Mathematical Analysis
& Convex Optimization
Vol. 5 (2024), No. 1, 9-20
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.5.1.2

Research Paper

ANOTHER VIEW OF THE LEVINSON INTEGRAL INEQUALITY

CHRISTOPHE CHESNEAU

Abstract. In this article, we extend the applicability of the Levinson integral inequality. In particular, we show that the ratio expression of the main integrated function can be avoided, as well as some monotonicity assumptions, and that a weight function appears naturally in the integral of the upper bound. These generalizations are made by revisiting an existing proof in the literature. Some examples and further integral inequalities based on the obtained weight function are given.

MSC(2010): 26D15; 33E20.
Keywords: Levinson integral inequality, Hardy integral inequality, exponential inequalities.

1. Introduction

In mathematical analysis, integral inequalities are crucial for establishing bounds on integrals involving functions of various kinds. They make it possible to analyze the behavior of solutions to differential equations and optimization problems. They are also used in other areas of applied mathematics, including statistics, engineering and mathematical physics. We can refer to [4, 2, 8, 1, 10]. Among these integral inequalities is the Levinson integral inequality, which was established in [5]. For the purposes of this article, let us briefly introduce and discuss it. We need $(a, b) \in (0, +\infty)^2$ with $a < b$, $f : (a, b) \mapsto (0, +\infty)$ an integrable function and $F : (a, b) \mapsto (0, +\infty)$ the corresponding primitive defined by

$$F(x) := \int_a^x f(t)dt.$$

In this framework, the Levinson integral inequality gives a relationship between the ratio function $F(x)/x$ raised to a power p, where $p \in (1, +\infty)$, and the integral of the function $f(x)$, also raised to the same power p. Formally, we have

$$\int_a^b \left[\frac{F(x)}{x}\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx, \tag{1.1}$$

provided that the integrals involved converge. It can be seen as a bounded interval version of the famous Hardy integral inequality, corresponding to $a = 0$ and $b = +\infty$ (see [4, 7, 6]). It is useful in several contexts, including the study of functional analysis and integral transforms. It allows complex integrals to be bound by simpler integrals. Note that the constant $[p/(p-1)]^p$ plays a crucial role in ensuring the tightness of the bound.

This Levinson integral inequality has been generalized in several references, including [7, 9, 3]. In particular, it was elegantly generalized in [9, Theorem 3.1.]. The main lines of this

---

Date: Received: September 11, 2023, Accepted: April 20, 2024.

generalization are given below. Adopting the above mathematical framework, we additionally consider $\eta \geq 0$ and a function $g : (a, b) \mapsto (0, +\infty)$, such that $(x+\eta-a)/g(x)$ is non-increasing. Then the following integral inequality holds:

$$\int_a^b \left[\frac{F(x)}{g(x)}\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b \left[(x+\eta-a)\frac{f(x)}{g(x)}\right]^p dx, \tag{1.2}$$

provided that the integrals involved converge. As noted in [9, Remark 3.3.], if we take $g(x) = x + \eta - a$ with $\eta \in [0, a)$, we get

$$\int_a^b \left[\frac{F(x)}{x}\right]^p dx \leq \int_a^b \left[\frac{F(x)}{x+\eta-a}\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x) dx,$$

so that the classical Levinson integral inequality in Equation (1.1) is obtained. The generality of [9, Theorem 3.1.] is thus characterized by the presence of the function $g(x)$ and the parameter η. The proof is based on a thorough use of the Hölder integral inequality combined with the non-increasing property of $(x+\eta-a)/g(x)$. When analyzing this proof, the following remarks can be made:

- **Remark 1.:** The ratio term $1/g^p(x)$ (including the exponent p) seems to play no role in the first (seven) steps of the proof; its ratio form seems to be justified to artificially correspond to that of the main integrated term in the original Levinson integral inequality, until the last steps when it is activated.
- **Remark 2.:** The non-increasing assumption on $(x+\eta-a)/g(x)$ is only used to extract it, in a sense, from an integral with a bounded interval; other, more general, alternatives are possible.
- **Remark 3.:** The penultimate step of the proof introduces a weight function in the upper bound which is roughly bounded by 1; there is no mention of it in the statement of [9, Theorem 3.1.]; some sharpness seems to be lost.

These three remarks suggest some improvements to this general Levinson integral inequality. More specifically, on a similar proof basis, we show that the ratio term of the main integrated term can indeed be relaxed and significantly generalized thanks to the use of a bivariate function. The assumption of monotonicity becomes unnecessary. Furthermore, we rehabilitate the weight function that appears in the proof in [9, Theorem 3.1.]. In particular, we discuss how it can be treated or bounded by various sharp functions to obtain original upper bounds on the main integral term. Some precise examples are given. The study is concluded with an integral inequality result dealing with a different kind of primitive of the main bivariate function.

The rest of the article is divided into two sections: Section 2, which presents the main theorem and secondary results, and Section 3, which is devoted to a conclusion.

2. Results

The new results are highlighted and proved in this section.

2.1. Main theorem.

The theorem below offers a different view of the Levinson integral inequality, by significantly generalizing it through the use of a bivariate function and a manageable weight function for the upper bound.

Theorem 2.1. *Let* $(a,b) \in (0,+\infty)^2$ *with* $a < b$, $f : (a,b)^2 \mapsto (0,+\infty)$ *be a bivariate integrable function and* $F : (a,b) \mapsto (0,+\infty)$ *be its primitive defined by*

$$F(x) := \int_a^x f(t;x)dt, \quad x \in (a,b).$$

We consider the function $f_\star : (a,b) \mapsto (0,+\infty)$ *defined by*

$$f_\star(x) := \sup_{u\in[x,b)} \left[(u-a)f(x;u)\right], \tag{2.1}$$

provided that it exists.

Then, for any $p \in (1,+\infty)$, *we have*

$$\int_a^b F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f_\star^p(x)w_p(x)dx,$$

where $w_p : (a,b) \mapsto (0,+\infty)$ *is given by*

$$w_p(x) := 1 - \left(\frac{x-a}{b-a}\right)^{1-1/p}. \tag{2.2}$$

It is understood that the integrals involved converge.

Proof. Let us set

$$A := \int_a^b F^p(x)dx. \tag{2.3}$$

Then we consider the following decomposition:

$$A = \int_a^b \left[\int_a^x f(t;x)dt\right]^p dx = \int_a^b \left[\int_a^x (t-a)^{(p-1)/p^2}(t-a)^{-(p-1)/p^2} f(t;x)dt\right]^p dx. \tag{2.4}$$

The Hölder integral inequality applied with the parameter p (and $q = p/(p-1)$) gives

$$\begin{aligned}
A &\leq \int_a^b \left\{\left[\int_a^x (t-a)^{1-1/p} f^p(t;x)dt\right]^{1/p} \left[\int_a^x (t-a)^{-1/p}dt\right\}^{1-1/p}\right\}^p dx \\
&= \int_a^b \left[\int_a^x (t-a)^{1-1/p} f^p(t;x)dt\right] \left[\int_a^x (t-a)^{-1/p}dt\right]^{p-1} dx \\
&= \int_a^b \left[\int_a^x (t-a)^{1-1/p} f^p(t;x)dt\right] \left[\frac{p}{p-1}(x-a)^{1-1/p}\right]^{p-1} dx \\
&= \left(\frac{p}{p-1}\right)^{p-1} \int_a^b \left[\int_a^x (t-a)^{1-1/p} f^p(t;x)dt\right] (x-a)^{(1-1/p)(p-1)}dx \\
&= B, \qquad (2.5)
\end{aligned}$$

where

$$B := \left(\frac{p}{p-1}\right)^{p-1} \int_a^b \int_a^x (t-a)^{1-1/p}(x-a)^{(1-1/p)(p-1)} f^p(t;x)dtdx.$$

Changing the order of integration according to the Fubini-Tonelli integral theorem (the integrated term is positive), we obtain

$$\begin{aligned} B &= \left(\frac{p}{p-1}\right)^{p-1} \int_a^b \left[\int_t^b (t-a)^{1-1/p}(x-a)^{(1-1/p)(p-1)} f^p(t;x)dx\right] dt \\ &= \left(\frac{p}{p-1}\right)^{p-1} \int_a^b (t-a)^{1-1/p} \left\{\int_t^b (x-a)^{1/p-2} \left[(x-a)f(t;x)\right]^p dx\right\} dt. \end{aligned}$$

Using, for any $x \in [t,b)$, $(x-a)f(t;x) \leq f_\star(t)$, where $f_\star(t)$ is given in Equation (2.1) and the definition of $w_p(x)$ given in Equation (2.2), we get

$$\begin{aligned} B &\leq \left(\frac{p}{p-1}\right)^{p-1} \int_a^b (t-a)^{1-1/p} f_\star^p(t) \left[\int_t^b (x-a)^{1/p-2} dx\right] dt \\ &= \left(\frac{p}{p-1}\right)^{p-1} \int_a^b (t-a)^{1-1/p} f_\star^p(t) \left\{\frac{p}{1-p}\left[(b-a)^{1/p-1} - (t-a)^{1/p-1}\right]\right\} dt \\ &= \left(\frac{p}{p-1}\right)^{p} \int_a^b f_\star^p(t) \left[1 - \frac{(t-a)^{1-1/p}}{(b-a)^{1-1/p}}\right] dt \\ &= \left(\frac{p}{p-1}\right)^{p} \int_a^b f_\star^p(t) w_p(t) dt = \left(\frac{p}{p-1}\right)^{p} \int_a^b f_\star^p(x) w_p(x) dx, \end{aligned} \tag{2.6}$$

where the notation has been standardized in the last step. Combining Equations (2.3) (2.4), (2.5) and (2.6), we end the proof of Theorem 2.1. □

The inequality in Theorem 2.1 includes those in Equations (1.1) and (1.2). More specifically, in the context of Equation (1.2), let us take

$$f(x;t) = \frac{1}{g(t)} f(x).$$

We recall that it is assumed that there is a constant $\eta \geq 0$ such that $(u+\eta-a)/g(u)$ is non-increasing. Based on Equation (2.1), for any $x \in (a,b)$, we have

$$\begin{aligned} f_\star(x) &= \sup_{u\in[x,b)} \left[(u-a)f(x;u)\right] = f(x) \sup_{u\in[x,b)} \left[(u-a)\frac{1}{g(u)}\right] \\ &\leq f(x) \sup_{u\in[x,b)} \left[(u+\eta-a)\frac{1}{g(u)}\right] = (x+\eta-a)\frac{f(x)}{g(x)}. \end{aligned}$$

Bounding $w_p(x)$ by 1 (details will be given in Proposition 2.2), Theorem 2.1 implies that

$$\begin{aligned} \int_a^b F^p(x)dx &\leq \left(\frac{p}{p-1}\right)^{p} \int_a^b f_\star^p(x) w_p(x) dx \\ &\leq \left(\frac{p}{p-1}\right)^{p} \int_a^b \left[(x+\eta-a)\frac{f(x)}{g(x)}\right]^p w_p(x) dx \\ &\leq \left(\frac{p}{p-1}\right)^{p} \int_a^b \left[(x+\eta-a)\frac{f(x)}{g(x)}\right]^p dx. \end{aligned}$$

This gives Equation (1.2).

Theorem 2.1 thus extends the scope of the Levinson integral inequality, where the function $f(x;u)$ can be chosen in an infinite number of ways.

2.2. **Secondary results.** Based on Theorem 2.1, the proposition below shows how the presence of the weight function $w_p(x)$ can be exploited to find an original upper bound for the main integral term. Seven proposals are given, including power, exponential and logarithmic-type upper bounds.

Proposition 2.2. *In the framework of Theorem 2.1, the upper bounds of the main upper bound presented below are valid. For any* $i = 1, 2, 3, 4, 5, 6$ *and* 7*, we have*

$$\int_a^b f_\star^p(x) w_p(x) dx \leq D_i,$$

where D_1, D_2, D_3, D_4, D_5, D_6 *and* D_7 *are given by*

$$D_1 := \int_a^b f_\star^p(x) dx, \quad D_2 := \int_a^b f_\star^p(x) \left(\frac{b-x}{b-a}\right)^{1-1/p} dx,$$

$$D_3 := \int_a^b f_\star^p(x) \exp\left[-\left(\frac{x-a}{b-a}\right)^{1-1/p}\right] dx, \quad D_4 := \int_a^b f_\star^p(x) \exp\left[-\left(1-\frac{1}{p}\right)\left(\frac{x-a}{b-a}\right)\right] dx,$$

$$D_5 := \sup_{x \in (a,b)} [f_\star^p(x)](b-a)\frac{p-1}{2p-1}, \quad D_6 := \sqrt{\int_a^b f_\star^{2p}(x) dx}\sqrt{(b-a)\frac{2(p-1)^2}{(2p-1)(3p-2)}}$$

and

$$D_7 := \left(1-\frac{1}{p}\right)\int_a^b f_\star^p(x) \log\left(\frac{b-a}{x-a}\right) dx.$$

Proof. Let us establish the upper bounds D_1, D_2, D_3, D_4, D_5, D_6 and D_7, in turns.

For D_1**:** Since $1 - 1/p \in (0, 1)$, for any $x \in (a, b)$, we have $[(x - a)/(b - a)]^{1-1/p} \in (0, 1)$, implying that $w_p(x) \leq 1$, and

$$\int_a^b f_\star^p(x) w_p(x) dx \leq \int_a^b f_\star^p(x) dx = D_1.$$

For D_2**:** Since $1 - 1/p \in (0, 1)$, using the inequality $1 - t^\gamma \leq (1 - t)^\gamma$ for any $t \in (0, 1)$ and $\gamma \in (0, 1)$, with $t = (x - a)/(b - a)$ and $\gamma = 1 - 1/p$, we get

$$w_p(x) = 1 - \left(\frac{x-a}{b-a}\right)^{1-1/p} \leq \left(1 - \frac{x-a}{b-a}\right)^{1-1/p} = \left(\frac{b-x}{b-a}\right)^{1-1/p}.$$

So we have

$$\int_a^b f_\star^p(x) w_p(x) dx \leq \int_a^b f_\star^p(x) \left(\frac{b-x}{b-a}\right)^{1-1/p} dx = D_2.$$

For D_3**:** Using the inequality $1 - t \leq \exp(-t)$ for any $t \in \mathbb{R}$, we have

$$w_p(x) = 1 - \left(\frac{x-a}{b-a}\right)^{1-1/p} \leq \exp\left[-\left(\frac{x-a}{b-a}\right)^{1-1/p}\right].$$

We therefore have

$$\int_a^b f_\star^p(x) w_p(x) dx \leq \int_a^b f_\star^p(x) \exp\left[-\left(\frac{x-a}{b-a}\right)^{1-1/p}\right] dx = D_3.$$

For D_4**:** Combining the techniques used for obtaining D_2 and D_3, we have

$$w_p(x) = 1 - \left(\frac{x-a}{b-a}\right)^{1-1/p} \leq \left(1 - \frac{x-a}{b-a}\right)^{1-1/p}$$
$$\leq \left\{\exp\left[-\left(\frac{x-a}{b-a}\right)\right]\right\}^{1-1/p} = \exp\left[-\left(1-\frac{1}{p}\right)\left(\frac{x-a}{b-a}\right)\right].$$

We therefore have

$$\int_a^b f_\star^p(x) w_p(x) dx \leq \int_a^b f_\star^p(x) \exp\left[-\left(1-\frac{1}{p}\right)\left(\frac{x-a}{b-a}\right)\right] dx = D_4.$$

For D_5**:** Taking the supremum of $f_\star^p(x)$, we have directly

$$\int_a^b f_\star^p(x) w_p(x) dx \leq \sup_{x\in(a,b)} [f_\star^p(x)] \int_a^b w_p(x) dx,$$

with

$$\int_a^b w_p(x) dx = \int_a^b \left[1 - \left(\frac{x-a}{b-a}\right)^{1-1/p}\right] dx = \int_a^b dx - \int_a^b \left(\frac{x-a}{b-a}\right)^{1-1/p} dx$$
$$= (b-a)\left(1 - \frac{p}{2p-1}\right) = (b-a)\frac{p-1}{2p-1}.$$

Combining the above results gives the upper bound D_5.

For D_6**:** It follows from the Cauchy-Schwarz integral inequality that

$$\int_a^b f_\star^p(x) w_p(x) dx \leq \sqrt{\int_a^b f_\star^{2p}(x) dx}\sqrt{\int_a^b w_p^2(x) dx}$$

with

$$\int_a^b w_p^2(x) dx = \int_a^b \left[1 - \left(\frac{x-a}{b-a}\right)^{1-1/p}\right]^2 dx$$
$$= \int_a^b dx - 2\int_a^b \left(\frac{x-a}{b-a}\right)^{1-1/p} dx + \int_a^b \left(\frac{x-a}{b-a}\right)^{2(1-1/p)} dx$$
$$= (b-a)\left[1 - \frac{2p}{2p-1} + \frac{p}{3p-2}\right] = (b-a)\frac{2(p-1)^2}{(2p-1)(3p-2)}.$$

Combining the above results gives the upper bound D_6.

For D_7**:** Using the inequality $\log(t) \leq t-1$ for any $t \in (-1,\infty)$, so that $1-t \leq -\log(t)$ for any $t \in (-1,\infty)$, with $t = [(x-a)/(b-a)]^{1-1/p}$, we obtain

$$w_p(x) = 1 - \left(\frac{x-a}{b-a}\right)^{1-1/p} \leq -\log\left[\left(\frac{x-a}{b-a}\right)^{1-1/p}\right]$$
$$= \left(1-\frac{1}{p}\right)\left\{-\log\left[\left(\frac{x-a}{b-a}\right)\right]\right\} = \left(1-\frac{1}{p}\right)\log\left(\frac{b-a}{x-a}\right).$$

We therefore have

$$\int_a^b f_\star^p(x) w_p(x) dx \leq \left(1-\frac{1}{p}\right)\int_a^b f_\star^p(x) \log\left(\frac{b-a}{x-a}\right) dx = D_7.$$

This concludes the proof of Proposition 2.2. □

Theorem 2.1 and Proposition 2.2 can be combined to obtain new upper bounds for the main term in Theorem 2.1. For instance, if we concentrate on D_4, D_5, D_6 and D_7 established in Proposition 2.2, in the framework of Theorem 2.1, we have

$$\int_a^b F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f_\star^p(x) w_p(x)dx$$
$$\leq \left(\frac{p}{p-1}\right)^p D_4 = \left(\frac{p}{p-1}\right)^p \int_a^b f_\star^p(x) \exp\left[-\left(1-\frac{1}{p}\right)\left(\frac{x-a}{b-a}\right)\right] dx,$$

$$\int_a^b F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f_\star^p(x) w_p(x)dx$$
$$\leq \left(\frac{p}{p-1}\right)^p D_5 = \left(\frac{p}{p-1}\right)^{p-1} \sup_{x\in(a,b)} [f_\star^p(x)](b-a)\frac{1}{p(2p-1)},$$

$$\int_a^b F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f_\star^p(x) w_p(x)dx$$
$$\leq \left(\frac{p}{p-1}\right)^p D_6 = \left(\frac{p}{p-1}\right)^{p-1} p\sqrt{\int_a^b f_\star^{2p}(x)dx}\sqrt{(b-a)\frac{2}{(2p-1)(3p-2)}}$$

and

$$\int_a^b F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f_\star^p(x) w_p(x)dx$$
$$\leq \left(\frac{p}{p-1}\right)^p D_7 = \left(\frac{p}{p-1}\right)^{p-1} \int_a^b f_\star^p(x) \log\left(\frac{b-a}{x-a}\right) dx.$$

To the best of our knowledge, these integral inequalities are new to the literature. Of course, other similar examples can be given.

2.3. **Another theorem.**

We complement Theorem 2.1 by another theorem, dealing with another kind of primitive of the main function.

Theorem 2.3. *Let $(a,b) \in (0,+\infty)^2$ with $a<b$, $f:(a,b)^2 \mapsto (0,+\infty)$ be a bivariate integrable function and $S:(a,b)\mapsto(0,+\infty)$ be defined by*

$$S(x) := \int_x^b f(t;x)dt, \quad x\in(a,b).$$

We consider the function $f_\dagger:(a,b)\mapsto(0,+\infty)$ defined by

$$f_\dagger(x) := \sup_{u\in(a,x]} \left[(b-u)f(x;u)\right], \tag{2.7}$$

provided that it exists.

Then, for any $p\in(1,+\infty)$, we have

$$\int_a^b S^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f_\dagger^p(x) z_p(x)dx,$$

where $z_p : (a,b) \mapsto (0,+\infty)$ *is given by*

$$z_p(x) := 1 - \left(\frac{b-x}{b-a}\right)^{1-1/p}. \tag{2.8}$$

It is understood that the integrals involved converge.

Proof. The proof follows the lines and arguments of the proof of Theorem 2.1. We consider

$$U := \int_a^b S^p(x)dx. \tag{2.9}$$

Then we decompose it as follows:

$$U = \int_a^b \left[\int_x^b f(t;x)dt\right]^p dx = \int_a^b \left[\int_x^b (b-t)^{(p-1)/p^2}(b-t)^{-(p-1)/p^2} f(t;x)dt\right]^p dx. \tag{2.10}$$

From this expression, it follows from the Hölder integral inequality applied with the parameter p (and $q = p/(p-1)$) that

$$\begin{aligned}
U &\leq \int_a^b \left\{\left[\int_x^b (b-t)^{1-1/p} f^p(t;x)dt\right]^{1/p} \left[\int_x^b (b-t)^{-1/p}dt\right\}^{1-1/p}\right\}^p dx \\
&= \int_a^b \left[\int_x^b (b-t)^{1-1/p} f^p(t;x)dt\right] \left[\int_x^b (b-t)^{-1/p}dt\right]^{p-1} dx \\
&= \int_a^b \left[\int_x^b (b-t)^{1-1/p} f^p(t;x)dt\right] \left[\frac{p}{p-1}(b-x)^{1-1/p}\right]^{p-1} dx \\
&= \left(\frac{p}{p-1}\right)^{p-1} \int_a^b \left[\int_x^b (b-t)^{1-1/p} f^p(t;x)dt\right] (b-x)^{(1-1/p)(p-1)}dx \\
&= V, \qquad (2.11)
\end{aligned}$$

where

$$V = \left(\frac{p}{p-1}\right)^{p-1} \int_a^b \int_x^b (b-t)^{1-1/p}(b-x)^{(1-1/p)(p-1)} f^p(t;x)dtdx.$$

Changing the order of integration according to the Fubini-Tonelli integral theorem (the integrated term is positive), we obtain

$$\begin{aligned}
V &= \left(\frac{p}{p-1}\right)^{p-1} \int_a^b \left[\int_a^t (b-t)^{1-1/p}(b-x)^{(1-1/p)(p-1)} f^p(t;x)dx\right] dt \\
&= \left(\frac{p}{p-1}\right)^{p-1} \int_a^b (b-t)^{1-1/p} \left\{\int_a^t (b-x)^{1/p-2}\left[(b-x)f(t;x)\right]^p dx\right\} dt.
\end{aligned}$$

Using, for any $x \in (a,t]$, $(b-x)f(t;x) \leq f_{\dagger}(t)$, where $f_{\dagger}(t)$ is given in Equation (2.7) and the definition of $z_p(x)$ given in Equation (2.8), we get

$$\begin{aligned} V &\leq \left(\frac{p}{p-1}\right)^{p-1} \int_a^b (b-t)^{1-1/p} f_{\dagger}^p(t) \left[\int_a^t (b-x)^{1/p-2} dx\right] dt \\ &= \left(\frac{p}{p-1}\right)^{p-1} \int_a^b (b-t)^{1-1/p} f_{\dagger}^p(t) \left\{\frac{p}{1-p}\left[(b-a)^{1/p-1} - (b-t)^{1/p-1}\right]\right\} dt \\ &= \left(\frac{p}{p-1}\right)^{p} \int_a^b f_{\dagger}^p(t) \left[1 - \frac{(b-t)^{1-1/p}}{(b-a)^{1-1/p}}\right] dt \\ &= \left(\frac{p}{p-1}\right)^{p} \int_a^b f_{\dagger}^p(t) z_p(t) dt = \left(\frac{p}{p-1}\right)^{p} \int_a^b f_{\dagger}^p(x) z_p(x) dx, \end{aligned} \tag{2.12}$$

(a standardization of the notation was applied in the last step). Combining Equations (2.9), (2.10), (2.11) and (2.12), we complete the proof of Theorem 2.3. □

In the context of Theorem 2.3, we can take

$$f(x;t) = \frac{1}{g(t)} f(x),$$

where $f : (a,b) \mapsto (0,+\infty)$ is an integrable function and $g : (a,b) \mapsto (0,+\infty)$ is a function, assuming that there exists a constant $\eta \geq 0$ such that $(b+\eta-u)/g(u)$ is non-decreasing. Based on Equation (2.7), we have

$$\begin{aligned} f_{\dagger}(x) &= \sup_{u\in(a,x]} \left[(b-u)f(x;u)\right] = f(x) \sup_{u\in(a,x]} \left[(b-u)\frac{1}{g(u)}\right] \\ &\leq f(x) \sup_{u\in(a,x]} \left[(b+\eta-u)\frac{1}{g(u)}\right] = (b+\eta-x)\frac{f(x)}{g(x)}. \end{aligned}$$

Bounding $z_p(x)$ by 1, Theorem 2.3 implies that

$$\begin{aligned} \int_a^b S^p(x)dx &\leq \left(\frac{p}{p-1}\right)^{p} \int_a^b f_{\dagger}^p(x) z_p(x) dx \\ &\leq \left(\frac{p}{p-1}\right)^{p} \int_a^b \left[(b+\eta-x)\frac{f(x)}{g(x)}\right]^p z_p(x) dx \\ &\leq \left(\frac{p}{p-1}\right)^{p} \int_a^b \left[(b+\eta-x)\frac{f(x)}{g(x)}\right]^p dx. \end{aligned}$$

In particular, if $\eta = 0$, we get the following variant of the Levinson inequality:

$$\int_a^b S^p(x)dx \leq \left(\frac{p}{p-1}\right)^{p} \int_a^b \left[(b-x)\frac{f(x)}{g(x)}\right]^p dx.$$

To the best of our knowledge, Theorem 2.3 is new to the literature, with the same perspectives of applications as Theorem 2.1.

2.4. **Secondary results.** Some bounds of $z_p(x)$ can be investigated in a similar way as in Proposition 2.2, leading to original upper bounds, as developed in the proposition below.

Proposition 2.4. *In the framework of Theorem 2.3, the upper bounds of the main upper bound presented below are valid. For any $i = 1, 2, 3, 4, 5, 6$ and 7, we have*

$$\int_a^b f_\dagger^p(x) z_p(x) dx \le E_i,$$

where E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are given by

$$E_1 := \int_a^b f_\dagger^p(x) dx, \quad E_2 := \int_a^b f_\dagger^p(x) \left(\frac{x-a}{b-a}\right)^{1-1/p} dx,$$

$$E_3 := \int_a^b f_\dagger^p(x) \exp\left[-\left(\frac{b-x}{b-a}\right)^{1-1/p}\right] dx, \quad E_4 := \int_a^b f_\dagger^p(x) \exp\left[-\left(1-\frac{1}{p}\right)\left(\frac{b-x}{b-a}\right)\right] dx,$$

$$E_5 := \sup_{x\in(a,b)} [f_\dagger^p(x)](b-a)\frac{p-1}{2p-1}, \quad E_6 := \sqrt{\int_a^b f_\dagger^{2p}(x) dx}\sqrt{(b-a)\frac{2(p-1)^2}{(2p-1)(3p-2)}}$$

and

$$E_7 := \left(1-\frac{1}{p}\right) \int_a^b f_\dagger^p(x) \log\left(\frac{b-a}{b-x}\right) dx.$$

Proof. Let us establish the upper bounds E_1, E_2, E_3, E_4, E_5, E_6 and E_7, in turns.

For E_1: Since $1 - 1/p \in (0, 1)$, for any $x \in (a, b)$, we have $[(b - x)/(b - a)]^{1-1/p} \in (0, 1)$, implying that $z_p(x) \le 1$, and

$$\int_a^b f_\dagger^p(x) z_p(x) dx \le \int_a^b f_\dagger^p(x) dx = E_1.$$

For E_2: Since $1 - 1/p \in (0, 1)$, using the inequality $1 - t^\gamma \le (1 - t)^\gamma$ for any $t \in (0, 1)$ and $\gamma \in (0, 1)$, with $t = (b - x)/(b - a)$ and $\gamma = 1 - 1/p$, we get

$$z_p(x) = 1 - \left(\frac{b-x}{b-a}\right)^{1-1/p} \le \left(1 - \frac{b-x}{b-a}\right)^{1-1/p} = \left(\frac{x-a}{b-a}\right)^{1-1/p}.$$

So we have

$$\int_a^b f_\dagger^p(x) z_p(x) dx \le \int_a^b f_\dagger^p(x) \left(\frac{x-a}{b-a}\right)^{1-1/p} dx = E_2.$$

For E_3: Using the inequality $1 - t \le \exp(-t)$ for any $t \in \mathbb{R}$, we have

$$z_p(x) = 1 - \left(\frac{b-x}{b-a}\right)^{1-1/p} \le \exp\left[-\left(\frac{b-x}{b-a}\right)^{1-1/p}\right].$$

We therefore have

$$\int_a^b f_\dagger^p(x) z_p(x) dx \le \int_a^b f_\dagger^p(x) \exp\left[-\left(\frac{b-x}{b-a}\right)^{1-1/p}\right] dx = E_3.$$

For E_4**:** Combining the techniques used for obtaining E_2 and E_3, we have

$$z_p(x) = 1 - \left(\frac{b-x}{b-a}\right)^{1-1/p} \leq \left(1 - \frac{b-x}{b-a}\right)^{1-1/p}$$
$$\leq \left\{\exp\left[-\left(\frac{b-x}{b-a}\right)\right]\right\}^{1-1/p} = \exp\left[-\left(1-\frac{1}{p}\right)\left(\frac{b-x}{b-a}\right)\right].$$

We therefore have

$$\int_a^b f_\dagger^p(x) z_p(x) dx \leq \int_a^b f_\dagger^p(x) \exp\left[-\left(1-\frac{1}{p}\right)\left(\frac{b-x}{b-a}\right)\right] dx = E_4.$$

For E_5**:** Taking the supremum of $f_\dagger^p(x)$, we have directly

$$\int_a^b f_\dagger^p(x) z_p(x) dx \leq \sup_{x\in(a,b)} [f_\dagger^p(x)] \int_a^b z_p(x) dx,$$

with

$$\int_a^b z_p(x) dx = \int_a^b \left[1 - \left(\frac{b-x}{b-a}\right)^{1-1/p}\right] dx = \int_a^b dx - \int_a^b \left(\frac{b-x}{b-a}\right)^{1-1/p} dx$$
$$= (b-a)\left(1 - \frac{p}{2p-1}\right) = (b-a)\frac{p-1}{2p-1}.$$

Combining the above results gives the upper bound E_5.

For E_6**:** It follows from the Cauchy-Schwarz integral inequality that

$$\int_a^b f_\dagger^p(x) z_p(x) dx \leq \sqrt{\int_a^b f_\dagger^{2p}(x) dx}\sqrt{\int_a^b z_p^2(x) dx}$$

with

$$\int_a^b z_p^2(x) dx = \int_a^b \left[1 - \left(\frac{b-x}{b-a}\right)^{1-1/p}\right]^2 dx$$
$$= \int_a^b dx - 2\int_a^b \left(\frac{b-x}{b-a}\right)^{1-1/p} dx + \int_a^b \left(\frac{b-x}{b-a}\right)^{2(1-1/p)} dx$$
$$= (b-a)\left[1 - \frac{2p}{2p-1} + \frac{p}{3p-2}\right] = (b-a)\frac{2(p-1)^2}{(2p-1)(3p-2)}.$$

Combining the above results gives the upper bound E_6.

For E_7**:** Using the inequality $\log(t) \leq t-1$ for any $t \in (-1,\infty)$, so that $1-t \leq -\log(t)$ for any $t \in (-1,\infty)$, with $t = [(b-x)/(b-a)]^{1-1/p}$, we get

$$z_p(x) = 1 - \left(\frac{b-x}{b-a}\right)^{1-1/p} \leq -\log\left[\left(\frac{b-x}{b-a}\right)^{1-1/p}\right]$$
$$= \left(1-\frac{1}{p}\right)\left\{-\log\left[\left(\frac{b-x}{b-a}\right)\right]\right\} = \left(1-\frac{1}{p}\right)\log\left(\frac{b-a}{b-x}\right).$$

We therefore have

$$\int_a^b f_\dagger^p(x) z_p(x) dx \leq \left(1 - \frac{1}{p}\right) \int_a^b f_\dagger^p(x) \log\left(\frac{b-a}{b-x}\right) dx = E_7.$$

This concludes the proof of Proposition 2.4. □

In some mathematical scenarios, some of these bounds may be preferable to others, depending on the functional structure of the integral of interest.

3. Conclusion

In this article, we have revisited the Levinson integral inequality. Our main result is a general theorem established by using a bivariate function and by rehabilitating a special weight function in the integral of the upper bound. Several examples and new integral inequalities are derived. Another theorem is proposed for a complementary type of primitive of the main function. The results obtained are attractive in the sense that they offer great adaptability and can be applied in various areas of applied mathematics.

Acknowledgments

The author thanks the reviewers for their constructive comments.

References

[1] D. Bainov and P. Simeonov, *Integral Inequalities and Applications. Mathematics and Its Applications*, vol. 57. Kluwer Academic, Dordrecht , 1992.

[2] E.F. Beckenbach and R. Bellman, R. *Inequalities*, Springer, Berlin, 1961.

[3] B. Benaissa, M. Sarikaya and A. Senouci, On some new Hardy-type inequalities, *Mathematical Methods in the Applied Sciences*, **43**: 8488–8495, 2020.

[4] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[5] N. Levinson, Generalizations of an inequality of Hardy, *Duke Mathematical Journal*, **31**: 389–394, 1964.

[6] B. Sroysang, More on some Hardy type integral inequalities, *Journal of Mathematical Inequalities*, **8**: 497–501, 2014.

[7] W.T. Sulaiman, Some Hardy type integral inequalities, *Applied Mathematics Letters*, **25**: 520–525, 2012.

[8] W. Walter, *Differential and Integral Inequalities*, Springer, Berlin, 1970.

[9] S. Wu, B. Sroysang and S. Li, A further generalization of certain integral inequalities similar to Hardy's inequality, *Journal of Nonlinear Sciences and Applications*, **9**: 1093–1102, 2016.

[10] B.C. Yang, *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates, 2009.

(Christophe Chesneau) DEPARTMENT OF MATHEMATICS, LMNO, UNIVERSITY OF CAEN-NORMANDIE, 14032 CAEN, FRANCE.

Email address: `christophe.chesneau@gmail.com`