



## A NEW METHOD FOR SOLVING NONLINEAR VOLTERRA-HAMMERSTEIN INTEGRAL EQUATIONS VIA SINGLE-TERM WALSH SERIES

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**ABSTRACT.** In this article, the properties of single-term Walsh series are presented and utilized for solving the nonlinear Volterra-Hammerstein integral equations of second kind. The interval  $[0, 1]$  is divided to  $m$  equal subintervals,  $m$  is a positive integer number. The mid-point of each subinterval is chosen as a suitable collocation point. By the method the computations of integral equations reduce into some nonlinear algebraic equations. The method is computationally attractive, and gives a continuous approximate solution. An analysis for the convergence of method is presented. The efficiency and accuracy of the method are demonstrated through illustrative examples. Some comparisons are made with the existing results.

**MSC(2010):** 65N06

**Keywords:** Collocation method, Integral equations, STWS method, Volterra-Hammerstein.

### 1. Introduction

Walsh functions (WF) have found wide applications in signal processing, communication and pattern recognition [15]. Hsiao and Chen [10] introduced the Walsh series operational matrix of integration to solve linear integral equations. Rao et. al [14] presented a method of extending computation beyond the limit of the initial normal interval in Walsh series analysis of dynamical systems. In the last method, various time functions in the system were first expanded as truncated WF with unknown coefficients. Then, by using the Kronecker product [13], the unknown coefficient of the rate variable was obtained by finding the inverse of an square matrix. It was shown that this method involves some numerical troubles if the dimension of this matrix is large. Rao et. al [14] introduced single-term Walsh series (STWS) to remove the inconveniences in WF technique. Furthermore, by using STWS the analysis of linear and nonlinear singular systems were given in [2, 3], and the direct method for the solution of nonlinear problems in the Calculus of variations was presented in [16]. In [18] a STWS method for nonlinear Volterra-Hammerstein equations is introduced and in [4], Balakumar and Murugusan developed the method for linear systems of Volterra integral equations. Also, Sepehrian introduced a STWS method for solving Volterra's population model in [17].

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*Date:* Received: October 9, 2020, Accepted: December 9, 2020.

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In the present article we are concerned with the numerical solution of nonlinear Volterra-Hammerstein integral equations of the form

$$(1.1) \quad x(t) = f(t) + \int_0^t \kappa(t, s)h(s, x(s))ds, \quad 0 \leq t < 1,$$

where  $f, h$  and  $\kappa$  are given continuous functions, with  $h(s, x)$  nonlinear in  $x$ . We assume that Eq. (1) has a unique solution  $x$  to be determined.

Several numerical methods for approximating the solution of integral equations are known. In [5], Brunner applied a collocation-type method to Eq. (1) and integro-differential equations, and discussed its connection with the iterated collocation method. The collocation methods for nonlinear Volterra-Fredholm integral equations were investigated in [6]. In [19] piecewise constant bounds on the solution of mixed nonlinear Volterra-Fredholm integral equations were presented. Guoqiang [9] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra-Hammerstein integral equations and a collocation-type method was developed in [12]. The methods in [9] and [12] transform a given integral equation into a system of nonlinear equations, which has to be solved with some kind of iterative methods. In [12], the definite integrals involved in the solution may be evaluated analytically only in favorable cases, while in [9] the integrals involved in the solution have to be evaluated at each time step of the iteration. In [1] the Hat functions were applied for Eq. (1) and a piecewise solution was obtained. Their method requires both operational matrix of integration, product operational matrix and solving large systems of nonlinear algebraic equations. However, in all of these methods solving the integral equations reduce to some systems of nonlinear algebraic equations. But, if the function  $h(s, x(s))$  in (1) is a complicated function of  $x(s)$ , these nonlinear systems are very complicated and solving them is very difficult especially for large ones. Even if these nonlinear systems are solvable by some kind of iteration methods, the computational cost can be very high. For this reason, most of the spectral methods are not so efficient for solving nonlinear integral equations with complicated  $h(s, x(s))$ . To overcome this barrier one candidate is STWS method. Because by STWS method the interval  $[0, 1]$  is divided to  $m$  equal intervals and in each interval just one algebraic equation (not a system) needs to be solved. In reference [18] the authors offered a Galerkin type STWS method to solve the nonlinear Volterra-Hammerstein integral equations of the form in Eq. (1). In [18] first by approximating  $f(t)$  and  $\kappa(t, s)$  by STWS, a block-pulse approximation of  $x(t)$  was calculated. Then by using the calculated block-pulse approximation a continuous approximation of unknown function  $x(t)$  was obtained. But, in general the Galerkin type methods are not so suitable for solving complex nonlinear equations and the method in [18] is applicable only for simple  $h(t, x(t))$  such as  $x^2(t)$  and  $x^3(t)$ , because the method in [18] needs to the STWS expansion of  $h(t, x(t))$ .

In the present paper, we apply the STWS with a collocation method to solve the nonlinear Volterra-Hammerstein integral equations given in Eq. (1). The method is first applied to an equivalent integral equation  $y(t) = h(t, x(t))$ ,  $t \in [0, 1]$  where the solution  $y$  is approximated by a STWS function with unknown coefficients. The properties of STWS together with a collocation method are then used to evaluate the unknown coefficients and find a continuous approximate solution for  $x(t)$ . The method presented here does not require the Kronecker product of matrices and there is no need for the operational matrix of integration and product operational matrix and solving large systems of nonlinear algebraic equations. However our method basically differs from the method in [18] and unlike that method, we do not need to approximate  $f(t)$  and  $\kappa(t, s)$  by STWS and, a continuous approximation of  $x(t)$  can be

calculated directly. So, as compared to [18], the present method is simpler and consumes less computer time.

The article is organized as follows: In Section 2, we describe the basic formulation of STWS required for our subsequent development. Section 3 is devoted to the solution of Eq. (1) by using STWS and a collocation method. The convergence of our method is investigated in Section 4. In Section 5, we report our numerical findings and demonstrate the accuracy of proposed scheme by considering numerical examples.

### 2. Properties of WF and STWS

A function  $f(t)$ , square integrable in  $[0,1)$ , may be approximated using WF as

$$(2.1) \quad f(t) = \sum_{i=0}^{\infty} f_i \phi_i(t),$$

where  $\phi_i(t)$  is the  $i$ -th WF and  $f_i$  is the corresponding coefficient. In practice, only the first  $m$ -term WF are considered, where  $m$  is an integral power of 2. Then from Eq. (2) we get

$$f(t) = \sum_{i=0}^{m-1} f_i \phi_i(t) = F^T \Phi(t),$$

where

$$F = (f_0, f_1, \dots, f_{m-1})^T,$$

and

$$\Phi(t) = (\phi_0(t), \phi_1(t), \dots, \phi_{m-1}(t))^T.$$

The coefficients  $f_i$  are chosen to minimize the mean integral square error

$$\epsilon = \int_0^1 (f(t) - F^T \Phi(t))^2 dt,$$

and are given by

$$f_i = \int_0^1 f(t) \phi_i(t) dt.$$

With the STWS approach, in the first interval, the given function is expanded as single-term Walsh series in the normalized interval  $\tau \in [0, 1)$ , which corresponds to  $t \in [0, \frac{1}{m})$  by defining  $\tau = mt$ ,  $m$  being any integer.

### 3. Solution of Nonlinear Volterra-Hammerstein Integral Equation

Consider the Volterra-Hammerstein integral equation given in Eq. (1). To solve for  $x(t)$ , we first approximate the solution not to the Eq. (1), but rather to an equivalent equation

$$(3.1) \quad y(t) = h(t, x(t)) \quad , \quad 0 \leq t < 1.$$

From Eqs. (1) and (3) we get

$$(3.2) \quad y(t) = h\left(t, f(t) + \int_0^t \kappa(t, s)y(s)ds\right).$$

In order to use STWS, we first divide interval  $[0, 1)$  to  $m$  equal subinterval,  $m$  is any positive integer. We then transform each interval  $\frac{i-1}{m} \leq s \leq t < \frac{i}{m}$ ,  $i = 1, \dots, m$ , into  $\tau_i \in [0, 1)$  and

$\lambda_i \in [0, 1]$  by means of transformation  $\tau_i = mt - (i - 1)$  and  $\lambda_i = ms - (i - 1)$  respectively and for  $t \in [\frac{i-1}{m}, \frac{i}{m})$  and  $s \in [\frac{j-1}{m}, \frac{j}{m})$  we define

$$(3.3) \quad y(t) = y_i(\tau_i), \quad f(t) = f_i(\tau_i), \quad h(t, x(t)) = h_i(\tau_i, x(\tau_i)),$$

and

$$(3.4) \quad \kappa(t, s) = \kappa_{i,j}(\tau_i, \lambda_j), \quad i, j = 1, \dots, m.$$

Thus, using Eqs. (4)-(6), in the first interval  $\tau_1$  we have

$$(3.5) \quad y_1(\tau_1) = h_1 \left( \tau_1, f_1(\tau_1) + \frac{1}{m} \int_0^{\tau_1} \kappa_{1,1}(\tau_1, \lambda_1) y(\lambda_1) d\lambda_1 \right),$$

and similarly in the  $i$ th interval we have

$$(3.6) \quad y_i(\tau_i) = h_i \left( \tau_i, f_i(\tau_i) + \frac{1}{m} \sum_{j=1}^{i-1} \int_0^1 \kappa_{i,j}(\tau_i, \lambda_j) y(\lambda_j) d\lambda_j + \frac{1}{m} \int_0^{\tau_i} \kappa_{i,i}(\tau_i, \lambda_i) y(\lambda_i) d\lambda_i \right).$$

Let  $y_i(\tau_i)$  be expressed by STWS as

$$(3.7) \quad y_i(\tau_i) = Y^{(i)} \phi_0(\tau_i),$$

in which,  $\phi_0(\tau_i) = 1$ .

To construct the approximations for  $y(t)$  we collocate Eqs. (7) and (8). For a suitable collocation point we choose the mid point of each interval as collocation point. For the these points we choose  $\tau_i = \frac{1}{2}, i = 1, 2, \dots, m$  and from Eqs. (7)-(9) we get

$$(3.8) \quad Y^{(1)} = h_1 \left( \frac{1}{2}, f_1\left(\frac{1}{2}\right) + \frac{1}{m} Y^{(1)} \int_0^{\frac{1}{2}} \kappa_{1,1}\left(\frac{1}{2}, \lambda_1\right) d\lambda_1 \right),$$

and

$$(3.9) \quad Y^{(i)} = h_i \left( \frac{1}{2}, f_i\left(\frac{1}{2}\right) + \frac{1}{m} \sum_{j=1}^{i-1} (Y^{(j)}) \int_0^1 \kappa_{i,j}\left(\frac{1}{2}, \lambda_j\right) d\lambda_j + \frac{1}{m} Y^{(i)} \int_0^{\frac{1}{2}} \kappa_{i,i}\left(\frac{1}{2}, \lambda_i\right) d\lambda_i \right),$$

for  $i = 2, \dots, m$ .

Eqs. (10) and (11) are nonlinear algebraic equations with unknown variables  $Y^{(1)}$  and  $Y^{(i)}, i = 2, \dots, m$ , respectively. By solving Eq. (10) for  $Y^{(1)}$  and Eq. (11) for  $Y^{(i)}, i = 2, \dots, m$ , STWS approximations of  $y(t)$  can be obtained. Then, by using Eqs. (1), (3) and (9) for  $t \in [0, \frac{1}{m})$ , we have

$$(3.10) \quad x(t) = f(t) + Y^{(1)} \int_0^t \kappa(t, s) ds,$$

and for  $t \in [\frac{i-1}{m}, \frac{i}{m})$ , we get

$$(3.11) \quad x(t) = f(t) + \sum_{j=1}^{i-1} \left( Y^{(j)} \int_{\frac{j-1}{m}}^{\frac{j}{m}} \kappa(t, s) ds \right) + Y^{(i)} \int_{\frac{i-1}{m}}^t \kappa(t, s) ds, \quad i = 2, \dots, m.$$

Equations (12) and (13) provide continuous approximations for  $x(t)$  at every  $t \in [0, 1)$ .

#### 4. Analysis of convergence

Let  $x_{exact}(t)$  be the exact solution of Eq. (1) and  $x_{approx.}(t)$  be the approximate solution given by Eqs. (12) and (13). We define  $y(s) = g(s, x_{exact}(s))$  and we suppose  $y(s)$  has continuous first order derivative in interval  $[0, 1]$ . Since  $Y^{(j)}$  is the coefficient of block-pulse approximation of  $y(s)$  in interval  $[\frac{j-1}{m}, \frac{j}{m}]$ , so

$$(4.1) \quad Y^{(j)} = m \int_{\frac{j-1}{m}}^{\frac{j}{m}} y(s) ds, \quad j = 1, 2, \dots, m.$$

Therefore, by mean value theorem for integral, there is a  $\mu_j \in [\frac{j-1}{m}, \frac{j}{m}]$  such that

$$(4.2) \quad Y^{(j)} = y(\mu_j), \quad j = 1, 2, \dots, m.$$

From Eqs. (1), (12) and (15), for  $0 \leq t \leq \frac{1}{m}$  we get

$$(4.3) \quad |x_{exact}(t) - x_{approx.}(t)| = \left| \int_0^t \kappa(t, s)(y(s) - y(\mu_1)) ds \right|.$$

Since  $\kappa(t, s)$  is continuous, therefore

$$(4.4) \quad |\kappa(t, s)| \leq M, \quad 0 \leq s \leq t \leq \frac{1}{m},$$

for a positive real number  $M$ .

By Eqs. (16) and (17) and mean value theorem, we get

$$|x_{exact}(t) - x_{approx.}(t)| \leq \frac{M}{m} \int_0^t |y'(\nu_s)| ds,$$

in which  $\nu_s$  is between  $s$  and  $\mu_1$ . So for  $0 \leq t \leq \frac{1}{m}$ ,

$$(4.5) \quad |x_{exact}(t) - x_{approx.}(t)| \leq \frac{M}{m^2} \max_{0 \leq s \leq \frac{1}{m}} |y'(s)| = \frac{M}{m^2} |y'(\xi_1)|,$$

for a  $\xi_1 \in [0, \frac{1}{m}]$ .

Similarly, by Eqs. (1), (13), (15) and (17), for  $\frac{i-1}{m} \leq t \leq \frac{i}{m}$  we can write

$$(4.6) \quad |x_{exact}(t) - x_{approx.}(t)| \leq \frac{iM}{m^2} |y'(\xi_i)|, \quad i = 2, \dots, m,$$

for a  $\xi_i \in [0, \frac{i}{m}]$ .

Thus, by Eqs. (18) and (19) we can generally conclude

$$(4.7) \quad |x_{exact}(t) - x_{approx.}(t)| \leq \frac{M}{m} |y'(\xi)|,$$

for a  $\xi \in [0, 1]$ . The above inequality shows that by increasing  $m$ , the approximate solutions converge to the exact solution.

#### 5. Illustrative Examples

We applied the method presented in this paper and solved 6 examples. The first and second examples have been given in [18]. These examples were carried out in [18] by using STWS, where a block-pulse approximation of  $x(t)$  was first calculated. Then by using the calculated block-pulse approximation a continuous approximation of unknown function  $x(t)$  was obtained. Example 3 has been solved in [1] by Hat functions which are another piecewise continuous functions. Thus these examples could be used as a basis for comparison. Other

three examples are devoted to equations in which  $h(s, x(s))$  are complicated functions of  $x(t)$  and cannot be solved by many other techniques (for example the methods in [1, 18]). We performed our computations using **Maple 12** software.

5.1. **Example 1.** Consider the nonlinear Volterra-Hammerstein integral equation [18]

$$(5.1) \quad x(t) = -\frac{15}{56}t^8 + \frac{13}{14}t^7 - \frac{11}{10}t^6 + \frac{9}{20}t^5 + t^2 - t + \int_0^t (t+s)x^3(s)ds, \quad t \in [0, 1).$$

By applying the method described in the section 3, Eq. (21) is solved. The computational results of  $x(t)$  with  $m = 20$  and  $m = 40$ , together with the results in [18] for continuous approximations (CA) with  $m = 40$  and the exact solution  $x(t) = t^2 - t$  are given in Table 1. Table 1 shows the numerical results of the new method with  $m = 20$  are more accurate than those of the method in [18] with  $m = 40$ . In Table 2, the least square errors for the present method together with block-pulse and CA in [18] with  $m = 20$  and  $m = 40$  are shown. The following is used for least square errors for the approximation  $\bar{x}(t)$  of  $x(t)$ .

$$\int_0^1 (x(t) - \bar{x}(t))^2 dt = \frac{1}{m} \sum_{i=1}^m \int_{\frac{i-1}{m}}^{\frac{i}{m}} (x(\tau_i) - \bar{x}(\tau_i))^2 d\tau_i.$$

In Figure 1, we plot the absolute error of the approximate solution resulted by the new method with  $m = 5$ . Figure 1 show that the absolute error for  $m = 5$  is less than  $0.3 \times 10^{-4}$ .

TABLE 1. Estimated values of  $x(t)$  using STWS with  $m = 10$  and  $m = 20$  for the present method together with continuous approximations in [18] with  $m = 40$  and exact values for Example 1.

$t$	CA in [18] ( $m = 40$ )	present ( $m = 10$ )	present ( $m = 20$ )	<i>Exact</i>
0.2	-0.16000	-0.15999	-0.16000	-0.16000
0.4	-0.24000	-0.23998	-0.24000	-0.24000
0.6	-0.24000	-0.24001	-0.24000	-0.24000
0.8	-0.16001	-0.16002	-0.16000	-0.16000
1.0	0.00001	0.00001	0.00000	0.00000

TABLE 2. Least Square Errors for the present method together with the block-pulse and continuous approximations in Example 1.

<i>Error</i>	$m = 20$	$m = 40$
block-pulse in [18]	$0.69 \times 10^{-4}$	$0.17 \times 10^{-4}$
CA in [18]	$0.46 \times 10^{-9}$	$0.29 \times 10^{-10}$
present method	$0.36 \times 10^{-11}$	$0.22 \times 10^{-12}$

5.2. **Example 2.** Consider the nonlinear Volterra-Hammerstein integral equation [18]

$$(5.2) \quad x(t) = 1 + \sin^2(t) - 3 \int_0^t \sin(t-s)x^2(s)ds, \quad t \in [0, 1).$$

By using the method in Section 3, the above equation is solved. The computational results of  $x(t)$  with  $m = 16$  and  $m = 60$  together with the results in [18] for CA with  $m = 80$ , and

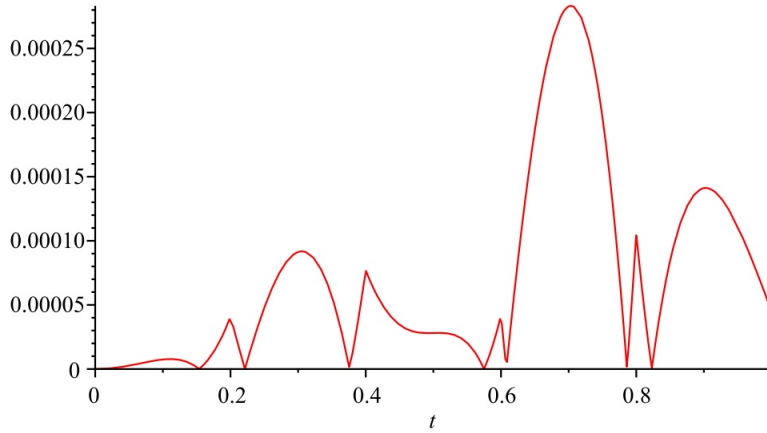


FIGURE 1. Plot of  $|x_{exact} - x_{approx.}|$  with  $m = 5$  for Example 1

the exact solution  $x(t) = \cos t$  are given in Table 3. Table 3 shows that the new method with  $m = 60$  gives more accurate results than the method in [18] with  $m = 80$ . In Table 4, the least square errors for the present method together with block-pulse approximation and CA in [18] with  $m = 60$  and  $m = 80$  are shown. In Figure 2, we plot the approximate solution obtained by the new method with  $m = 4$  together with the exact solution of the example.

TABLE 3. Estimated values of  $x(t)$  using STWS with  $m = 16$  and  $m = 60$  for the present method together with continuous approximations in [25] with  $m = 80$ , and exact values for Example 2.

$t$	present ( $m = 16$ )	present ( $m = 60$ )	CA in [18] ( $m = 80$ )	<i>Exact</i>
0.2	0.98008	0.98007	0.98006	0.98007
0.4	0.92110	0.92107	0.92105	0.92106
0.6	0.82546	0.82534	0.82531	0.82533
0.8	0.69688	0.69669	0.69667	0.69671
1.0	0.54048	0.54031	0.54028	0.54030

TABLE 4. Least Square Errors for the present method together with the block-pulse and continuous approximations in Example 2.

<i>Error</i>	$m = 60$	$m = 80$
block-pulse in [18]	$0.36 \times 10^{-5}$	$0.16 \times 10^{-5}$
CA in [18]	$0.25 \times 10^{-8}$	$0.50 \times 10^{-9}$
present method	$0.21 \times 10^{-10}$	$0.45 \times 10^{-11}$

5.3. **Example 3.** Consider [1]

$$(5.3) \quad x(t) = \frac{-1}{2}e^{-2t} + \frac{3}{2} - \int_0^t (x^2(s) + x(s))ds, \quad t \in [0, 1).$$

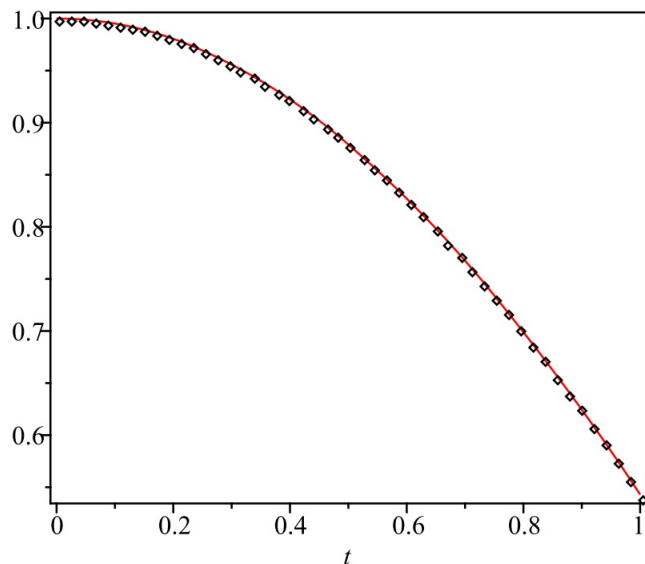


FIGURE 2. Plot of approximate solution (points) with  $m = 4$  and exact solution of Example 2

TABLE 5.  $E_2$  errors for method in [1] and our method for Example 3.

Methods	$E_2$ Error
Hat function method [1]	
$m=8$	4.6e-3
$m=16$	1.3e-3
Present method	
$m=8$	2.6e-3
$m=16$	1.1e-3

The exact solution is  $x(t) = e^{-t}$ . This equation has been solved in [1] by Hat functions. By their method the interval  $[0, 1]$  is divided to  $m$  equal subintervals and the unknown function,  $x(t)$ , is approximated by a Galerkin type method and with  $m$  hat functions. A hat function is as a special linear function in two of the subintervals and is equal to zero in other subintervals. We solved Eq. (23) by our method. A comparison between the  $E_2$  errors of our method and the method in [1] is shown in Table 5. The following is used for  $E_2$  errors for the approximation  $\bar{x}(t)$  of  $x(t)$ .

$$E_2 = \left( \int_0^1 (x(t) - \bar{x}(t))^2 dt \right)^{\frac{1}{2}}.$$

In Table 6, the numerical results obtained by present method with  $m=50$ , 100 and 150 are reported. Our method with  $m=150$  was performed in 3.57 seconds.

5.4. **Example 4.** Consider

$$(5.4) \quad x(t) = \frac{1}{t} - 2t + \int_0^t (5 - sx(s))^{\frac{1}{2}} ds, \quad t \in (0, 1).$$

TABLE 6. Numerical approximations obtained by our method for Example 3.

$t$	$m = 50$	$m = 100$	$m = 150$	$Exact$
0.2	0.81869	0.81872	0.81873	0.81873
0.4	0.67027	0.67031	0.67031	0.67032
0.6	0.54876	0.54880	0.54881	0.54881
0.8	0.44928	0.44932	0.44932	0.44933
1.0	0.36784	0.36787	0.36788	0.36788

The exact solution is  $x(t) = \frac{1}{t}$ . The function  $f(t) = \frac{1}{t} - 2t$  and the exact solution  $x(t) = \frac{1}{t}$  are not square integrable in  $(0, 1)$ , so the equation can not be solved by many methods (e. g. methods in [1,18]). But equation (24) can be solved by presented method. By our method with each values of  $m$ , the approximate solution is  $x_{approx.}(t) = \frac{1}{t}$  which is the same exact solution.

5.5. **Example 5.** Consider

$$(5.5) \quad x(t) = 1 + t - e^t + \int_0^t e^{x(s)} ds, \quad t \in [0, 1].$$

The exact solution is  $x(t) = t$ . The numerical solutions obtained by presented method with  $m=30, 60$  and  $100$  are shown in Table 7. For  $m=100$ , the method was performed in 1.72 seconds.

TABLE 7. Numerical approximations obtained by our method for Example 5.

$t$	$m = 30$	$m = 60$	$m = 100$	$Exact$
0.2	0.20003	0.20001	0.20000	0.20000
0.4	0.40008	0.40002	0.40000	0.40000
0.6	0.60018	0.60005	0.60002	0.60000
0.8	0.80039	0.80010	0.80003	0.80000
1.0	1.00082	1.00021	1.00007	1.00000

5.6. **Example 6.** Consider

$$(5.6) \quad x(t) = -1 + \frac{t^2}{2} - \int_0^t \frac{x(s)}{(s - x(s))^{\frac{1}{3}}} ds, \quad t \in [0, 1].$$

The exact solution is  $x(t) = t - 1$ . We solved the equation by the new method with  $m=10, 20$  and  $50$ . The computer time for  $m=50$  was 0.4 seconds. Table 8 depicts our numerical findings.

## 6. CONCLUSION

The properties of STWS are applied to solve the nonlinear Volterra-Hammerstein integral equations and to obtain a continuous approach for the solution of these problems. By using the properties of STWS and choosing mid point of each intervals as collocation points, the solution of integral equations are converted to the solution of algebraic equations. The method is convergent, if the first order derivative of  $y(t) = h(t, x_{exact}(t))$  be continuous in interval  $[0, 1]$ . Our method is computationally attractive and occupies less memory space and consumes less computer time than the methods in [1, 18]. Because in [1, 18], the functions  $f(t)$  and  $\kappa(t, s)$

TABLE 8. Numerical approximations obtained by our method for Example 6.

$t$	$m = 10$	$m = 20$	$m = 50$	<i>Exact</i>
0.2	-0.79984	-0.79996	-0.79999	-0.80000
0.4	-0.59968	-0.59992	-0.59999	-0.60000
0.6	-0.39954	-0.39988	-0.39998	-0.40000
0.8	-0.19941	-0.19985	-0.19998	-0.20000
1.0	0.00071	0.00018	0.00003	0.00000

are approximated by Hat functions [1] and STWS [18]. These needs  $m$  simple integrations and  $\frac{m(m+1)}{2}$  double integrations. But in the new method, we do not need to approximate  $f(t)$  and  $\kappa(t, s)$  by STWS. Furthermore, unlike the Hat approach in [1] and other spectral methods, the new method does not require operational matrices of integration, product operational matrices and solving large nonlinear algebraic systems. Illustrative examples demonstrate the validity and the applicability of the technique even for integral equations with complex  $h(t, x(t))$ . Moreover, our numerical findings show that the new method gives more accurate results than the methods in [1, 18].

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