the following Neumann problem, originated from a capillary phenomena,

$$\begin{cases} -div\left(\left(1 + \dfrac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary of class C^1, ν is the outer unit normal to $\partial\Omega, \lambda > 0, \mu \geq 0, a \in L^\infty, f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are L^1- Carathéodory functions and $p \in C^0(\overline{\Omega})$ were discussed. In [11] the existence of at least one non-trivial weak solution for a parametric version of the problem (1.1) containing the parameter $\lambda > 0$, under an asymptotical behaviour of the nonlinear datum at zero was obtained.

For further information on the subject we refer the interested reader to the recent papers [10, 5, 6, 7].

Our results here are motivated by the papers [1, 2]. In the present paper we are interested in ensuring the existence of at least two weak solutions and infinitely many weak solutions for the problem (1.1).

The present paper is organized as follows. In Section 2, we recall some basic definitions and our main tools. In Section 3, we state and prove the main Theorem of the paper and finally, we give two examples to show the application of our results.

2. Preliminaries and Basic Notation

In the present section, we introduce some definitions and results used in the next section. First, we introduce some theories of Lebesgue-Sobolev spaces with variable exponent. We assume that $p \in C([0,1], \mathbb{R})$ satisfies condition

$$1 < p^- := \min_{x\in[0,1]} p(x) \leq p^+ := \max_{x\in[0,1]} p(x). \tag{2.1}$$

We define the variable-exponent Lebesgue space

$$L^{p(x)}([0,1]) = \{u : [0,1] \to \mathbb{R} : u \,\text{is measurable and} \int_0^1 |u|^{p(x)}dx < +\infty\},$$

equipped with norm

$$\|u\|_{L^{p(x)}([0,1])} := \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\}.$$

We define the variable-exponent Sobolev space $W^{1,p(x)}([0,1])$ as follows:

$$W^{1,p(x)}([0,1]) := \{u : u \in L^{p(x)}([0,1]), u' \in L^{p(x)}([0,1])\},$$

endowed with norm

$$\|u\|_{W^{p(x)}([0,1])} := \|u\|_{L^{p(x)}([0,1])} + \|\nabla u\|_{L^{p(x)}([0,1])}. \tag{2.2}$$

Both $L^{p(x)}([0,1])$ and $W^{1,p(x)}([0,1])$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces. Moreover, since $\alpha \in L^\infty([0,1])$, with $\alpha_- := ess\inf_{x\in[0,1]} \alpha(x) > 0$ is assumed, the norm

$$\|u\|_\alpha := \inf\{\sigma > 0 : \int_0^1 \left(|\frac{\nabla u(x)}{\sigma}|^{p(x)} + \alpha(x)|\frac{u(x)}{\sigma}|^{p(x)}\right)dx \leq 1\},$$

on $W^{1,p(x)}([0,1])$ is equivalent to that introduced in (2.2). Let $X = W^{1,p(x)}([0,1])$, for any $u \in X$, set

$$\Phi(u) := \int_0^1 \frac{1}{p(x)}\left(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\right)dx$$

and

$$\Psi(u) := \int_0^1 F(x, u(x))dx + [G(u(0)) + H(u(1))]$$

where

$$F(x,t)=\int_0^t f(x,\xi)d\xi \qquad \text{for all } (x,t)\in[0,1]\times\mathbb{R},$$

$$G(t)=\int_0^t g(\xi)d\xi \qquad \text{for all } t\in\mathbb{R}$$

and

$$H(t)=\int_0^t h(\xi)d\xi \qquad \text{for all } t\in\mathbb{R}.$$

We define the functional $\varphi(u)$ on X as

$$\varphi(u)=\Phi(u)-\Psi(u).$$

Proposition 2.1. [2, proposition 2.1] *For all $u\in X$, one has*

$$\|u\|_{C^0([0,1])}\le m\|u\|_\alpha, \tag{2.3}$$

where

$$\begin{cases} 2\left[\dfrac{1}{\alpha_-^{\frac{p^+}{p^-(1-P^+)}}+1}\right]^{1/P^+}+\left[1-\dfrac{1}{\alpha_-^{\frac{p^+}{p^-(1-P^+)}}+1}\right]^{1/P^+}\dfrac{2}{\alpha_-^{1/P^-}}, & \text{if } \alpha_-<1,\\ 2\left[\dfrac{1}{\alpha_-^{\frac{1}{1-p^+}}+1}\right]^{1/P^+}+\left[1-\dfrac{1}{\alpha_-^{\frac{1}{1-p^+}}+1}\right]^{1/P^+}\dfrac{2}{\alpha_-^{1/P^+}} & \text{if } \alpha_-\ge 1.\end{cases}$$

Definition 2.2. We say that $u:[0,1]\to\mathbb{R}$ is a weak solution to (1.1) if $u\in X$ and

$$\int_0^1 |\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)dx+\int_0^1 \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)dx$$
$$-\int_0^1 f(x,u(x))v(x)dx-[g(u(0))v(0)+h(u(1)v(1)]=0,$$

for all $v\in X$.

In the sequel, we will mention some auxiliary results used through the paper.

Definition 2.3. Let E be a real reflexive Banach space. If any sequence $\{u_k\}\subset E$ for which $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k)\to 0$ as $k\to+\infty$ possesses a convergent subsequence, then we say φ satisfies Palais-Smale condition.

Theorem 2.4. [13, Theorem 4.4] *Let X be a Banach space and $\varphi: X\to\mathbb{R}$ be a function bounded from below and differentiable on X. If φ satisfies the $(PS)_c$-condition with $c=\inf_X\varphi$, then φ has a minimum on X.*

It is clear that the (PS)-condition implies the $(PS)_c$-condition for each $c\in\mathbb{R}$.

Theorem 2.5. [13, Theorem 4.10] *Let $\varphi\in C^1(X,\mathbb{R})$, and φ satisfies the Palais-Smale condition. Assume that there exist $u_0,u_1\in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1\notin\Omega$ and*

$$\inf_{\nu\in\partial\Omega}\varphi(\nu)>\max\{\varphi(u_0),\varphi(u_1)\},$$

then there exists a critical point u of φ, i.e. $\varphi'(u)=0$ with $\varphi(u)>\max\{\varphi(u_0),\varphi(u_1)\}$.

Theorem 2.6. [14, Theorem 9.12] *Let E be an infinite dimensional real Banach space. Let $\varphi \in C^1(E, \mathbb{R})$ be an even functional which satisfies the (PS) condition, and $\varphi(0) = 0$. Suppose that $E = V \bigoplus X$, where V is finite dimensional, and φ satisfies that*

(i) *There exist $\alpha > 0$ and $\rho > 0$ such that $\varphi(u) \geq \alpha$ for all $u \in X$ with $\| u \| = \rho$,*
(ii) *For any finite dimensional subspace $W \subset E$ there is $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \setminus B_R$.*

Then φ possesses an unbounded sequence of critical values.

Theorem 2.7. [18, Theorem 38] *For the functional $F : M \subseteq X \longrightarrow [-\infty, +\infty]$ with $M \neq \emptyset, \min_{u\in M} F(u) = \alpha$ has a solution in case the following conditions hold:*

(h_1) *X is a real reflexive Banach space,*
(h_2) *M is bounded and weak sequentially closed,*
(h_3) *F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $F(u) \leq \lim_{n\to\infty} \inf F(u_n)$ holds.*

We refer to the paper [1, 19] in which Theorems 2.5 and 2.6 have been successfully employed to prove the multiple solutions of nonlinear impulsive differential equations with Dirichlet boundary conditions and the existence of solutions for a class of degenerate nonlocal problems involving sub-linear nonlinearities, respectively.

3. Main results

We utilize the following assumptions throughout this paper.

(A_1) there exist constants $\nu > p^+$ and $T > 0$ such that $0 < \nu F(x, t) \leq t f(x, t)$, $|t| > T$.
(f_0) $f(x, t) = o(|t|)$, $t \longrightarrow 0$, for $x \in [0, 1]$ uniformly.
(G_0) there exist constants $K_1 > 0$, $\vartheta_1 > 1$ and $\mu \geq \nu$ such that $G(t) \leq \frac{1}{\mu} g(t)t$, for $|t| > K_1$ and $G(t) < |t|^{\vartheta_1}$, for $|t| \leq K_1$.
(H_0) there exist constants $K_2 > 0$, $\vartheta_2 > 1$ and $\theta \geq \nu$ such that $H(t) \leq \frac{1}{\theta} h(t)t$, for $|t| > K_2$ and $H(t) < |t|^{\vartheta_2}$, for $|t| \leq K_2$.

The main result of this paper is the following theorem.

Theorem 3.1. *Assume that the assumptions (A_1), (f_0), (G_0) and (H_0) hold. Then: if $f(x, t) \geq 0$ for all $(x, t) \in [0, 1] \times \mathbb{R}$, the problem (1.1) has at least two weak solutions.*

Theorem 3.2. *Assume that the assumptions (A_1), (G_0) and (H_0) hold. Then: if $f(x, t)$ is odd about t, the problem (1.1) has infinitely many weak solutions.*

We need the following lemma to prove our main results.

Lemma 3.3. *Assume that (A_1), (G_0) and (H_0) hold. Then $\varphi(u)$ satisfies the (PS)-condition.*

Proof. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|\varphi(u_n)| \leq c_0$ and $|\varphi'(u_n)| \leq c_0$ for all $n \in \mathbb{N}$. Since $\mu \geq \nu$, from G_0 implies $\nu G(u_n(0)) < g(u_n(0))u_n(0)$, also, since $\theta \geq \nu$, from H_0 implies $\nu H(u_n(1)) < h(u_n(1))u_n(1)$, Therefore, from the definition of φ' and the assumption (A_1) and for some $c_1 > 0$ we have,

$$
\begin{aligned}
c_0 + c_1\|u_n\| \geq \nu\varphi(u_n) - \varphi'(u_n)(u_n) \\
&\geq (\frac{\nu}{p^+} - 1)\|u_n\|_\alpha - \nu\int_0^t F(x, u_n(x))dx \\
&+ \int_0^t f(x, u_n(x))u_n dx - \nu G(u_n(0)) - \nu H(u_n(1)) \\
&+ g(u_n(0))u_n(0) + h(u_n(1))u_n(1) \\
&\geq (\frac{\nu}{p^+} - 1)\|u_n\|_\alpha.
\end{aligned}
$$

Since $\nu > p^+$ this implies that (u_n) is bounded. Consequently, since X is a reflexive Banach space we have, up to a subsequence,

$$u_n \rightharpoonup u \ \ in \ \ X.$$

By $\varphi'(u_n) \to 0$ and $u_n \rightharpoonup u$ in X, we obtain

$$(\varphi'(u_n) - \varphi'(u)(u_n - u) \to 0.$$

From the continuity of f, g and h, we have

$$
\begin{aligned}
\int_0^1 (f(x, u_n(x)) - f(x, u(x))(u_n(x) - u(x))dx \to 0, \\
(g(u_n(0)) - g(u(0))(u_n(0) - u(0)) \to 0
\end{aligned}
$$

and

$$(h(u_n(1)) - h(u(1))(u_n(1) - u(1)) \to 0.$$

Moreover, an easy computation shows

$$
\begin{aligned}
&(\varphi'(u_n) - \varphi'(u)(u_n - u) = \int_0^1 \Big((\nabla u_n(x)|^{p(x)-2}\nabla u_n(x))(u_n''(x) - u''(x)) \\
&+ \alpha(x)|u_n(x)|^{p(x)-2}u_n(x)(u_n''(x) - u''(x))\Big)dx \\
&- \int_0^1 \Big(\nabla u(x)|^{p(x)-2}\nabla u(x))(u_n''(x) - u''(x)) \\
&- \alpha(x)|u(x)|^{p(x)-2}u(x)(u_n(x) - u(x))\Big)dx \\
&- \int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))dx \\
&- (g(u_n(0)) - g(u(0))(u_n(0) - u(0)) - (h(u_n(1)) - h(u(1))(u_n(1) - u(1)) \\
&\geq \beta\|u_n - u\|,
\end{aligned}
$$

where $\beta > 0$. Therefore, $\{u_n\}$ converges strongly to u in X. Consequently, φ satisfies the (PS)-condition. □

To obtain main results is needed that we prove the following Lemma.

Lemma 3.4. *([20],Lemma 2.2) If condition (A_1) holds, then for every $x \in [0,1]$, the following inequalities hold:*

$$F(x,t) \le F(x,\frac{t}{|t|})|t|^{\nu}, if \ \ 0<|t|\le 1;$$

$$F(x,t) \ge F(x,\frac{t}{|t|})|t|^{\nu}, if \ \ |t|\ge 1.$$

In view Lemma 3.4 and f_0 implies that for every $x \in [0,1]$,

$$F(x,t) \le a_3|t|^{\nu}, if \ \ |t|\le 1,$$

$$F(x,t) \ge a_1|t|^{\nu}, if \ \ |t|\ge 1, \tag{3.1}$$

where $a_3 = \max_{x\in[0,1],|t|=1} F(x,t)$ and $a_1 = \min_{x\in[0,1],|t|=1} F(x,t)$. By assumption (f_0), we imply $a_1, a_3 > 0$. In addition, since $F(x,t) - a_1|t|^{\nu}$ is continuous on $[0,1]\times[0,T]$, there exists a constant $a_2 > 0$ such that

$$F(x,t) \ge a_1|t|^{\nu} - a_2 \ \ for \ \ all \ \ (x,t)\in[0,1]\times[0,T]. \tag{3.2}$$

so it follows from (3.1) and (3.2) that

$$F(x,t) \ge a_1|t|^{\nu} - a_2 \ \ for \ \ all \ \ (x,t)\in[0,1]\times\mathbb{R}. \tag{3.3}$$

In this paper the main results are the following.

3.1. **The proof of Theorem 3.1.**

Proof. In our case it is clear that $\varphi(0) = 0$. Lemma 3.7 shows that φ satisfies the (PS)-condition.

Step 1. We will show that there exists $M > 0$ such that the functional φ has a local minimum $u_0 \in B_M = \{u \in X; \|u\|_{\alpha} < M\}$. Let $\{u_n\} \subseteq \overline{B}_M$ and $u_n \rightharpoonup u$ as $n \to \infty$, by Mazur Theorem [12], there exists a sequence of convex combinations

$$v_n = \sum_{j=1}^{n} a_{n_j} u_j, \quad \sum_{j=1}^{n} a_{n_j} = 1, \qquad a_{n_j} \ge 0, \ j \in N$$

such that $v_n \to u$ in X. Since $\overline{B}_M$ is a closed convex set, we have $\{v_n\} \subseteq \overline{B}_M$ and $u \in \overline{B}_M$. Noting that φ is weak sequentially lower semi-continuous on $\overline{B}_M$ and X is a reflexive Banach space. Then by Theorem 2.7 we can know that φ has a local minimum $u_0 \in \overline{B}_M$. We assume that $\varphi(u_0) = min_{u\in\overline{B}_M}\varphi(u)$. Now we will show that

$$\varphi(u_0) < inf_{u\in\partial B_M}\varphi(u).$$

Since $\nu > p^+ > 1$ and

$$\|u\|_{\infty} \le m\|u\|_{\alpha}, \quad u \in X.$$

When $\|u\|_{\alpha} \to 0$, from (f_0) implies there exits $\varepsilon > 0$ such that

$$F(x,t) \le \varepsilon|t|^2, \tag{3.4}$$

therefore

$$\begin{aligned}\varphi(u) &\ge \frac{1}{p^+}\|u\|_{\alpha} - \varepsilon\int_0^1 |u|^2 - [G(u(0)) + H(u(1))] \\ &\ge \frac{1}{p^+}\|u\|_{\alpha} - \varepsilon m^2\|u\|_{\alpha}^2 - \|u\|_{\infty}^{\vartheta_1} - \|u\|_{\infty}^{\vartheta_2},\end{aligned}$$

since $\vartheta_1, \vartheta_2 > 1$. Therefore, there exist $r > 0$, $\delta > 0$ such that $\varphi(u) \geq \delta > 0$ for every $\|u\|_\alpha = r$, We choosing $M = r$, so $\varphi(u) > 0 = \varphi(0) \geq \varphi(u_0)$ for $u \in \partial B_M$. Hence $u_0 \in B_M$ and $\varphi'(u_0) = 0$.

Step 2. Since u_0 is a minimum point of φ on X, we can consider $M > 0$ sufficiently large such that $\varphi(u_0) \leq 0 < \inf_{u\in\partial B_M} \varphi(u)$, where $B_M = \{u \in X; \|u\|_\alpha < M\}$. Now we will illustrate that there exists u_1 with $\|u_1\|_\alpha > M$ such that $\varphi(u_1) < \inf_{\partial B_M} \varphi(u)$. For this, let $e_1(x) \in X$ and $u_1 = re_1, r > 0$ and $\|e_1\|_\alpha = 1$. By (3.3) there exist constants $a_1, a_2, b_1, b_2, c_1, c_2 > 0$ such that $F(x,t) \geq a_1|t|^\nu - a_2$, $G(t) \geq b_1|t|^\mu - b_2$ and $H(t) \geq c_1|t|^\theta - c_2$ for all $x \in [0,1]$. Thus

$$
\begin{aligned}
\varphi(u_1) &= (\Phi - \Psi)(re_1) \\
&\leq \frac{1}{p^-}\|re_1\|_\alpha - \int_0^1 F(x, re_1(x))dx - [G(re_1(0)) + H(re_1(1))] \\
&\leq \frac{1}{p^-} r\|e_1\|_\alpha - r^\nu a_1 \int_0^1 |e_1|^\nu dx + a_2 - b_1 r^\mu |e_1(1)|^\mu + b_2 \\
&- c_1 r^\theta |e_1(0)|^\theta + c_2.
\end{aligned}
$$

Since $\nu > p^+ > 1$, there exists sufficiently large $r > M > 0$ so that $\varphi(re_1) < 0$. Hence, $\max\{\varphi(u_0), \varphi(u_1)\} < \inf_{\partial B_M} \varphi(u)$. Then, Theorem 2.5 gives the critical point u^*. Therefore, u_0 and u^* are two critical points of φ, which are two solutions of the problem (1.1). □

Now, we give an illustrating example for Theorem 3.1.

Example 3.5. Consider $p(x) = x^4 + x + 2$, therefore, $p(x) \in C([0,1], \mathbb{R})$, $p^- = 2$, and $p^+ = 4$. Let

$$
f(x,t) = \begin{cases} 5t^4 & |t| > 1, \\ 5t^2 & |t| \leq 1. \end{cases}
$$

For $(x,t) \in [0,1] \times \mathbb{R}$. By the expression of f, we have

$$
F(x,t) = \begin{cases} t^5 & |t| > 1, \\ \frac{5}{3}t^5 - \frac{2}{3} & |t| \leq 1. \end{cases}
$$

Moreover, $f(x,t) = O(|t|)$ as $t \to 0$ and $f(x,t) > 0$ for $t \in \mathbb{R}$. Also $\lim_{|t|\to\infty} \frac{tf(x,t)}{F(x,t)} = 5$, by choosing $\nu = 5$ and $T = 1$ we have $\nu > p^+$ and $5F(x,t) \leq tf(x,t)$ for $|t| > 1$. Let $g(t) = h(t) = t^6$ we have,

$$
G(t) = H(t) = \frac{t^7}{7}.
$$

By selecting $K_1 = K_2 = 1$, $\vartheta_1 = \vartheta_2 = 7 > 1$ and $\mu = \theta = 7 \geq 5 = \nu$, we have $G(t) \leq \frac{1}{7}g(t).t$ for $|t| > 1$, and $G(t) \leq |t|^7$ for $|t| \leq 1$. Also $H(t) \leq \frac{1}{7}h(t).t$ for $|t| > 1$, and $H(t) \leq |t|^7$ for $|t| \leq 1$. So we see that all conditions in Theorem 3.1 are satisfied, therefore, the problem

$$
\begin{cases} -(|u'(x)|^{x^2+x}u'(x))' + \alpha(x)|u(x)|^{x^2+x}u(x) = f(x,u), & x \in]0,1[\\ u'(0) = -(u(0))^6, \\ |u'(1)|^2 u'(1) = -(u(1))^6, \end{cases}
$$

has at least two weak solutions.

3.2. **The proof of Theorem 3.2.**

Proof. According to definitions of the functionals Φ and Ψ, it is clear that $\varphi(u)$ is even and $\varphi(0) = 0$.
Step 1. We will depict that φ satisfies condition (i) in Theorem 2.6. Since, φ is coercive and also satisfies (PS)-condition, by the minimization theorem [13, Theorem 4.4] the functional φ has a minimum critical point u with $\varphi(u) \geq \alpha > 0$ and $\|u\|_\alpha = \rho$ for $\rho > 0$ small enough.
Step 2. We will show that φ satisfies condition (ii) in Theorem 2.6. Let $W \subset X$ be a finite dimensional subspace. By (3.3) there exist constants $a_1, a_2, b_1, b_2, c_1, c_2 > 0$ such that $F(x,t) \geq a_1|t|^\nu - a_2$, $G(t) \geq b_1|t|^\mu - b_2$ and $H(t) \geq c_1|t|^\theta - c_2$ for all $x \in [0,1]$. Now, For every $r > 0$ and $u \in W \setminus \{0\}$ with $\|u\|_\alpha = 1$, one has

$$\begin{aligned}\varphi(ru) &= (\Phi - \Psi)(ru)\\ &\leq \frac{1}{p^-}\|ru\|_\alpha - \int_0^1 F(x, ru(x))dx - [G(ru(0)) + H(ru(1))]\\ &\leq \frac{1}{p^-} r\|u\|_\alpha - r^\nu a_1 \int_0^1 |u|^\nu dx - a_2 - b_1 r^\mu |u(1)|^\mu + b_2\\ &- c_1 r^\theta |u(0)|^\theta + c_2 \to -\infty, \ \ r \to +\infty.\end{aligned}$$

The above inequality implies that there exists r_0 such that $\|ru\|_\alpha > \rho$ and $\varphi(ru) < 0$ for every $r \geq r_0 > 0$. Since W is a finite dimensional subspace, there exists $R = R(W) > 0$ such that $\varphi(u) \leq 0$ on $W \setminus B_{R(W)}$. According to Theorem 2.6, the functional $\varphi(u)$ possesses infinitely many critical points, i.e., the problem (1.1) has infinitely many weak solutions. □

We end this paper by presenting the following illustrating example for Theorem 3.2.

Example 3.6. Let $p(x) = x^2 + 2$ then $p \in C([0,1], \mathbb{R})$, $p^- = 2$ and $p^+ = 3$. By considering

$$f(x,t) = \begin{cases} t^5, & |t| > 1,\\ \sin(\frac{\pi}{2}t), & |t| \leq 1.\end{cases}$$

for $(x,t) \in [0,1] \times \mathbb{R}$ we have,

$$F(x,t) = \begin{cases} \frac{t^6}{6}, & |t| > 1,\\ \frac{\pi}{2}\cos(\frac{\pi}{2}t) + \frac{1}{6}, & |t| \leq 1.\end{cases}$$

By choosing $\nu = 6$ and $T = 1$ we have $\nu > p^+$ and $6F(x,t) \leq tf(x,t)$ for $|t| > 1$. Let $h(t) = g(t) = t^8$ thus $G(t) = H(t) = \frac{t^9}{9}$, by selecting $K_1 = K_2 = 1$ and $\theta = \mu = 9 \geq 6 = \nu$ we have $G(t) \leq \frac{1}{9}g(t)t$ for $|t| > 1$, and $H(t) \leq \frac{1}{9}h(t)t$ for $|t| > 1$. Also by electing $\vartheta_1 = \vartheta_2 = 8 > 1$, we have $G(t) \leq |t|^8$ for $|t| \leq 1$ and $H(t) \leq |t|^8$ for $|t| \leq 1$. Also f, g, h are three continuous functions and f is odd about t. We see that all conditions in Theorem 3.2 are satisfied. Thus, the problem

$$\begin{cases} -(|u'(x)|^{x^2}u'(x))' + \alpha(x)|u(x)|^{x^2}u(x) = f(x,u), \ \ x \in]0,1[\\ u'(0) = -(u(0))^8,\\ |u'(1)|u'(1) = -(u(1))^8,\end{cases}$$

has infinitely many weak solutions.

Theorem 3.7. *Assume that (A_1) and the following assumptions hold.*

(G_1) *there exist constants* $K_1' > 0$ *and* $\mu' \geq \nu$ *such that* $G(t) \leq \frac{1}{\mu} g(t)t$, *for* $|t| > K_1'$.

(H_1) *there exist constants* $K_2' > 0$ *and* $\theta' \geq \nu$ *such that* $H(t) \leq \frac{1}{\theta} h(t)t$, *for* $|t| > K_2'$.

Then: if $f(x,t)$ *is odd about* t*, the problem* (1.1) *has infinitely many weak solutions.*

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