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Research Paper

RECONSTRUCTION OF DISCONTINUOUS STURM-LIOUVILLE PENCILS WITH THE EIGENVALUE IN THE BOUNDARY CONDITION ON THE HALF-LINE

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Abstract. This work considers the discontinuous differential pencil on the half-line with the spectral boundary condition. We establish some uniqueness theorems on the potentials for the Sturm-Liouville pencil by the incomplete inverse problem and the interior inverse problem methods. We determine the potentials by only a set of eigenvalues knowing the coefficients $\beta_0, \beta_1, \beta_2, \beta_3$ and potentials $q_0(x), q_1(x)$ on $(0, a)$. We also establish the potentials by a set of values of eigenfunctions at some internal point $x = a$ and eigenvalues.

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1. Introduction

Inverse spectral problems that are recovering the operator from its given spectral data are used in various branches of the physical science and engineering. Boundary value problems with discontinuities often appear in branches of sciences like mathematics, mechanics, physics, geophysics (see [3, 16]).

In this article, we investigate the boundary value problem $B := B(q_0, q_1, \beta_0, \beta_1, \beta_2, \beta_3)$ for the Sturm-Liouville pencil

$$y''(x) + (\rho^2 + i\rho\, q_1(x) + q_0(x))y(x) = 0, \quad x \geq 0, \tag{1.1}$$

where complex-valued functions $q_0(x)$ and $q_1(x)$ satisfy $q_l^{(\iota)}(x) \in L(0, \infty)$ as $0 \leq \iota \leq l \leq 1$, $q_1(x)$ is absolutely continuous and ρ is the spectral parameter, and with the spectral boundary condition

$$U(y) := (\beta_1\rho + \beta_0)y'(0) + (\beta_3\rho + \beta_2)y(0) = 0, \tag{1.2}$$

where the coefficients $\beta_0, \beta_1, \beta_2, \beta_3$ are complex numbers, and with the transmission conditions

$$y(a + 0, \rho) = a_1 y(a - 0, \rho), \quad y'(a + 0, \rho) = a_1^{-1} y'(a - 0, \rho) + a_2 y(a - 0, \rho), \tag{1.3}$$

where $a_1 (a_1 \neq \pm 1, \pm i)$ and a_2 are complex numbers.

Hochstadt-Lieberman showed that one spectrum is suffice to determine the potential $q(x)$ on the whole interval provided that $q(x)$ is known a priori on the half of the interval [8]. Taking this method, some researchers investigated inverse problems for the Sturm-Liouville

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operator [10, 17, 22]. In [4, 18] it can be also seen the incomplete inverse problem that recovers the potential $q(x)$ using a part of the spectrum prescribed that the potential $q(x)$ is known on a part of the interval. Mochizuki and Trooshin discussed interior inverse problems and showed that one spectrum and a part of information on eigenfunctions at some interior point suffice to determine the potential $q(x)$ [12]. This technique has been used in many articles for studying the inverse problem for Sturm-Liouville equations [15, 19, 20, 21]. Inverse spectral problems for the Sturm-Liouville operator with eigenvalue dependent boundary conditions were investigated in [1, 19, 20]. Inverse problems for Sturm-Liouville equations with discontinuities inside the interval were also studied in [13, 18]. Some aspects of the inverse problem for Sturm-Liouville pencils were investigated in [4, 9, 20, 21]. However, in this work, we try to prove inverse problems for the discontinuous Sturm-Liouville pencil with the spectral boundary condition on the half line. We establish the incomplete inverse problem for the problem B and recover B by taking eigenvalues provided that the potentials are known a priori on a part of the interval. In the other theorem, we prove the interior inverse problem for B and show that a set of values of eigenfunctions at some interior point $x = a$ and eigenvalues are sufficient to reconstruct the problem B.

The paper is organized as follows. In Sec. 2, some properties of spectral characteristics are given. In Sec. 3, we give two uniqueness theorems for B taking the incomplete and interior inverse problem methods. The techniques employed in this paper are a extension of Hochstadt-Lieberman's and Mochizuki-Trooshin's theorems [8, 12].

2. Preliminaries

Let $\lambda = \rho^2$, and let for definiteness $\Im\rho \geq 0$. Denote by Π the λ-plane with the cut $\lambda > 0$. Then Π corresponds to the domain $\Pi_+ := \{\rho : \Im\rho > 0\}$ and $\Pi := \overline{\Pi_+} \setminus \{0\}$. We know from the technique in [6, 11] that the Jost solution of (1.1) satisfying the condition

$$\lim_{x\to\infty} e(x,\rho) = 0, \quad \rho \in \Pi, \tag{2.1}$$

has the following form for uniformly in $x > a$ and sufficiently large $\rho \in \Pi$,

$$e^{(m)}(x,\rho) = (i\rho)^m exp(i\rho x - Q(x))\lceil 1 \rfloor, \quad m = 0,1, \tag{2.2}$$

where $Q(x) = \frac{1}{2}\int_0^x q_1(t)dt$ and $\lceil 1 \rfloor := 1 + O\left(\rho^{-1}\right).$

For every fixed $x > a$, the functions $e^{(m)}(x,\rho)$, $m = 0,1$, are holomorphic and continuous for $\rho \in \Pi_+$ and $\rho \in \overline{\Pi_+}$, respectively. Also these functions are continuously differentiable for $\rho \in \Pi$.

Moreover the integral form of the Jost solution has the following presentation

$$e(x,\rho) = \exp(i\rho x - Q(x)) + \int_x^\infty \mathcal{M}_1(x,t)\exp(i\rho t)dt, \quad x \geq a, \tag{2.3}$$

for a bounded function $\mathcal{M}_1(x,t)$ [2, 21].

Using the Birkhoff fundamental system of solutions

$$E_k^{(m)}(x,\rho) = ((-1)^{k-1}i\rho)^m \exp((-1)^{k-1}(i\rho x - Q(x)))\lceil 1 \rfloor, \ \rho \in \Pi, \quad x \geq 0,$$

for $k = 1, 2$ [14, 23], we can write

$$\begin{aligned} e^{(m)}(x,\rho) &= \frac{(i\rho)^m}{2a_1}\exp(i\rho a - Q(a))\Big((a_1^2+1)\exp\big(i\rho(x-a)-(Q(x)-Q(a))\big)\lceil 1\rfloor \\ &\quad +(-1)^m(a_1^2-1)\exp\big(-i\rho(x-a)+(Q(x)-Q(a))\big)\lceil 1\rfloor\Big), \quad x\in[0,a) \end{aligned} \tag{2.4}$$

(see [9] for more details). The properties of the Jost solution $e(x, \rho)$ in the interval $[0, a)$ remain true.
The integral form of this solution has the following presentation

$$\begin{aligned} e(x,\rho) &= \frac{1}{2a_1}\exp(i\rho a - Q(a))\Big((a_1^2+1)\exp\big(i\rho(x-a)-(Q(x)-Q(a))\big) \\ &\quad +(a_1^2-1)\exp\big(-i\rho(x-a)+(Q(x)-Q(a))\big)\Big) \\ &\quad +\int_0^x \mathcal{M}_2(x,t)\exp(i\rho t)dt + \int_0^x \mathcal{M}_3(x,t)\exp(i\rho(2a-t))dt, \quad 0\le x\le a, \end{aligned} \tag{2.5}$$

for bounded functions $\mathcal{M}_2(x, t)$ and $\mathcal{M}_3(x, t)$ [2, 21].

The eigenvalues of B coincide with the roots of the characteristic function

$$\Delta(\rho) = (\beta_1\rho + \beta_0)e'(0, \rho) + (\beta_3\rho + \beta_2)e(0, \rho), \tag{2.6}$$

in the plane Π_+. This function is entire in $\rho \in \Pi_+$ and taking (2.4), we will have for sufficiently large $\rho \in \Pi$,

$$\begin{aligned} \Delta(\rho) &= \frac{\beta_1 i\rho^2}{2a_1}\exp(i\rho a - Q(a))\Big((a_1^2+1)\exp(-i\rho a + Q(a))\lceil 1\rfloor \\ &\quad -(a_1^2-1)\exp(i\rho a - Q(a))\lceil 1\rfloor\Big). \end{aligned} \tag{2.7}$$

By the Rouche's theorem [5] and the known technique [7], one can give that these roots have the form

$$\rho_n = \frac{1}{a}\left(n\pi - iQ(a) + \kappa_1\right) + O\left(n^{-1}\right), \tag{2.8}$$

for large enough n, wherein

$$\kappa_1 = \frac{1}{2i} ln\frac{a_1^2+1}{a_1^2-1}.$$

Put $\Lambda_+ = \{\rho \in \Pi_+; \Delta(\rho) = 0\}$. For $\rho_n \in \Lambda_+$, we denote $\lambda_n = \rho_n^2$ and $y_n(x, \rho) = y(x, \rho_n)$ the eigenvalues and the eigenfunctions of B, respectively.

Denote $G_\delta := \{\rho \in \Pi_+; \mid \rho - \rho_n \mid \geq \delta,\ \forall n\}$ for $\delta > 0$ and a constant $C > 0$. Applying (2.7) and the known method [7], one can prove that

$$|\Delta(\rho)| \geq C|\rho^2|, \tag{2.9}$$

for sufficiently large $\rho \in G_\delta$.

To show the uniqueness theorems in the next sections, alongside $B := B(q_1, q_0, \beta_3, \beta_2, \beta_1, \beta_0)$, a boundary value problem $\widetilde{B} := B(\widetilde{q}_1, \widetilde{q}_0, \beta_3, \beta_2, \beta_1, \beta_0)$ of the similar form (1.1)-(1.3) is considered. We suppose that if α signifies an object relevant to B, then $\widetilde{\alpha}$ will signify the similar object relevant to $\widetilde{B}$.

3. Main results

In this section, we state the main results of this work in two theorems. We will prove the Hochstadt-Lieberman type theorem and the Mochizuki-Trooshin type theorem of (1.1)-(1.3).

In the following theorem, we first give the so-called incomplete inverse problem for B and show that the potentials $q_1(x), q_0(x)$ are determined from eigenvalues provided that the potentials $q_1(x), q_0(x)$ are known a priori on $(0, a)$.

Theorem 3.1. *Let $q_1(x) = \widetilde{q}_1(x)$ and $q_0(x) = \widetilde{q}_0(x)$ a.e. on $(0, a)$. Then the set of the eigenvalues uniquely establishes $q_1(x)$ and $q_0(x)$ a.e. on $x \geq 0$.*

Proof. Let $e(x, \rho)$ be the solution of (1.1) and $\widetilde{e}(x, \rho)$ be the solution of this equation with tilde. Multiplying (1.1) by $\widetilde{e}(x, \rho)$ and the secondary equation by $e(x, \rho)$ and subtracting, we will have

$$\big(q_0(x) - \widetilde{q}_0(x) + i\rho(q_1(x) - \widetilde{q}_1(x))\big)e(x,\rho)\widetilde{e}(x,\rho) = e'(x,\rho)\widetilde{e}(x,\rho) - e(x,\rho)\widetilde{e}'(x,\rho). \tag{3.1}$$

Now by integrating the above equation on $[0, \infty)$, one gets

$$\int_0^\infty \big(q_0(x) - \widetilde{q}_0(x) + i\rho(q_1(x) - \widetilde{q}_1(x))\big)e(x,\rho)\widetilde{e}(x,\rho)dx = (e'(x,\rho)\widetilde{e}(x,\rho) - e(x,\rho)\widetilde{e}'(x,\rho))|_0^a + |_a^\infty.$$

Taking the hypothesis of the theorem and (2.1), we can infer that

$$\begin{aligned} H_a(\rho) &:= \int_a^\infty \big(Q_0(x) + i\rho Q_1(x,\rho)\big)e(x,\rho)\widetilde{e}(x,\rho)dx \\ &= e(0,\rho)\widetilde{e}'(0,\rho) - e'(0,\rho)\widetilde{e}(0,\rho), \end{aligned} \tag{3.2}$$

where $Q_0(x) = q_0(x) - \widetilde{q}_0(x)$ and $Q_1(x) = q_1(x) - \widetilde{q}_1(x)$. Therefore

$$H_a(\rho) = e(0,\rho)\widetilde{e}'(0,\rho) - e'(0,\rho)\widetilde{e}(0,\rho).$$

Using the assumption of the theorem, we will have $H_a(\rho_n) = 0,\ \rho_n \in \Lambda_+$. Thus it is sufficient to show that $H_a(\rho) = 0$ for other ρ in Π_+.

Let $\mathcal{Q}_+(x) = Q(x) + \widetilde{Q}(x)$. The relation (2.3) gives that

$$\begin{aligned} e(x,\rho)\widetilde{e}(x,\rho) &= \exp(2i\rho x - \mathcal{Q}_+(x)) \\ &+ \int_x^\infty \mathbf{M}_1(x,t)\exp(2i\rho t - \mathcal{Q}_+(t))dt, \qquad x \geq a, \end{aligned} \tag{3.3}$$

and therefore we have

$$|e(x,\rho)\widetilde{e}(x,\rho)| \leq \mathcal{C}\exp(-2|\Im\rho|x), \quad x \geq a,$$

for some constant $\mathcal{C} > 0$. Taking (3.2) and this result, we can write that

$$|H_a(\rho)| \leq \mathbf{C}|\rho|\exp(-2\Im\rho a), \tag{3.4}$$

for some constant $\mathbf{C} > 0$. Now considering the function

$$G(\rho) = \frac{H_a(\rho)}{\Delta(\rho)},$$

which is analytic for $\rho \in \Pi_+ \setminus \Lambda_+$ and the relations (2.9) and (3.4), we will have that $G(\rho) = 0$ and therefore $H_a(\rho) = 0$ for other ρ in Π_+.

From (3.2) and $H_a(\rho) = 0$, we obtain for all ρ,

$$\int_a^\infty \big(Q_0(x) + i\rho Q_1(x)\big)e(x,\rho)\widetilde{e}(x,\rho)dx = 0. \tag{3.5}$$

Substituting (3.3) in (3.5) and then rewriting it, we get for $t > a$,

$$\int_a^\infty \exp(2i\rho t - \mathcal{Q}_+(t)) \left(Q_0(t) + \int_a^t Q_0(x)\mathbf{M}_1(x,t)dx\right) dt$$
$$+i\rho \int_a^\infty \exp(2i\rho t - \mathcal{Q}_+(t)) \left(Q_1(t) + \int_a^t Q_1(x)\mathbf{M}_1(x,t)dx\right) dt = 0.$$

Taking Riemann-Lebesgue lemma, one gives

$$\int_a^\infty \exp(2i\rho t - \mathcal{Q}_+(t)) \left(Q_\nu(t) + \int_a^t Q_\nu(x)\mathbf{M}_1(x,t)dx\right) dt = 0, \quad \nu = 0, 1,$$

for sufficiently large ρ. From the completeness of the exponential function [7, 22], we can write

$$Q_\nu(t) + \int_a^t Q_\nu(x)\mathbf{M}_1(x,t)dx = 0, \quad \nu = 0, 1,$$

which are homogeneous Volterra integral equations with the trivial solution, i.e., $Q_0(x) = Q_1(x) = 0$, for $x > a$. Thus we result that $q_1(x) = \widetilde{q}_1(x)$ and $q_0(x) = \widetilde{q}_0(x)$ a.e. on $x > a$. The proof is completed. □

Remark 3.2. Consider the functions $y(x,\rho)$ and $z(x,\rho)$ as the solutions of the boundary value problem B. The Wronskian of the functions $y(x,\rho)$ and $z(x,\rho)$ at the neighborhood of $x = a$ is equal. In other words, $< y(x,\rho), z(x,\rho) >_{|a-0} = < y(x,\rho), z(x,\rho) >_{|a+0}$.

In the following theorem, we state the interior inverse problem for B and prove the uniqueness theorem using its eigenvalues and the properties of the eigenfunctions in an interior point $x = a$.

Theorem 3.3. *Let for the eigenvalues λ_n,*

$$\lambda_n = \widetilde{\lambda}_n, \quad < e_n(x,\rho), \widetilde{e}_n(x,\rho) >_{|a-0} = 0.$$

Then $q_1(x) = \widetilde{q}_1(x)$ and $q_0(x) = \widetilde{q}_0(x)$ a.e. on $x \geq 0$.

Proof. Integrating (3.1) on $[0,a)$, we have

$$\int_0^a \big(q_0(x) - \widetilde{q}_0(x) + i\rho(q_1(x) - \widetilde{q}_1(x))\big)e(x,\rho)\widetilde{e}(x,\rho)dx = \big(e'(x,\rho)\widetilde{e}(x,\rho) - e(x,\rho)\widetilde{e}'(x,\rho)\big)|_0^a.$$

So

$$\begin{aligned} H_0(\rho) &:= \int_0^a \big(Q_0(x) + i\rho Q_1(x)\big)e(x,\rho)\widetilde{e}(x,\rho)dx \\ &= \big(e'(a,\rho)\widetilde{e}(a,\rho) - e(a,\rho)\widetilde{e}'(a,\rho)\big) - \big(e'(0,\rho)\widetilde{e}(0,\rho) - e(0,\rho)\widetilde{e}'(0,\rho)\big). \end{aligned} \tag{3.6}$$

On the base of the assumption of the theorem, we result that

$$H_0(\rho_n) = (e_n'(a,\rho)\widetilde{e}_n(a,\rho) - e_n(a,\rho)\widetilde{e}_n'(a,\rho)) - (e_n'(0,\rho)\widetilde{e}_n(0,\rho) - e_n(0,\rho)\widetilde{e}_n'(0,\rho)) = 0,$$

for $\rho_n \in \Lambda_+$. Now we have to show that $H_0(\rho) = 0$ for other $\rho \in \Pi_+$.
Let $\mathcal{Q}_-(x) = Q(x) - \widetilde{Q}(x)$. The Jost solution (2.5) gives

$$\begin{aligned} e(x,\rho)\widetilde{e}(x,\rho) &= \frac{1}{4a_1^2}\exp(2i\rho a - \mathcal{Q}_+(a))\Big((a_1^2+1)^2\exp(2i\rho(x-a) - (\mathcal{Q}_+(x) - \mathcal{Q}_+(a))) \\ &\quad + (a_1^2-1)^2\exp(-2i\rho(x-a) + (\mathcal{Q}_+(x) - \mathcal{Q}_+(a))) \\ &\quad + 2(a_1^4-1)cosh(\mathcal{Q}_-(x) - \mathcal{Q}_-(a))\Big) \\ &\quad + \int_0^x \mathbf{M}_2(x,t)\exp(2i\rho t - \mathcal{Q}_+(t))dt \\ &\quad + \int_0^x \mathbf{M}_3(x,t)\exp(2i\rho(2a-t) - (2\mathcal{Q}_+(a) - \mathcal{Q}_+(t)))dt, \quad x \in [0,a], \end{aligned} \tag{3.7}$$

and then

$$|e(x,\rho)\widetilde{e}(x,\rho)| \le \mathcal{C}\exp(-2|\Im\rho|x), \quad 0 \le x \le a.$$

Repeating the similar arguments in the proof of Theorem 3.1, we can get that $H_0(\rho) = 0$ for other $\rho \in \Pi_+$.

Now this result and the relations (3.6) and (3.7) give for all ρ,

$$\begin{aligned} &\int_0^a \frac{2(a_1^4-1)}{4a_1^2}\exp(2i\rho a - \mathcal{Q}_+(a))cosh(\mathcal{Q}_-(t) - \mathcal{Q}_-(a))(Q_0(t) + i\rho Q_1(t))dt \\ &+ \int_0^a \frac{(a_1^2+1)^2}{4a_1^2}\exp(2i\rho t - \mathcal{Q}_+(t))(\mathcal{Q}_{02}(t) + i\rho\mathcal{Q}_{12}(t))dt \\ &+ \int_0^a \frac{(a_1^2-1)^2}{4a_1^2}\exp(2i\rho(2a-t) - (2\mathcal{Q}_+(a) - \mathcal{Q}_+(t)))(\mathcal{Q}_{03}(t) + i\rho\mathcal{Q}_{13}(t))dt = 0, \end{aligned}$$

where $\mathcal{Q}_{\nu 2} = Q_\nu(t) + \int_t^a Q_\nu(x)\mathbf{M}_2(x,t)dx$ and $\mathcal{Q}_{\nu 3} = Q_\nu(t) + \int_t^a Q_\nu(x)\mathbf{M}_3(x,t)dx$, $\nu = 0,1$. Using Riemann-Lebesgue lemma for continuous functions, we can get as large enough ρ,

$$\int_0^a \exp(2i\rho t - \mathcal{Q}_+(t))\mathcal{Q}_{\nu 2}(t)dt = 0,$$
$$\int_0^a \exp(2i\rho(2a-t) - (2\mathcal{Q}_+(a) - \mathcal{Q}_+(t)))\mathcal{Q}_{\nu 3}(t)dt = 0.$$

From the completeness of the exponential function [7, 22], we can write for $0 \le t < a$,

$$\mathcal{Q}_{\nu 2}(t) = \mathcal{Q}_{\nu 3}(t) = 0, \quad \nu = 0,1,$$

that is

$$Q_\nu(t) + \int_t^a Q_\nu(x)\mathbf{M}_2(x,t)dx = 0, \qquad Q_\nu(t) + \int_t^a Q_\nu(x)\mathbf{M}_3(x,t)dx = 0.$$

These homogeneous Volterra integral equations have only the trivial solution. Therefore $Q_0(x) = Q_1(x) = 0$ on $0 \le x < a$. Thus $q_1(x) = \widetilde{q}_1(x)$ and $q_0(x) = \widetilde{q}_0(x)$ a.e. on $[0,a]$.

Based on Remark 3.2 and using (2.1), we can give for $\rho_n \in \Lambda_+$,

$$H_a(\rho_n) = e_n(a,\rho)\widetilde{e}'_n(a,\rho) - e'_n(a,\rho)\widetilde{e}_n(a,\rho) = 0.$$

Similar to the pervious section, one can prove that $H_a(\rho) = 0$ for other $\rho \in \Pi_+$ and therefore by repeating the last discussion, we can show that $q_0(x) = \widetilde{q}_0(x)$ and $q_1(x) = \widetilde{q}_1(x)$ a.e. on

$[a, \infty)$.

Thus $q_0(x) = \widetilde{q}_0(x)$ and $q_1(x) = \widetilde{q}_1(x)$ a.e. on $x \geq 0$. The proof is completed. $\square$

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References

[1] R.Kh. Amirov, A.S. Ozkan and B. Keskin, Inverse problems for impulsive Sturm-Liouville operator with spectral parameter linearly contained in boundary conditions. *Integral Transforms and Special Functions*, **20**(8): 607-618, 2009.

[2] E. Bairamov and C. Coskun, Jost solutions and the spectrum of the system of difference equations. *Applied Mathematics Letters*, **17**: 1039-1045, 2004.

[3] P.E. Bulavin and V.M. Kascheev, Solution of the non homogeneous heat conduction equation for multi-layered bodies. *Int Chem Eng*, **5**: 112-115, 1965.

[4] S.A. Buterin and C.T. Shieh, Incomplete inverse spectral and nodal problems for differential pencils. *Results Math*, **62**: 167-179, 2012.

[5] J.B. Conway, *Functions of One Complex Variable*, 2nd ed., Vol. I, Springer-Verlag, New York, 1995.

[6] G. Freiling and V. Yurko, Inverse spectral problems for singular non-selfadjoint differential operators with discontinuities in an interior point. *Inverse Problems*, **18**(3): 757, 2002.

[7] G. Freiling and V.A. Yurko, *Inverse Sturm-Liouville Problems and their Applications*, NOVA Scince Publ, New York, 2001.

[8] H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed given data. *SIAM J Appl Math*, **34**: 676-680, 1978.

[9] Y. Khalili and D. Baleanu, A uniqueness result for differential pencils with discontinuities from interior spectral data. *Analysis*, **38**(4): 195-202, 2018.

[10] Y. Khalili, M. Yadollahzadeh and M.K. Moghadam, Half inverse problems for the impulsive operator with eigenvalue-dependent boundary conditions. *Electronic J Differential Equations*, **2017**(190): 1-5, 2017.

[11] V. Marchenko, *Sturm-Liouville Operators and their Applications*, Izd Nauk Dumka, Kiev, 8, 1977.

[12] K. Mochizuki and I. Trooshin, Inverse problem for interior spectral data of Sturm-Liouville operator. *J Inverse Ill-Posed Problems*, **9**: 425-433, 2001.

[13] O.Sh. Mukhtarov, M. Kadakal and F.S. Muhtarov, On discontinuous Sturm-Liouville problems with transmission conditions. *J Math Kyoto Univ*, **44**: 779-798, 2004.

[14] M.A. Naimark, *Linear Differential Operators*, Parts I, II, Ungar, New York, 1968.

[15] A. Neamaty and Y. Khalili, Determination of a differential operator with discontinuity from interior spectral data. *Inver Probl Sci Eng*, **22**(6): 1002-1008, 2013.

[16] M.N. Ozisik, *Boundary Value Problems of Heat Conduction*, Dover, New York, 1989.

[17] A.S. Ozkan, B. Keskin and Y. Cakmak, Uniqueness of the solution of half inverse problem for the impulsive Sturm-Liouville operator. *Bull Korean Math Soc*, **50**(2): 499-506, 2013.

[18] C.T. Shieh and V.A. Yurko, Inverse nodal and inverse spectral problems for discontinuous boundary value problems. *J Math Anal Appl*, **347**(1): 266-272, 2008.

[19] Y.P. Wang, Inverse problems for Sturm-Liouville operators with interior discontinuities and boundary conditions dependent on the spectral parameter. *Mathematical Methods in the Applied Sciences*, **36**(7): 857-868, 2013.

[20] Y.P. Wang, The inverse problem for differential pencils with eigenparameter dependent boundary conditions from interior spectral data. *Applied Mathematics Letters*, **25**: 1061-1067, 2012.

[21] C.F. Yang and Y.X. Guo, Determination of a differential pencil from interior spectral data. *J Math Anal Appl*, **375**: 284-293, 2011.

[22] C.F. Yang and A. Zettl, Half inverse problems for quadratic pencils of Sturm-Liouville operators. *Taiwanese J Mathematics*, **16**(5): 1829-1846, 2012.

[23] V. Yurko, Inverse spectral problems for differential pencils on the half-line with turning points. *J Math Anal Appl*, **320**: 439-463, 2006.

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