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Research Paper

ON LANDSBERG AND MEAN LANDSBERG CURVATURES OF TWO-DIMENSIONAL FINSLER MANIFOLDS

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Abstract. In this paper, we study the Landsberg and mean Landsberg curvatures of two-dimensional Finsler manifolds. First, we prove that a two-dimensional Finsler metric is a generalized Landsberg metric if and only if it is a stretch metric. Then, we study Finsler surfaces with isotropic main scalar and find the necessary and sufficient condition under which these spaces has vanishing $\tilde{\mathbf{J}}$-curvature.

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1. Introduction

The class of two-dimensional Finsler spaces has some extraordinary geometric properties which shows these spaces have more attentions with respect to the spaces with higher dimensions. As a first example, all of Finsler surfaces are C-reducible. Also, every two-dimensional Finsler space is of scalar flag curvature, i.e., the flag curvature of any two-dimensional Finsler space is independent of flag $\mathbf{K} = \mathbf{K}(x, y)$. In [4], Berwald studied two-dimensional spaces and obtained many interesting results for this class of Finsler spaces.

In order to study of two-dimensional Finsler spaces, one can consider their non-Riemannian curvature properties. Among these non-Riemannian curvatures, Landsberg and mean Landsberg curvatures are deep geometric meaning. Let (M, F) be a Finsler manifold. The second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ are inner products $\mathbf{g}_y$ and symmetric trilinear forms $\mathbf{C}_y$ on T_xM, respectively. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The Cartan torsion is a non-Riemannian quantity in Finsler geometry which was first introduced by Finsler and emphased by Cartan [17]. A Finsler metric reduces to a Riemannian metric if and only if it has vanishing Cartan torsion. The Landsberg curvature $\mathbf{L}$ measure the rate of changes of the Cartan torsion $\mathbf{C}$ along geodesics. A Finsler metric with vanishing Landsberg curvature is called a Landsberg metric [6][**?**].

There are some weaker notions of Landsberg metrics, namely stretch metrics and generalized Landsberg metrics. In 1924, at the annual meeting of the Mathematical Society of Germany in Innsbruck, Berwald gave the definition of stretch curvature as a generalization of Landsberg curvature [3]. He showed that a Finsler metric has vanishing stretch curvature if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. In [2], Bajancu-Farran introduced another weaker notion of

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Landsberg metrics, namely generalized Landsberg metrics. This class of Finsler metrics contains the class of Landsberg metrics as a special case. A Finaler metric F is a generalized Landsberg metric if the h-curvatures of the Berwald and Chern connections of F coincide in the sense of Bajancu-Farran.

In this paper, we study the stretch curvature of the class of Finsler surfaces and prove the following result.

Theorem 1.1. *Let (M, F) be a Finsler surface. Then F is a generalized Landsberg metric if and only if it is a stretch metric.*

In [4], Berwald decided to made a special frame for the class of two-dimensional Finsler spaces, called by Berwald's frame. The study of Berwald's frame shows that a well-known function appears such that distinguishes each metric from the other metrics, called by the main scalar of the Finsler surface and denoted by $\mathcal{I} = \mathcal{I}(x, y)$. Berwald characterized two-dimensional Finsler metrics with isotropic main scalar $\mathcal{I} = \mathcal{I}(x)$. Using this characterization, he found the classification of projectively flat Finsler surfaces with isotropic main scalar [4]. In [8], Matsumoto gave some geometrical meanings of the main scalar of Finsler surfaces. These studies shows that the class of Finsler surfaces with isotropic main scalars has important position in the class of two-dimensional Finsler metrics.

Taking a trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}_y$. In [5], Deicke proves that a positive definite Finsler metric F is Riemannian if and only if the mean Cartan torsion vanishes. The mean Landsberg curvature $\mathbf{J}$ measure the rate of changes of the mean Cartan torsion $\mathbf{I}$ along the geodesics in a general Finsler space. Indeed, $\mathbf{J} = \nabla_0 \mathbf{I}$, where ∇_0 denotes the horizontal derivation along Finslerian geodesics. A Finsler metric F is called weakly Landsberg metric if it has vanishing mean Landsberg curvature $\mathbf{J} = 0$. This non-Riemannian curvature has been observed in many situations, including when working with the Gauss-Bonnet theorem in the Finslerian setting. In [1], Bao and Shen proved that the volume function is a constant for every weakly Landsberg metric. Let us define

$$\bar{\mathbf{J}}_{ij} := J_{i,j} + J_{j,i},$$

which is a $(0, 2)$-type symmetric tensor and "," denotes the vertical covariant derivatives with respect to the Berwald connection. It is easy to show that $\bar{\mathbf{J}} = 0$ if and only $\mathbf{J} = 0$. In [18], Xia introduced an extension of mean Landsberg curvature. Let us denote by $\tilde{\mathbf{J}}$ the horizontal covariant differentiation of $\bar{\mathbf{J}}$ along Finslerian geodesics. More precisely, one can define $\tilde{\mathbf{J}} = \tilde{J}_{ij} dx^i \otimes dx^j$, where

$$\tilde{J}_{ij} := (J_{i,j} + J_{j,i})_{|m} y^m.$$

In this paper, we study the class of Finsler surfaces with isotropic main scalar. We find the necessary and sufficient condition under which these spaces has vanishing $\tilde{\mathbf{J}}$-curvature.

Theorem 1.2. *Let (M, F) be a Finsler surface with isotropic main scalar $\mathcal{I} = \mathcal{I}(x)$. Then $\tilde{\mathbf{J}} = 0$ if and only if the main scalar of F satisfies following*

$$(\mu^2 F - \mu')\Big[2(F_j I_i + F_i I_j) + F(I_{i,j} + I_{j,i})\Big] + 2\mu F(I_{i|j} + I_{j|i}) = 0, \tag{1.1}$$

where $\mu := -2\mathcal{I}_{|1}/\mathcal{I}$ and $\mu' := \mu_{|s} y^s$.

2. Preliminary

Let M be an n-dimensional C^∞ manifold, $TM=\bigcup_{x\in M}T_xM$ the tangent space and $TM_0:=TM-\{0\}$ the slit tangent space of M. A Finsler structure on manifold M is a function $F:TM\to[0,\infty)$ with the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x,\lambda y)=\lambda F(x,y)$, $\forall\lambda>0$; (iii) The quadratic form $\mathbf{g}_y:T_xM\times T_xM\to\mathbb{R}$ is positive-definite on T_xM

$$\mathbf{g}_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y+su+tv)\Big]_{s=t=0},\quad u,v\in T_xM.$$

Then, the pair (M,F) is called a Finsler manifold.

Let $x\in M$ and $F_x:=F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y:T_xM\times T_xM\times T_xM\to\mathbb{R}$ by

$$\mathbf{C}_y(u,v,w):=\frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0},\quad u,v,w\in T_xM.$$

The family $\mathbf{C}:=\{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}=0$ if and only if F is Riemannian.

For $y\in T_xM_0$, define $\mathbf{I}_y:T_xM\to\mathbb{R}$ by

$$\mathbf{I}_y(u):=\sum_{i=1}^{n}g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x\in M$. The family $\mathbf{I}:=\{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by

$$\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i},$$

where $G^i=G^i(x,y)$ are local functions on TM given by

$$G^i:=\frac{1}{4}g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^k\partial y^l}y^k-\frac{\partial[F^2]}{\partial x^l}\Big\},\quad y\in T_xM.$$

$\mathbf{G}$ is called the associated spray to (M,F).

Define $\mathbf{B}_y:T_xM\times T_xM\times T_xM\to T_xM$ by $\mathbf{B}_y(u,v,w):=B^i{}_{jkl}(y)u^jv^kw^l\partial/\partial x^i|_x$, where

$$B^i{}_{jkl}:=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature and F is called a Berwald metric if $\mathbf{B}=\mathbf{0}$.

For $y\in T_xM$, define the Landsberg curvature $\mathbf{L}_y:T_xM\times T_xM\times T_xM\to\mathbb{R}$ by

$$\mathbf{L}_y(u,v,w):=-\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u,v,w),y\big).$$

In local coordinates, $\mathbf{L}_y(u,v,w):=L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk}:=-\frac{1}{2}y_lB^l{}_{ijk}.$$

$\mathbf{L}$ is called the Landsberg curvature and F is called a Landsberg metric if $\mathbf{L} = 0$. Also, F is called of relatively isotropic Landsberg curvature if

$$L_{ijk} = cFC_{ijk},$$

where $c = c(x)$ is a scalar function on M.

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk}L_{ijk}.$$

The quantity $\mathbf{J}$ is called the mean Landsberg curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J} = 0$. By definition, every Landsberg metric is a weakly Landsberg metric.

Let (M, F) be an n-dimensional Finsler manifold. Let $\{e_j\}$ be a local frame for π^*TM, $\{\omega^i, \omega^{n+i}\}$ be the corresponding local coframe for $T^*(TM_0)$ and $\{\omega_j^i\}$ be the set of local Berwald connection forms with respect to $\{e_j\}$. In local coordinate system, the Berwald connection determined by following

$$d\omega^i = \omega^j \wedge \omega_j^i,$$
$$dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k},$$

where $\omega^i := dx^i$ and $\omega^{n+k} := dy^k + y^j\omega^k{}_j$. Thus

$$g_{ij|k} = -2L_{ijk}, \qquad g_{ij,k} = 2C_{ijk}.$$

For a tensor $\mathbf{T} = T_{i\cdots k}\omega^i \otimes \cdots \otimes \omega^k$, we have $T_{i\cdots k\cdot m} = \frac{\partial T_{i\cdots k}}{\partial y^m}$. For a non-zero vector $y \in T_xM$, the tensor $\mathbf{T}$ induces a multi-linear form $\mathbf{T}_y(u, \cdots, w) := T_{i\cdots k}(x, y)u^i \cdots w^k$ on T_xM. Let $\sigma(t)$ denote the geodesic with $\dot\sigma(0) = y$. We have

$$\frac{d}{dt}\Big[\mathbf{T}_{\dot\sigma(t)}\Big(U(t), \cdots, W(t)\Big)\Big] = T_{i\cdots k|m}(\sigma(t), \dot\sigma(t))\dot\sigma^m(t)U^i(t)\cdots W^k(t),$$

where $U(t) = U^i(t)\partial/\partial x^i|_{\sigma(t)}, \cdots, W(t) = W^k(t)\partial/\partial x^k|_{\sigma(t)}$ are linearly parallel vector fields along σ.

By the above mentioned explanations, one can find that the Landsberg curvature can be expressed as follows

$$L_{ijk} = C_{ijk|m}y^m.$$

3. PROOF OF THEOREM 1.1

Let $(V, \mathcal{F})$ be a Minkowskian plane. For any non-zero vector $v \in V$, there is a non-zero vector $w \in V$ such that is orthogonal to v with respect to the fundamental tensor $\mathfrak{g}$ raised by Minkowski functional $\mathcal{F}$. The Berwald frame was founded by Berwald to study of two-dimensional Finsler spaces [4]. It works under the assumption that the fundamental tensor is positive-definite.

For a two-dimensional Finsler manifold (M, F) and every $\mathbf{y} \in T_xM$, $x \in M$, there is a vector $\mathbf{y}^\perp \in T_xM_0$ such that

$$\mathbf{g}(\mathbf{y}, \mathbf{y}^\perp) = 0, \quad \mathbf{g}(\mathbf{y}^\perp, \mathbf{y}^\perp) = F(\mathbf{y}).$$

The pair $\{\mathbf{y}, \mathbf{y}^\perp\}$ is called the Berwald frame at $\mathbf{y}$.

Based on the Berwald frame, the Cartan torsion can be determined by a scalar function on slit tangent bundle. Let us define

$$\mathcal{I}(\mathbf{y}) := \frac{\mathbf{C}_{\mathbf{y}}(\mathbf{y}^{\perp}, \mathbf{y}^{\perp}, \mathbf{y}^{\perp})}{F(\mathbf{y})} = \mathcal{I}(\mathbf{y}^{\perp}).$$

One can see that $\mathcal{I}(\lambda\mathbf{y}) = \mathcal{I}(\mathbf{y})$ holds for $\forall\lambda > 0$ and $\forall\mathbf{y} \in T_xM_0$. We call $\mathcal{I}$ the main scalar of Finsler metric F.

In some of literature of Finsler geometry, the special notation (ℓ, m) was used instead of $\{\mathbf{y}, \mathbf{y}^{\perp}\}$. Then, for a scalar $T = T(x, y)$, we define the horizontal scalar derivatives $(T_{|1}, T_{|2})$ and vertical scalar derivatives $(T_{,1}, T_{,2})$ as follows

$$T_{|i} := T_{|1}\ell_i + T_{|2}m_i, \quad FT_{,i} := T_{,1}\ell_i + T_{,2}m_i,$$

where

$$T_{|i} := \frac{\partial T}{\partial x^i} - G^j{}_i\frac{\partial T}{\partial y^j}, \qquad FT_{,i} := F\frac{\partial T}{\partial y^i}, \qquad G^j{}_i := \frac{\partial G^i}{\partial y^j},$$

denote the horizontal and vertical derivations with respect to the Berwald connection of F.

In order to prove Theorem 1.1, we need to know the special form of Berwald curvature of Finsler surface. We remark that the following identity holds

$$B^p_{jkl} = g^{ip}\Big\{C_{ijl|k} + C_{ikl|j} - C_{jkl|i} + L_{ijk,l}\Big\}. \tag{3.1}$$

See (10.19) at page 145 in [13].

On the other hand, the Cartan torsion of a Finsler surface (M, F) has no components in the direction ℓ^i, i.e., $C_{ijk}y^i = 0$. Then it can be written in the Berwald frame (ℓ, m) as follows

$$FC_{ijk} = \mathcal{I}m_im_jm_k. \tag{3.2}$$

Proof of Theorem 1.1: By definition of a generalized Landsberg metric, we have

$$L^i{}_{jl|k} - L^i{}_{jk|l} + L^i{}_{sk}L^s{}_{jl} - L^i{}_{sl}L^s{}_{jk} = 0,$$

where "|" denotes the horizontal derivation with respect to the Berwald connection of F. Fix k and l and put

$$Q_{ij} := L_{ijl|k} - L_{ijk|l} + L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk}.$$

One can write

$$Q_{ij} := Q^s_{ij} + Q^a_{ij},$$

where

$$Q^s_{ij} := \frac{1}{2}\Big[Q_{ij} + Q_{ji}\Big], \quad \text{and} \quad Q^a_{ij} := \frac{1}{2}\Big[Q_{ij} - Q_{ji}\Big].$$

It is easy to see that $Q_{ij} = 0$ if and only if $Q^s_{ij} = 0$ and $Q^a_{ij} = 0$. On the other hand, we have

$$\begin{aligned} Q_{ji} =&\ L_{jil|k} - L_{jik|l} + L_{jsk}L^s{}_{il} - L\ {}_{jsl}L^s{}_{ik} \\ =&\ L_{ijl|k} - L_{ijk|l} + L^s{}_{jk}L_{sil} - L^s{}_{jl}L_{sik}. \end{aligned}$$

Hence

$$Q^s_{ij} = L_{ijl|k} - L_{ijk|l},$$

and consequently

$$Q^a_{ij} = L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk}.$$

Thus

$$Q^a = Q^s = 0,$$

which is equal to

$$L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk} = 0, \tag{3.3}$$

$$L_{ijl|k} - L_{ijk|l} = 0. \tag{3.4}$$

On the other hand, we have

$$FC_{ijk} = \mathcal{I}m_im_jm_k, \tag{3.5}$$

Taking a horizontal derivation of (3.5) implies that

$$FC_{ijk|s} = \big(\mathcal{I}_{|1}\ell_s + \mathcal{I}_{|2}m_s\big)m_im_jm_k. \tag{3.6}$$

Contracting (3.6) with y^s yields

$$L_{ijk} = \mathcal{I}_{|1}m_im_jm_k. \tag{3.7}$$

We have

$$L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk} = (\mathcal{I}_{|1})^2 m_im_sm_km^sm_jm_l - (\mathcal{I}_{|1})^2 m_im_sm_lm^sm_jm_k = 0. \tag{3.8}$$

(3.8) shows that the relation (3.3) trivially holds for two dimensional Finsler manifold. □

4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. First, we remark that by putting (3.6) and (3.7) in (3.1), we get

$$FB^i{}_{jkl} = \Big\{ -2\mathcal{I}_{|1}\ell^i + \big(\mathcal{I}_{|1,2} + \mathcal{I}_{|2}\big)m^i\Big\}m_jm_km_l. \tag{4.1}$$

Let us put $\mathcal{I}_2 := \mathcal{I}_{|1,2} + \mathcal{I}_{|2}$. Thus the Berwald curvature of Finsler surfaces is given by

$$B^i{}_{jkl} = \frac{1}{F}\Big(\mathcal{I}_2 m^i - 2\mathcal{I}_{|1}\ell^i\Big)m_jm_km_l. \tag{4.2}$$

By (3.2) and (4.2), we have

$$B^i{}_{jkl} = -\frac{2\mathcal{I}_{,1}}{\mathcal{I}}C_{jkl}\ell^i + \frac{\mathcal{I}_2}{3F}\Big\{h_{jk}h^i_l + h_{kl}h^i_j + h_{lj}h^i_k\Big\}, \tag{4.3}$$

where $\mathbf{h} = h_{ij}dx^idx^j$ denotes the angular metric. Then for a Finsler surface, the Berwald curvature can be written as follows

$$B^i{}_{jkl} = \mu C_{jkl}\ell^i + \lambda\Big(h^i_jh_{kl} + h^i_kh_{jl} + h^i_lh_{jk}\Big), \tag{4.4}$$

where

$$\mu := -\frac{2}{\mathcal{I}}\mathcal{I}_{|1}, \qquad \lambda := \frac{1}{3}\mathcal{I}_2.$$

Contracting (4.4) with y_i implies that

$$L_{jkl} + \frac{1}{2}\mu FC_{jkl} = 0. \tag{4.5}$$

Taking a trace of (4.5) implies that

$$J_i = -\frac{1}{2}\mu FI_i. \tag{4.6}$$

Proof of Theorem 1.2: By assumption, F has isotropic main scalar $\mathcal{I} = \mathcal{I}(x)$. Thus, we get

$$\mu = -\frac{2}{\mathcal{I}(x)}\mathcal{I}_{|1}(x).$$

which shows that

$$\mu = \mu(x).$$

In this case, by taking a vertical derivation of (4.6), we obtain

$$J_{i,j} = -\frac{1}{2}\mu\Big(F_j I_i + F I_{i,j}\Big). \tag{4.7}$$

By (4.7), we get

$$J_{i,j} + J_{j,i} = -\frac{1}{2}\mu\Big(F_j I_i + F_i I_j\Big) - \frac{1}{2}\mu F\Big(I_{i,j} + F I_{j,i}\Big). \tag{4.8}$$

Taking a horizontal derivation of (4.8) along Finslerian geodesic yields

$$\begin{aligned}\Big(J_{i,j} + J_{j,i}\Big)_{|s} y^s = -\frac{1}{2}\mu'\Big(F_j I_i + F_i I_j\Big) - \frac{1}{2}\mu\Big(F_j J_i + F_i J_j\Big) - \frac{1}{2}\mu' F\Big(I_{i,j} + I_{j,i}\Big)\\ -\frac{1}{2}\mu F\Big(I_{i,j|s} + I_{j,i|s}\Big)y^s.\end{aligned} \tag{4.9}$$

where $\mu' := \mu_{x^s} y^s$. The following Ricci identity holds

$$I_{i,j|s} = I_{i|s,j} + I_r B^r{}_{ijs}. \tag{4.10}$$

Multiplying (4.10) with y^s implies that

$$I_{i,j|s} y^s = I_{i|s,j} y^s. \tag{4.11}$$

Also, the following holds

$$J_{i,j} = (I_{i|s} y^s)_{,j} = I_{i|s,j} y^s + I_{i|j}, \tag{4.12}$$

which gives us

$$I_{i|s,j} y^s = J_{i,j} - I_{i|j}. \tag{4.13}$$

Comparing (4.11) and (4.13) yields

$$I_{i,j|s} y^s = J_{i,j} - I_{i|j}. \tag{4.14}$$

Putting (4.7) in (4.14) implies

$$I_{i,j|s} y^s = -\frac{1}{2}\mu\Big(F_j I_i + F I_{i,j}\Big) - I_{i|j}. \tag{4.15}$$

By putting (4.6), (4.15) in (4.9), one can obtain

$$\begin{aligned}\Big(J_{i,j} + J_{j,i}\Big)_{|s} y^s = \frac{1}{2}\Big(\mu^2 F - \mu'\Big)(F_j I_i + F_i I_j) + \frac{1}{4}F\Big(\mu^2 F - 2\mu'\Big)\Big(I_{i,j} + I_{j,i}\Big)\\ +\frac{1}{2}\mu F\Big(I_{i|j} + I_{j|i}\Big).\end{aligned} \tag{4.16}$$

By (4.16) and the assumption, we get (1.1). □

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