Mathematical Analysis
& Convex Optimization

Vol. 5 (2024), No. 1, 29-38
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.5.1.4

Research Paper

INEQUALITIES INVOLVING HADAMARD PRODUCT FOR SECTOR MATRICES

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Abstract. In this paper, we investigate new inequalities for the Hadamard product of sector matrices. Using operator theory techniques and concepts of operator means, we establish new inequalities for the Hadamard product in this class of matrices. In particular, for matrices $A_i, B_i \in S_\theta$ satisfying $I(A_i\sigma B_i) \circ I(A_i\sigma^{\perp} B_i) \leq 0$, the following inequality holds:

$$\begin{aligned}&\left(\sum_{i=1}^{k} R(A_i \# B_i)\right) \circ \left(\sum_{i=1}^{k} R(A_i \# B_i)\right)\\ &\qquad\leq \sec^4(\theta) R\left(\left(\sum_{i=1}^{k} A_i\sigma B_i\right) \circ \left(\sum_{i=1}^{k} A_i\sigma^{\perp} B_i\right)\right).\end{aligned}$$

Furthermore, we extend existing inequalities for positive operators to the class of sector matrices and prove determinant inequalities for the Hadamard product of sector matrices.

MSC(2010): 15A60, 15B48, 47A64

Keywords: Hadamard product, Positive linear map, Sector matrices.

1. Introduction and Background

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathbb{M}_n$, we write that $A \geqslant 0$ if A is positive semidefinite, i.e. if $x^*Ax \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. We also write $A > 0$ if A is positive definite, i.e. if $x^*Ax > 0$ for all vectors $x \in \mathbb{C}^n$, and $A \geqslant B$ if $A - B \geqslant 0$. The set of all positive definite matrices will be denoted by $\mathbb{M}_n^+$. A matrix $A \in \mathbb{M}_n$ is called accretive if in its Cartesian (or Toeplitz) decomposition, $A = \mathcal{R}A + i\mathcal{I}A$, $\mathcal{R}A$ is positive definite, where $\mathcal{R}A = \frac{A+A^*}{2}$, $\mathcal{I}A = \frac{A-A^*}{2i}$.

The numerical range of a matrix $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

A matrix $A \in \mathbb{M}_n$ is said to be sectorial if $W(A) \subset S_\theta$ for some $0 \leq \theta < \frac{\pi}{2}$, where S_θ denote the sector regions in the complex plane as follows:

$$S_\theta = \{z \in \mathbb{C} : \mathcal{R}z \geq 0, |\mathcal{I}z| \leq (\mathcal{R}z)\tan\theta\}.$$

Clearly, A is positive semidefinite if and only if $W(A) \subset S_0$, and if $W(A)$, $W(B) \subset S_\theta$ for some $\theta \in [0, \frac{\pi}{2})$, then $W(A+B) \subset S_\theta$. Moreover, $W(A) \subset S_\theta$ implies $W(X^*AX) \subset S_\theta$ for any nonzero $n \times n$ matrix X; thus $W(A^{-1}) \subset S_\theta$. The smallest such θ is called the sectorial index of A. When $W(A) \subset S_\theta$, we will write $A \in S_\theta$.

Date: Received: January 15, 2023, Accepted: May 9, 2024.

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Kubo-Ando [2] defined an operator mean σ by an operator monotone function $f:(0,\infty)\longrightarrow(0,\infty)$ with $f(1)=1$ as follows:

$$A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2},$$

where A and B are positive invertible operators. The function f is referred to as the representing function of σ. Recently, Bedrani et al. [1] extended this definition to accretive operators

Some important operator means are presented as follows:

- r-weighted arithmetic mean:
$$A\nabla_r B = rA+(1-r)B, \quad (0\le r\le 1).$$
- r-weighted geometric mean:
$$A\sharp_r B = A^{1/2}(A^{-1/2}BA^{-1/2})^r A^{1/2}, \quad (0\le r\le 1).$$
- r-weighted harmonic mean:
$$A\ !_r\ B = \left(rA^{-1}+(1-r)B^{-1}\right)^{-1}, \quad (0\le r\le 1).$$

The operator defined by the function $tf(t^{-1})$ is called the transpose of σ and is denoted as σ^0.

$$A\sigma^0 B = B\sigma A;$$

and the operator mean represented by the function $\frac{t}{f(t)}$ is referred to as the dual of σ and is denoted by $\sigma^{\perp}$.

$$A\sigma^{\perp}B = \left(A^{-1}\sigma B^{-1}\right)^{-1}.$$

For example, $\sharp_r^{\perp} = \sharp_{1-r} = \sharp_r^0$ and $!_{1-r}^{\perp} = \nabla_r$.

Bedrani et al. also derived the following inequalities regarding means for sector matrices:

Lemma 1.1. [1] *Let $A,B\in\mathbb{M}_n$ are accretive matrices such that $W(A),W(B)\subset S_\theta$, for some $0\le\theta<\frac{\pi}{2}$. Then*

$$\mathcal{R}A\sigma\mathcal{R}B \le \mathcal{R}(A\sigma B) \le \sec^2(\theta)(\mathcal{R}A\sigma\mathcal{R}B). \tag{1.1}$$

Notably, If $A=[a_{ij}]\in\mathbb{M}_{m\times n}$ and $B=[b_{ij}]\in\mathbb{M}_{p\times q}$, their Kronecker or tensor product $A\otimes B$ is a block matrix in $\mathbb{M}_{mp\times nq}$, defined as:

$$A\otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

It is established that the following formulas hold for A,B,C and D of appropriate sizes:

- $(A\otimes B)^* = A^*\otimes B^*$,
- $(A\otimes B)(C\otimes D) = (AC\otimes BD)$,
- $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$(if A and B be invertible).

A notable property of the tensor product is that if $A\ge C\ge 0$ and $B\ge D\ge 0$, then it follows that: $A\otimes B\ge C\otimes D\ge 0$. Furthermore, if A,B,C and D are positive invertible operators with appropriate sizes, then for all $r\in[0,1]$, the following relation holds:

$$(A\otimes B)\sharp_r(C\otimes D) = (A\sharp_r C)\otimes(B\sharp_r D).$$

Pecaric et al. [4] defined the Hadamard product as follows.

Definition 1.2. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis of $\mathbb{C}^n$, and define the isometry $U : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ by $Ue_j = e_j \otimes e_j$. The Hadamard product $A \circ B$ with respect to the basis $\{e_j\}$ can be expressed as

$$A \circ B = U^*(A \otimes B)U.$$

This means that if the matrices $A = [a_{ij}] \in \mathbb{M}_n$ and $B = [b_{ij}] \in \mathbb{M}_n$ are associated with the same basis, then their Hadamard product $A \circ B$ is related to the matrix $A \circ B = [a_{ij}b_{ij}] \in \mathbb{M}_n$.

Lemma 1.3. [4, Theorem 6.1] *If A and B are positive matrices, then their Hadamard product $A \circ B$ is also a positive matrix. More generally, if $A_1 \geq A_2 \geq 0$ and $B_1 \geq B_2 \geq 0$, then $A_1 \circ B_1 \geq A_2 \circ B_2 \geq 0$.*

Additionally, $A \otimes B \geq 0$ if and only if $A \circ B \geq 0$. It is also clear that

$$\mathcal{R}(A \circ B) = U^* \mathcal{R}(A \otimes B)U.$$

Thus, $\mathcal{R}(A \otimes B) \geq 0$ if and only if $\mathcal{R}(A \circ B) \geq 0$. For more information, please refer to [6, 4].

At the end of this section we bring the some lemmas that we need to prove the main results.

Lemma 1.4. [9] *Let $A, B \in \mathbb{M}_n^+$. Then*

$$\det(A \circ B) \geq \det A \det B. \tag{1.2}$$

Lemma 1.5. [4, Theorem 6.6] *If A and B are positive matrices, then*

$$A \circ B \geq (A \sharp_r B) \circ (A \sharp_{1-r} B) \quad \text{and} \quad A \circ B \geq (A !_{1-r} B) \circ (A \nabla_r B) \tag{1.3}$$

for all $r \in [0, 1]$.

Recall that if I is an interval in $\mathbb{R}$, a function $f \in \mathcal{C}(I)$ is said to be super-multiplicative on I if the inequality $f(xy) \geq f(x)f(y)$ holds for all $x, y \in I$.

Pecaric et al. [4] derived the following extension from the first expression of inequality (1.3), too. If σ is an operator mean with a super-multiplicative representing function f, then

$$A \circ B \geq (A \sigma B) \circ (A \sigma^0 B), \tag{1.4}$$

holds for positive operators A and B.

2. Main Results

The Hadamard product differs from the usual product in several ways. The most important is the commutativity of Hadamard multiplication:

$$A \circ B = B \circ A,$$

for all $A, B \in \mathbb{M}_n$. The following formulas are well known.

$$\alpha A \circ B = A \circ \alpha B = \alpha(A \circ B),$$

and

$$\alpha A \circ \beta B = \alpha\beta(A \circ B), \tag{2.1}$$

where $\alpha, \beta \in \mathbb{C}$.

Lemma 2.1. [6] *If $A, B \in S_\theta$ such that $\mathcal{I}A \circ \mathcal{I}B \leq 0$, then*

$$\mathcal{R}(A \circ B) = \mathcal{R}A \circ \mathcal{R}B - \mathcal{I}A \circ \mathcal{I}B \geq \mathcal{R}A \circ \mathcal{R}B. \tag{2.2}$$

Thus using (2.2), we obtain an extension of (2.1).

$$\mathcal{R}(\alpha A \circ \beta B) \geq \mathcal{R}(\alpha A) \circ \mathcal{R}(\beta B) = \alpha\beta(\mathcal{R}A \circ \mathcal{R}B).$$

For $A, B \in \mathbb{M}_n$ simple calculations imply

$$A \circ B = (\mathcal{R}A \circ \mathcal{R}B - \mathcal{I}A \circ \mathcal{I}B) + i(\mathcal{R}A \circ \mathcal{I}B + \mathcal{I}A \circ \mathcal{R}B).$$

Hence, we have the following lemma.

Lemma 2.2. [6] *Let $A, B \in S_\theta$. Then*

- *If $\mathcal{I}A \circ \mathcal{I}B \geq 0$, then*

$$\mathcal{R}(A \circ B) \leq \mathcal{R}A \circ \mathcal{R}B.$$

- *If $-\mathcal{I}A \circ \mathcal{I}B \geq 0$, then*

$$\mathcal{R}A \circ \mathcal{R}B \leq \mathcal{R}(A \circ B).$$

Singh et al.[7] obtained some inequalities for positive definite matrices, which we extend to sector matrices in this section.
They also proved that if A_i, B_i, for $i = 1, ..., n$, are positive definite matrices in $\mathbb{M}_n$, then

$$\left(\sum_{i=1}^{n} A_i \sharp B_i\right) \circ \left(\sum_{i=1}^{n} A_i \sharp B_i\right) \leq \left(\sum_{i=1}^{n} A_i\right) \circ \left(\sum_{i=1}^{n} B_i\right), \tag{2.3}$$

and

$$\left(\sum_{i=1}^{n} A_i \sharp B_i\right) \circ \left(\sum_{i=1}^{n} A_i \sharp B_i\right) \leq \left(\sum_{i=1}^{n} A_i \sigma B_i\right) \circ \left(\sum_{i=1}^{n} A_i \sigma^{\perp} B_i\right). \tag{2.4}$$

Matharu at el.[8] proved that if $A_1 \geq ... \geq A_n \geq 0$ and $B_1 \geq ... \geq B_n \geq 0$, then

$$\left(\sum_{i=1}^{n} p_i A_i\right) \circ \left(\sum_{i=1}^{n} p_i B_i\right) \leq \left(\sum_{i=1}^{n} p_i\right) \left(\sum_{i=1}^{n} p_i (A_i \circ B_i)\right), \tag{2.5}$$

where $p_i \geq 0$, $i = 1, ..., n$, are weights.
For further details on this topic, refer to [5, 8].

Proposition 2.3. *If $A_i, B_i \in S_\theta$, with $\mathcal{R}A_1 \geq \ldots \geq \mathcal{R}A_n \geq 0$ and $\mathcal{R}B_1 \geq \ldots \geq \mathcal{R}B_n \geq 0$, such that $\mathcal{I}A_i \circ \mathcal{I}B_i \leq 0$, then*

$$\mathcal{R}\left(\sum_{i=1}^{n} p_i A_i\right) \circ \mathcal{R}\left(\sum_{i=1}^{n} p_i B_i\right) \leq \left(\sum_{i=1}^{n} p_i\right) \mathcal{R}\left(\sum_{i=1}^{n} p_i (A_i \circ B_i)\right), \tag{2.6}$$

where $p_i \geq 0$ for $i = 1, \ldots, n$ are weights.

Proof. By inequalities (2.2) and (2.5), we get

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i\right)\mathcal{R}\left(\sum_{i=1}^{n} p_i(A_i\circ B_i)\right) &= \left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i\mathcal{R}(A_i\circ B_i)\right)\\ &\geq \left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i(\mathcal{R}A_i\circ \mathcal{R}B_i)\right)\\ &\geq \left(\sum_{i=1}^{n} p_i\mathcal{R}A_i\right)\circ\left(\sum_{i=1}^{n} p_i\mathcal{R}B_i\right)\\ &= \mathcal{R}\left(\sum_{i=1}^{n} p_iA_i\right)\circ\mathcal{R}\left(\sum_{i=1}^{n} p_iB_i\right).\end{aligned}$$

□

The next theorem is an extension of inequality (2.4).

Theorem 2.4. *Let* $A_i, B_i, \in S_\theta$ *for all* $i = 1, ..., n$, *such that* $\mathcal{I}(A_i\sigma B_i)\circ\mathcal{I}(A_i\sigma^\perp B_i)\leq 0$. *Then*

$$\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\circ\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\leq \sec^4(\theta)\mathcal{R}\left((\sum_{i=1}^{k}A_i\sigma B_i)\circ(\sum_{i=1}^{k}A_i\sigma^\perp B_i)\right).$$

Proof. Using Lemma 1.3 and the appropriate inequalities, we have

$$\begin{aligned}&\mathcal{R}\left(\left(\sum_{i=1}^{k}A_i\sigma B_i\right)\circ\left(\sum_{i=1}^{k}A_i\sigma^\perp B_i\right)\right)\\ &\geq \sum_{i=1}^{k}\mathcal{R}(A_i\sigma B_i)\circ\sum_{i=1}^{k}\mathcal{R}(A_i\sigma^\perp B_i) && \text{(by (2.2))}\\ &\geq \left(\sum_{i=1}^{k}\mathcal{R}A_i\sigma\mathcal{R}B_i\right)\circ\left(\sum_{i=1}^{k}\mathcal{R}A_i\sigma^\perp\mathcal{R}B_i\right) && \text{(by (1.1))}\\ &\geq \left(\sum_{i=1}^{k}\mathcal{R}A_i\sharp\mathcal{R}B_i\right)\circ\left(\sum_{i=1}^{k}\mathcal{R}A_i\sharp\mathcal{R}B_i\right) && \text{(by (2.4))}\\ &\geq \left(\cos^2(\theta)\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\circ\left(\cos^2(\theta)\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right) && \text{(by (1.1))}\\ &= \cos^4(\theta)\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\circ\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right). && \text{(by (2.1))}\end{aligned}$$

□

The next corollary can be obtained by taking $\sigma = !_{1-r}, \sharp_r$ where $r\in[0,1]$.

Corollary 2.5. *Let* $r\in[0,1]$.

(*i*) *If* $\mathcal{I}(A_i!_{1-r}B_i)\circ\mathcal{I}(A_i\nabla_r B_i)\leq 0$, *then*

$$\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\circ\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\leq \sec^4(\theta)\mathcal{R}\left((\sum_{i=1}^{k}A_i!_{1-r}B_i)\circ(\sum_{i=1}^{k}A_i\nabla_r B_i)\right);$$

(*ii*) *If* $\mathcal{I}(A_i\sharp_r B_i)\circ\mathcal{I}(A_i\sharp_{1-r}B_i)\leq 0$, *then*

$$\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\circ\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\leq \sec^4(\theta)\mathcal{R}\left((\sum_{i=1}^{k}A_i\sharp_r B_i)\circ(\sum_{i=1}^{k}A_i\sharp_{1-r}B_i)\right);$$

for all $A_i, B_i\in S_\theta,\ i=1,...,n.$

In [**?**], Lin proved the following inequality for $A\in S_\theta$:

$$\det(\mathcal{R}A)\leq\mid\det A\mid\leq\sec^n(\theta)\det(\mathcal{R}A). \tag{2.7}$$

We will now use the inequality mentioned above to prove the following theorem:

Theorem 2.6. *If* $A_i, B_i\in S_\theta$, $\mathcal{R}A_1\geq ...\geq\mathcal{R}A_n\geq 0$ *and* $\mathcal{R}B_1\geq ...\geq\mathcal{R}B_n\geq 0$ *such that* $\mathcal{I}A_i\circ\mathcal{I}B_i\leq 0$, *then*

$$\left(\sum_{i=1}^{n}p_i^n\mid\det A_i\mid\right)\left(\sum_{i=1}^{n}p_i^n\mid\det B_i\mid\right)\leq\sec^{2n}(\theta)\left(\sum_{i=1}^{n}p_i\right)^n\det\left(\sum_{i=1}^{n}p_i\mathcal{R}(A_i\circ B_i)\right).$$

where $p_i\geq 0$, $i=1,...,n$, *are weights.*

Proof. Applying determinant properties for for positive definite matrices, we conclude that

$$\begin{aligned}
\left(\sum_{i=1}^{n}p_i\right)^n\det\left(\sum_{i=1}^{n}p_i\mathcal{R}(A_i\circ B_i)\right)&=\det\left(\sum_{i=1}^{n}p_i\sum_{i=1}^{n}p_i\mathcal{R}(A_i\circ B_i)\right)\\
&\geq\det\left((\sum_{i=1}^{n}p_i\mathcal{R}A_i)\circ(\sum_{i=1}^{n}p_i\mathcal{R}B_i)\right) && \text{(by (2.6))}\\
&\geq\det\left(\sum_{i=1}^{n}p_i\mathcal{R}A_i\right)\det\left(\sum_{i=1}^{n}p_i\mathcal{R}B_i\right) && \text{(by (1.2))}\\
&\geq\sum_{i=1}^{n}p_i^n\det(\mathcal{R}A_i)\sum_{i=1}^{n}p_i^n\det(\mathcal{R}B_i)\\
&\geq\sum_{i=1}^{n}p_i^n\cos^n(\theta)\mid\det A_i\mid\sum_{i=1}^{n}p_i cos^n(\theta)\mid\det A_i\mid && \text{(by (2.7))}\\
&=cos^{2n}(\theta)\left(\sum_{i=1}^{n}p_i^n\mid\det A_i\mid\right)\left(\sum_{i=1}^{n}p_i^n\mid\det B_i\mid\right).
\end{aligned}$$

□

The subsequent theorem extends the inequality presented in inequality (2.3).

Theorem 2.7. *Let* $A_i, B_i\in S_\theta$ *for all* $i=1,...,n$ *such that* $\mathcal{I}A_i\circ\mathcal{I}B_i\leq 0$. *Then*

$$\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\circ\left(\sum_{i=1}^{k}\mathcal{R}(A_i\sharp B_i)\right)\leq\sec^4(\theta)\mathcal{R}\left(\sum_{i=1}^{k}A_i\circ\sum_{i=1}^{k}B_i)\right).$$

Proof. According to the assumption of the theorem and Lemma 2.1, we have

$$
\begin{aligned}
\mathcal{R}\left(\sum_{i=1}^{k} A_i \circ \sum_{i=1}^{k} B_i)\right) &\geq \left(\mathcal{R}(\sum_{i=1}^{k} A_i)\right) \circ \left(\mathcal{R}(\sum_{i=1}^{k} B_i)\right) \\
&= \left(\sum_{i=1}^{k} \mathcal{R}A_i\right) \circ \left(\sum_{i=1}^{k} \mathcal{R}B_i\right) \\
&\geq \left(\sum_{i=1}^{k} \mathcal{R}A_i \sharp \mathcal{R}B_i\right) \circ \left(\sum_{i=1}^{k} \mathcal{R}A_i \sharp \mathcal{R}B_i\right) && \text{(by (2.3))} \\
&\geq \left(\sum_{i=1}^{k} \cos^2(\theta)\mathcal{R}(A_i \sharp B_i)\right) \circ \left(\sum_{i=1}^{k} \cos^2(\theta)\mathcal{R}(A_i \sharp B_i)\right) && \text{(by (1.1))} \\
&= \cos^4(\theta)\left(\sum_{i=1}^{k} \mathcal{R}(A_i \sharp B_i) \circ (\sum_{i=1}^{k} \mathcal{R}(A_i \sharp B_i)\right) && \text{(by (2.1)).}
\end{aligned}
$$

$\square$

In the following proposition, we present the sectoral case of inequality (1.4).

Proposition 2.8. *If $A, B \in S_\theta$, such that $\mathcal{I}A \circ \mathcal{I}B \leq 0$, and σ is an operator mean with a super-multiplicative representation function, then*

$$\mathcal{R}(A\sigma B) \circ \mathcal{R}(A\sigma^{\circ} B) \leq \sec^4(\theta)\mathcal{R}(A \circ B).$$

Proof. By Lemma 1.1 and inequality (1.4), we get

$$
\begin{aligned}
\mathcal{R}(A \circ B) &\geq \mathcal{R}A \circ \mathcal{R}B \\
&\geq \big(\mathcal{R}A\sigma\mathcal{R}B\big) \circ \big(\mathcal{R}A\sigma^0\mathcal{R}B\big) \\
&\geq \big(\cos^2(\theta)\mathcal{R}(A\sigma B)\big) \circ \big(\cos^2(\theta)\mathcal{R}(A\sigma^0 B)\big) \\
&= \cos^4(\theta)\Big(\mathcal{R}(A\sigma B) \circ \mathcal{R}(A\sigma^0 B)\Big).
\end{aligned}
$$

$\square$

Using the Proposition 2.8, we prove the following practical theorem.

Theorem 2.9. *If $A_i, B_i \in S_\theta$, $\mathcal{R}A_1 \geq ... \geq \mathcal{R}A_n \geq 0$ and $\mathcal{R}B_1 \geq ... \geq \mathcal{R}B_n \geq 0$ such that $\mathcal{I}A_i \circ \mathcal{I}B_i \leq 0$, and σ is an operator mean with a super-multiplicative representation function, then*

$$\left(\sum_{i=1}^{n} p_i\mathcal{R}(A_i\sigma B_i)\right) \circ \left(\sum_{j=1}^{n} p_j\mathcal{R}(A_j\sigma^{\circ} B_j)\right) \leq \sec^4(\theta)\Big(\sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i\mathcal{R}(A_i \circ B_i)\Big),$$

where $p_i \geq 0$, $i = 1, ..., n$, are weights.

Proof. By Lemmas 1.1, 1.3 and 2.2, and inequality (2.5), we have

$$\begin{aligned}
&\sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i \sec^4(\theta)\mathcal{R}(A_i \circ B_i) - \left(\sum_{i=1}^{n} p_i \mathcal{R}(A_i \sigma B_i)\right) \circ \left(\sum_{j=1}^{n} p_j \mathcal{R}(A_j \sigma^{\circ} B_j)\right)\\
&\geq \sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i \sec^4(\theta)\mathcal{R}(A_i \circ B_i)\\
&- \left(\sec^2(\theta)\sum_{i=1}^{n} p_i (\mathcal{R}A_i \sigma \mathcal{R}B_i)\right) \circ \left(\sec^2(\theta)\sum_{j=1}^{n} p_j (\mathcal{R}A_j \sigma^{\circ} \mathcal{R}B_j)\right)\\
&\geq \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \sec^4(\theta)\mathcal{R}(A_i \circ B_i)\\
&- \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \sec^4(\theta)\Big((\mathcal{R}A_i \sigma \mathcal{R}B_i) \circ (\mathcal{R}A_j \sigma^{\circ} \mathcal{R}B_j)\Big)\\
&\geq \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \Big(\mathcal{R}(A_i \sigma B_i) \circ \mathcal{R}(A_i \sigma^{\circ} B_i)\Big) - \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \Big(\mathcal{R}(A_i \sigma B_i) \circ \mathcal{R}(A_j \sigma^{\circ} B_j)\Big)\\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \left[\Big(\mathcal{R}(A_i \sigma B_i) \circ \mathcal{R}(A_i \sigma^{\circ} B_i) - \mathcal{R}(A_i \sigma B_i) \circ \mathcal{R}(A_j \sigma^{\circ} B_j)\Big)\right]\\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \left[p_i p_j \Big(\mathcal{R}(A_i \sigma B_i) \circ \mathcal{R}(A_i \sigma^{\circ} B_i)\Big) - p_i p_j \Big(\mathcal{R}(A_i \sigma B_i) \circ \mathcal{R}(A_j \sigma^{\circ} B_j)\Big)\right]\\
&+ p_i p_j \Big(\mathcal{R}(A_j \sigma B_j) \circ \mathcal{R}(A_j \sigma^{\circ} B_j)\Big) - p_i p_j \Big(\mathcal{R}(A_j \sigma B_j) \circ (A_i \sigma^{\circ} B_i)\Big)\\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \Big(\mathcal{R}(A_i \sigma B_i) - \mathcal{R}(A_j \sigma B_j)\Big) \circ \Big(\mathcal{R}(A_i \sigma^{\circ} B_i) - \mathcal{R}(A_j \sigma^{\circ} B_j)\Big) \geq 0.
\end{aligned}$$

This completes the proof. □

The next corollary follows on taking $\sigma = \sharp_r$ where $r \in [0, 1]$.

Corollary 2.10. *If $A_i, B_i \in S_\theta$, $\mathcal{R}A_1 \geq ... \geq \mathcal{R}A_n \geq 0$ and $\mathcal{R}B_1 \geq ... \geq \mathcal{R}B_n \geq 0$ such that $\mathcal{I}A_i \circ \mathcal{I}B_i \leq 0$, then*

$$\sec^4(\theta)\Big(\sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i \mathcal{R}(A_i \circ B_i)\Big) \geq \left(\sum_{i=1}^{n} p_i \mathcal{R}(A_i \sharp_r B_i)\right) \circ \left(\sum_{j=1}^{n} p_j \mathcal{R}(A_j \sharp_{1-r} B_j)\right)$$

where $r \in [0, 1]$.

Theorem 2.11. *Let $A_i, B_i \in S_\theta$ for all $i = 1, ..., n$ such that $\mathcal{I}A_i \circ \mathcal{I}B_i \leq 0$. Then*

$$\sum_{i=1}^{n} |\det(A_i \sharp B_i)|^2 \leq \sec^{2(n+2)}(\theta)\det\left(\Big(\sum_{i=1}^{n} A_i\Big) \circ \Big(\sum_{i=1}^{n} B_i\Big)\right),$$

where $p_i \geq 0$, $i = 1, ..., n$, are weights.

Proof. By Theorem 2.7, we obtain that

$$\begin{aligned}
\sec^4(\theta)\det\left(\mathcal{R}(\sum_{i=1}^n A_i \circ \sum_{i=1}^n B_i)\right) &\geq \det\left(\left(\sum_{i=1}^n \mathcal{R}(A_i\sharp B_i)\right)\circ\left(\sum_{i=1}^n \mathcal{R}(A_i\sharp B_i)\right)\right)\\
&\geq \left(\det\left(\sum_{i=1}^n \mathcal{R}(A_i\sharp B_i)\right)\right)^2 \quad (\text{by } (1.2))\\
&\geq \left(\sum_{i=1}^n \det\mathcal{R}(A_i\sharp B_i)\right)^2\\
&\geq \left(\sum_{i=1}^n \mid\det(A_i\sharp B_i)\mid \cos^n(\theta)\right)^2 \quad (\text{by } (2.7))\\
&\geq \cos^{2n}(\theta)\sum_{i=1}^n \mid\det(A_i\sharp B_i)\mid^2 \quad (\text{by } (2.5)).
\end{aligned}$$

□

Using Theorems 2.6 and 2.11, we have the next results.

Corollary 2.12. *Let $A, B \in S_\theta$ such that $\mathcal{I}A\circ\mathcal{I}B \leq 0$. Then*

$$\sec(\theta)\det\mathcal{R}(A\circ B) \geq \mid\det A\mid\mid\det B\mid .$$

Corollary 2.13. *Let $A, B \in S_\theta$ such that $\mathcal{I}A\circ\mathcal{I}B \leq 0$. Then*

$$\mid\det(A\sharp B)\mid^2 \leq \sec^6(\theta)\det\mathcal{R}(A\circ B)$$

Theorem 2.14. *If $A, B \in S_\theta$ such that $\mathcal{I}A\circ\mathcal{I}B \leq 0$, then*

(*i*) $\mathcal{R}(A\sharp_r B)\circ\mathcal{R}(A\sharp_{1-r}B) \leq \sec^4(\theta)\big(\mathcal{R}(A\circ B)\big)$,
(*ii*) $\mathcal{R}(A!_{1-r}B)\circ\mathcal{R}(A\nabla_r B) \leq \sec^4(\theta)\big(\mathcal{R}(A\circ B)\big)$;

for all $r \in [0,1]$.

Proof. Using Lemmas 1.1, 1.5 and 2.2, we have

$$\begin{aligned}
\cos^4(\theta)\big(\mathcal{R}(A\sharp_r B)\circ\mathcal{R}(B\sharp_r A)\big) &\leq (\mathcal{R}A\sharp_r\mathcal{R}B)\circ(\mathcal{R}B\sharp_r\mathcal{R}A)\\
&\leq \mathcal{R}A\circ\mathcal{R}B\\
&\leq \mathcal{R}(A\circ B),
\end{aligned}$$

and

$$\begin{aligned}
\cos^4(\theta)\big(\mathcal{R}(A!_{1-r}B)\circ\mathcal{R}(A\nabla_r B)\big) &\leq (\mathcal{R}A!_{1-r}\mathcal{R}B)\circ(\mathcal{R}A\nabla_r\mathcal{R}B)\\
&\leq \mathcal{R}A\circ\mathcal{R}B\\
&\leq \mathcal{R}(A\circ B).
\end{aligned}$$

□

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