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Research Paper

BEST ω-PROXIMITY POINT FOR ω-PROXIMAL QUASI CONTRACTION MAPPINGS IN MODULAR METRIC SPACES

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Abstract. In this paper we introduce ω-proximal quasi contraction mapping and best ω-proximity point in modular metric spaces. In fact, we show that every ω-proximal quasi contraction mapping has unique best ω-proximity point in modular metric spaces. Finally, we give an example to illustrate the applications of our results.

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1. Introduction

The classical contraction mapping principle of Banach states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, then T has a unique fixed point. This principle has attracted the attention of many authors to extend and refine the metric fixed point theory.

For this aim, the authors considered to extend metric fixed point theory to different abstract spaces such as modular metric spaces, that was introduced by Chistykov in [3, 4] as a generalization of modular spaces and Cho in [5] developed the fixed point theory to modular metric spaces.

In this case, as a generalization of Banach contraction principle, Ćirić, in 1974, in [6], has introduced the concept of quasi-contraction mapping and investigated the fixed point result similar to the Banach contraction fixed point Theorem. And Choin in [5] introduced the concept of quasi-contraction mappings in modular metric spaces.

The study of the existence of fixed point for non-self mapping on various abstract spaces is also very interesting. More precisely, for a given nonempty closed subsets A and B of a complete metric space (X, d), a contraction nonself mapping $T : A \to B$ does not necessarily yields a fixed point, that is, $d(Tx, x) \neq 0$. In this case, it is quite natural to investigate an element $x \in X$ such that $d(x, Tx)$ is minimum, that is, the points x and Tx are close proximity to each other.

The generally accepted point of view in this domain, let A and B be closed subsets of a metric space (X, d) and $T : A \to B$ be a nonself mapping. A point a in A for which $d(a, Ta) = d(A, B)$ is called a best proximity point of T . If $A \cap B \neq \varnothing$ then the best proximity point becomes a fixed point of T. The best proximity point Theorems are natural

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generalizations of the BCMP. A classical best approximation theorem was introduced by Fan in [7], and several authors have derived the extensions of the Fan's Theorem in many directions, for example see [9] and [12]. M. Jleli, E. Karapinar proved best proximity point in modular spaces in [8].

In this paper, we first introduce the notions of best ω-proximity point and the ω-proximal quasi-contraction mapping in modular metric spaces. Next, we show that every ω-proximal quasi-contraction mapping in modular metric spaces has best ω-proximity point. We also give an example of our main result. We begin by recalling some terminology.

Let X be a nonempty set. Throughout this paper, for a given function

$$\omega : (0, +\infty) \times X \times X \to [0, +\infty]$$

we will denote $\omega_\lambda(x, y) = \omega(\lambda, x, y)$, for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a modular metric on X if it satisfies in the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If (i) is replaced by following condition,

(i′) $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$,

then ω is said to be a pseudomodular (metric) on X. Also a modular on X is said to be regular if (i) was replaced by the following weaker condition;

(i′′) $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for some $\lambda > 0$.

For more information see for example [3, 4] and the references therein. Finally, ω is said to be convex if for any $\lambda, \mu > 0$ and $x, y, z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda + \mu}\omega_\mu(z, y).$$

Note that for a metric pseudomodular ω on a set X and any $x, y \in X$, the function $\lambda \to \omega_\lambda(x, y)$ is nonincreasing on $(0, \infty)$. Indeed if $0 < \mu < \lambda$, then

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

Let ω be a pseudomodular on X and x_0 be a fixed element of X, then

$$X_\omega = X_\omega(x_0) = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x, x_0) = 0\},$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0; \quad \omega_\lambda(x, x_0) < \infty\}$$

are said to be modular metric spaces around x_0. It is clear that $X_\omega \subset X^*_\omega$, but this inclusion may be proper in general. If ω is a convex modular on X, according to [3] the two modular spaces coincide, i.e, $X_\omega = X^*_\omega$. It is obviously to see that every metric space is modular metric space.

Note. Let (X_ω) be a modular metric space.

(1) a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to a point $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$, for some $\lambda > 0$.

(2) a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-couchy if $\omega_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$, for some $\lambda > 0$.
(3) A subset C of X_ω is said to be ω-closed if the ω-limit of ω-convergent sequence of C always belong to C.
(4) A subset C of X_ω is said to be ω-complete if every ω-cauchy sequence in C is ω-convergent and its ω-limit is in C.

The metric modular ω on X has the Fatou's property if and only if

$$\omega_\lambda(x, y) \leq \liminf_{n\to\infty} \omega_\lambda(x_n, y), \tag{1.1}$$

whenever $\{x_n\} \subseteq X_\omega$ and ω-convergent to x for all $y \in X_\omega$.

2. Main results

We adopt throughout the convention that modular metric spaces (X, ω) has the Fatou's property. Suppose that A, B are nonempty subsets of X_ω. For all $\lambda > 0$, we define:

$$\gamma(\lambda, A, B) = \gamma_\lambda(A, B) = \inf\{\omega_\lambda(x, y) : (x, y) \in A \times B\},$$

$$A_0 = A_0(\lambda) = \{a \in A : \omega_\lambda(a, b) = \gamma_\lambda(A, B)\ for\ some\ b \in B\},$$

$$B_0 = B_0(\lambda) = \{b \in B : \omega_\lambda(a, b) = \gamma_\lambda(A, B)\ for\ some\ a \in A\}.$$

Definition 2.1. Let A, B be nonempty subsets of X_ω and $T : A \to B$ be a given nonself mapping. An element $z \in A_0$ is said to be a best ω-proximity point of T if for every positive number λ,

$$\omega_\lambda(z, Tz) = \gamma_\lambda(A, B).$$

It is easy to see that if $A = B$, then the best ω-proximity point becomes a fixed point of T.

Definition 2.2. A nonself mapping $T : A \to B$ is said to be ω-proximal quasi contraction if there exists a number $q \in (0, 1)$ such that

$$\omega_\lambda(u, v) \leq q \max\{\omega_\lambda(x, y), \omega_\lambda(x, u), \omega_\lambda(y, v), \omega_\lambda(x, v), \omega_\lambda(y, u)\},$$

whenever $\omega_\lambda(u, Tx) = \gamma_\lambda(A, B)$ and $\omega_\lambda(v, Ty) = \gamma_\lambda(A, B)$, for all $x, y, u, v \in A$ and $\lambda \in (0, \infty)$.

If T is a self-mapping on A, then the requirement in the preceding definition, reduces to the ω-quasi contraction, that is there exists $q \in (0, 1)$ such that for any $x, y \in A$ we have

$$\omega_\lambda(Tx, Ty) \leq \max\{\omega_\lambda(x, y), \omega_\lambda(x, Tx), \omega_\lambda(x, Ty), \omega_\lambda(y, Ty), \omega_\lambda(y, Ty)\}.$$

Lemma 2.3. Let $T : A \to B$ be a non-self mapping, $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$. Then for any $a \in A_0$, there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_0 = a$ and $\omega_\lambda(x_{n+1}, Tx_n) = \gamma_\lambda(A, B)$ for any $n \in \mathbb{N}$.

Proof. Let $a \in A_0$, By using (2) we get, $T(a) \in B_0$. From the definition of the set B_0, there exists $x_1 \in A_0$ such that $\omega_\lambda(x_1, Ta) = \gamma_\lambda(A, B)$
Again, we have $Tx_1 \in B_0$, which implies that there exists $x_2 \in A_0$ such that $\omega_\lambda(x_2, Tx_1) = \gamma_\lambda(A, B)$. Continuing this process, by using the induction, we obtain $\{x_n\} \subseteq A_0$ such that

$$\omega_\lambda(x_{n+1}, Tx_n) = \gamma_\lambda(A, B), \qquad \forall n \in \mathbb{N}$$

$\square$

Definition 2.4. Under the assumption of lemma 2.3, any sequence $\{x_n\} \subseteq A_0$ satisfying $\omega_\lambda(x_{n+1}, Tx_n) = \gamma_\lambda(A, B)$ for any $n \in \mathbb{N}$ and $\lambda > 0$ and $x_0 = a$ is called an proximal Picard sequence associated to $a \in A_0$.
We denote by $pp(a)$ the set of all proximal Picard sequences associated to $a \in A_0$.

Definition 2.5. Under the assumption of lemma 2.3, we say that A_0 is proximal T-orbitally ω-complete if every ω-Cauchy sequence $\{x_n\} \in pp(a)$ for some $a \in A_0$, ω-converges to an element in A_0.

Let $a \in A_0$ and $\{x_n\} \in pp(a)$. For all $n \in \mathbb{N}$, we denote

$$\delta_{\omega_\lambda}(x_n) = \sup\{\omega_\lambda(x_{n+s}, x_{n+r}) : r, s \in \mathbb{N}\}.$$

Since $x_0 = a$, then

$$\delta_{\omega_\lambda}(a) = \sup\{\omega_\lambda(x_s, x_r) : r, s \in \mathbb{N}\}.$$

Theorem 2.6. Let X_ω be a modular metric space. Let (A, B) be a non-empty pair of subsets of X_ω and $T : A \to B$ be a ω-proximal quasi-contraction mapping such that A_0 is proximal T-orbitally ω-complete and $T(A_0) \subseteq B_0$ and $\delta_{\omega_\lambda}(a) < \infty$ for some $a \in A_0$. Then:

(i) each $\{x_n\} \in pp(a)$, ω-converges to some $z \in A_0$; Moreover there exist $t \in A_0$ such that $\omega_\lambda(t, Tz) = \gamma_\lambda(A, B)$.
(ii) if $\omega_\lambda(z, t) < \infty$ and $\omega_\lambda(a, t) < \infty$ for any $\lambda > 0$, then $z \in A_0$ is a best ω-proximity point of T;
(iii) if u is a best ω-proximity point of T and $\omega_\lambda(z, u) < \infty$, then $z = u$.

Proof. (i) Let $\{x_n\} \in pp(a)$ and $s, r \in \mathbb{N}$. We have

$$\omega_\lambda(x_{n+s}, x_{n+s-1}) = \omega_\lambda(x_{n+r}, x_{n+r-1}) = \gamma_\lambda(A, B).$$

Since T is a ω-proximal quasi-contraction,

$$\begin{aligned} \omega_\lambda(x_{n+s}, x_{n+r}) &\leq q \max\{\omega_\lambda(x_{n+s-1}, x_{n+r-1})\omega_\lambda(x_{n+s-1}, x_{n+s}) \\ &\quad \omega_\lambda(x_{n+r-1}, x_{n+r})\omega_\lambda(x_{n+s-1}, x_{n+r})\omega_\lambda(x_{n+r-1}, x_{n+s})\} \\ &\leq q\delta_\omega(x_{n-1}). \end{aligned}$$

This implies immediately that $\delta_\omega(x_n) \leq q\delta_\omega(x_{n-1})$, for all $n \geq 1$. Now by induction on n, we have

$$\delta_\omega(x_n) \leq q\delta_\omega(x_{n-1}) \leq q^2\delta_\omega(x_{n-2}) \leq \ldots \leq q^n\delta_\omega(x_0) = q^n\delta_\omega(a). \tag{2.1}$$

Thus, $\{x_n\}$ is ω_λ-Cauchy, since $\delta_{\omega_\lambda}(a) < \infty$.
By using the proximal T-orbitally ω-completeness of A_0, let $z \in A_0$ be a ω-limit of $\{x_n\}$.
So $Tz \in B_0$, and by definition there exists some $t \in A_0$ such that $\omega_\lambda(t, Tz) = \gamma_\lambda(A, B)$.
(ii) From definition of $\{x_n\}$, we have $\omega_\lambda(x_1, Ta) = \gamma_\lambda(A, B)$. Since T is ω-proximal quasi-contraction then:

$$\begin{aligned} \omega_\lambda(t, x_1) &\leq q \max\{\omega_\lambda(a, z), \omega_\lambda(z, t), \omega_\lambda(a, x_1), \omega_\lambda(z, x_1), \omega_\lambda(a, t)\} \\ &\leq \max\{q\delta_{\omega_\lambda}(a), q\omega_\lambda(z, t), q\omega_\lambda(a, t)\}. \end{aligned}$$

With the same method we have

$$\begin{aligned}
\omega_\lambda(t,x_2) &\leq q\max\{\omega_\lambda(z,x_1),\omega_\lambda(z,t),\omega_\lambda(x_1,x_2),\omega_\lambda(z,x_2),\omega_\lambda(x_1,t)\}\\
&\leq q\max\{q\delta_{\omega_\lambda}(a),\omega\lambda(z,t),\delta_{\omega_\lambda}(x_1),q^2\delta_{\omega_\lambda(a)},\omega_\lambda(x_1,t)\}\\
&\leq q\max\{q\delta_{\omega_\lambda}(a),\omega_\lambda(z,t),q\delta_{\omega_\lambda}(a),q^2\delta_{\omega_\lambda}(a),\omega_\lambda(x_1,t)\}\\
&= q\max\{q\delta_{\omega_\lambda}(a),\omega_\lambda(z,t),\omega_\lambda(x_1,t)\}\\
&\leq \max\{q^2\delta_{\omega_\lambda}(a),q\omega_\lambda(z,t),q^2,\omega_\lambda(a,t)\}.
\end{aligned}$$

Continuing this process, by induction, we circumvent that

$$\omega_\lambda(t,x_n)\leq \max\{q^n\delta_{\omega_\lambda}(a),q\omega_\lambda(z,t),q^n\omega\lambda(a,t)\},$$

for all $n\geq 1$. Therefore,

$$\limsup_{n\to\infty}\omega_\lambda(x_n,t)\leq q\omega_\lambda(z,t),\qquad \lambda>0.$$

By the Fatou's property, we have

$$\omega_\lambda(z,t)\leq q\omega_\lambda(z,t),$$

and this shows that $\omega_\lambda(z,t)=0$; that is $z=t$, since $q<1$. So this proves that z is a best ω-proximity point of T, that is, $\omega_\lambda(z,Tz)=\gamma_\lambda(A,B)$, for all $\lambda>0$.

(iii) Suppose that u is a best ω-proximity point of T and $\omega_\lambda(z,u)<\infty$.
Since T is ω-proximal quasi-contraction map,

$$\begin{aligned}
\omega_\lambda(z,u) &\leq q\max\{\omega_\lambda(z,u),\omega_\lambda(z,z),\omega_\lambda(u,u),\omega_\lambda(z,u),\omega_\lambda(u,z)\}\\
&= q\omega_\lambda(z,u),
\end{aligned}$$

so $u=z$, because $q<1$. Thus z is the unique best ω-proximal point of T and this completes the proof. ☐

Consider now the case $A=B$, then the best proximity point of $T:A\to B$ will be a fixed point of self mapping T.

Corollary 2.7. Let X_ω be a modular metric space and A be a non-empty subset of X_ω which is ω-complete. If $T:A\to A$ is a quasi-contraction mapping and there exist $a\in A$ such that $\delta_\omega(a)<\infty$. Then the Picard iteration sequence $\{T^n a\}$ is ω-convergent to a fixed point $z\in A$. If u is a fixed point of T such that $\omega_\lambda(z,u)<\infty$, then $z=u$.

Example 2.8. Consider $\mathbb{R}^2$, and define d_∞ on $\mathbb{R}^2$ by

$$d_\infty((x_1,x_2),(y_1,y_2))=\max\{|x_1-y_1|,|x_2-y_2|\} \tag{2.2}$$

Let $\lambda_0>0$ and $a>0$, define $g(\lambda)$ by:

$$g(\lambda)=\begin{cases} a & \text{if } 0<\lambda<\lambda_0,\\ 0 & \text{if } \lambda\geq\lambda_0.\end{cases}$$

The modular ω on $\mathbb{R}^2$ is of the form:

$$\omega_\lambda(x,y)=g(\lambda)d_\infty(x,y)=\begin{cases} 0 & \text{if } x=y,\ \lambda>0,\\ ad_\infty(x,y) & \text{if } x\neq y,\ 0<\lambda<\lambda_0,\\ 0 & \text{if } x\neq y,\ \lambda\geq\lambda_0.\end{cases} \tag{2.3}$$

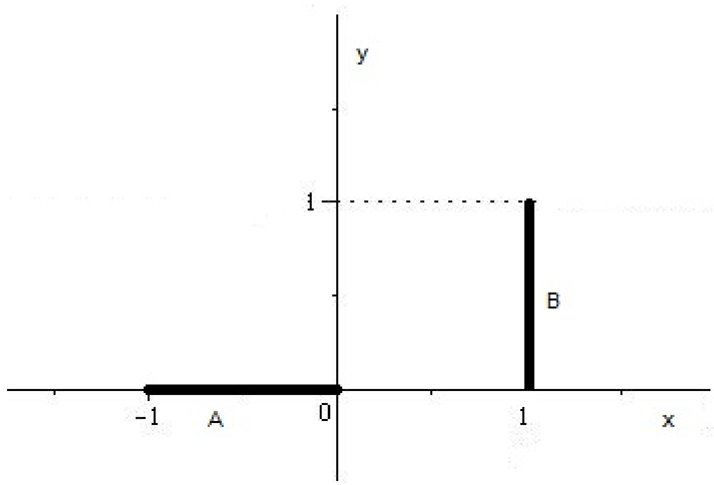

FIGURE 1. The pair (A, B) of closed subsets of $(\mathbb{R}^2, d)$.

Hence $(\mathbb{R}^2, \omega_\lambda)$ is modular space and $\mathbb{R}^2_\omega = \mathbb{R}^2$. Assume that

$$A = \{(x,0)| -1 \leq x \leq 0\} \ and \ B = \{(1,y)|0 \leq y \leq 1\}$$

See Figure 1. It is clearly

$$\gamma_\lambda(A,B) = \inf\{\omega_\lambda((x,0),(1,y))| -1 \leq x \leq 0\ ,\ 0 \leq y \leq 1\} = \begin{cases} a & \text{if } 0 < \lambda < \lambda_0, \\ 0 & \text{if } \lambda \geq \lambda_0. \end{cases}$$

Also

$$\begin{aligned} A_0 &= \{(x,0) \in A | \omega_\lambda((x,0),(1,b)) = \gamma_\lambda(A,B)\ for\ some\ (1,b) \in B\} \\ &= \begin{cases} \{(0,0)\} & \text{if } 0 < \lambda < \lambda_0, \\ A & \text{if } \lambda \geq \lambda_0. \end{cases} \end{aligned}$$

and

$$B_0 = \{(1,y) \in B | \omega_\lambda((a,0),(1,y)) = a\ for\ some\ (a, 0 \in A)\} = B.$$

Define $T : A \to B$ by $T(x,0) = (1, \varphi(x))$ where φ is any function from $[-1,0]$ to $[a,b]$ such that $0 \leq a < b \leq 1$. It is obviously T is ω-proximal quasi contraction, since there is one and only one point $u = (0,0)$ such that $\omega_\lambda(u,v) = a$ for all $u \in B$. So $u = (0,0)$ is the unique best ω-proximity point 1of T.

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