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Research Paper

POLYNOMIAL DIFFERENTIAL QUADRATURE METHOD FOR NUMERICAL SOLUTION OF THE GENERALIZED BLACK-SCHOLES EQUATION

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Abstract. In this paper, the polynomial differential quadrature method (PDQM) is implemented to find the numerical solution of the generalized Black-Scholes partial differential equation. The PDQM reduces the problem into a system of first order non-linear differential equations and then, the obtained system is solved by optimal four-stage, order three strong stability-preserving time-stepping Runge-Kutta (SSP-RK43) scheme. Numerical examples are given to illustrate the efficiency of the proposed method.

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Keywords: Option pricing, Generalized Black-Scholes equation, Numerical solutions, Polynomial differential quadrature method (PDQM), Runge-Kutta method.

1. Introduction

An option is a financial contract that gives its owner the right to buy or sell a specified amount of a particular asset at a fixed price, called the exercise price, on or before a specified date, called the maturity date. Options that can be exercised at any time up to the maturity are called American, while options that can only be exercised on the maturity date are European. Options which provide the right to buy the underlying asset are known as calls, whereas options conferring the right to sell the underlying asset are referred to as puts. It was shown by Fischer Black and Myron Scholes in 1973 that these option prices satisfy a second-order partial differential equation with respect to the time horizon t and the underlying asset price [1]. This equation is now known as the Black-Scholes equation, and can be solved exactly when the coefficients are constant or space-independent. However, in many practical situations, numerical solutions are normally sought. Therefore, efficient and accurate numerical algorithms are essential for solving this problem accurately.

In the past several decades, many researchers have spent a great deal of effort to compute the numerical and analytical solution of the Black-Scholes equation using various numerical methods. The first numerical approach to the Black-Scholes equation was the lattice technique proposed in [2] and improved in [3]. That approach is equivalent to an explicit time-stepping scheme. Other numerical schemes based on classical finite difference methods applied to constant coefficient heat equation have also been developed (see, [4-13]). The reason for this is that when the coefficients of the Black-Scholes equation are constant or space independent,

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the equation can be transformed in to a diffusion equation. In this case the problem is said to be path-independent. However, when a problem is space dependent, one cannot transform this to the standard heat equation and thus the Black-Scholes equation in the original form needs to be solved.

Since the Black-Scholes equation becomes degenerate at the underlying asset price $x = 0$, classical finite difference methods may fail to give accurate approximations when x is small. To overcome this difficulty, some authors suggest to solve the differential equation in a truncated space interval excluding the point singularity $x = 0$ (see [13]). Others tend to use a transformation technique which transforms the space interval $(0, X]$ into a semi-infinite interval (see [9]). Obviously, neither of these approaches resolves the singularity. Furthermore, if the final condition of the problem is chosen to be a step or delta function, an interior layer appears in the solution due to the singularity in the final condition. In this case, the gradient near the layer is very large so that classical methods may fail to yield accurate approximations.

To the best of the author's knowledge, the differential quadrature method (DQM), where approximations of the spatial derivatives have been based on a polynomial of high degree, has not been implemented for the Black-Scholes equation so far. The DQM is an efficient discretization technique in solving initial and/or boundary value problems accurately that firstly, was introduced by Bellman et al. [17] in 1972. After that, many authors employed this method for the solutions of many problems in applied sciences (for example see the references [18-34] and the papers therein) .

In this paper, we propose a numerical scheme based on polynomial differential quadrature method (PDQM) to find the numerical solution of the generalized Black-Scholes equation. The PDQM reduces the problem into a system of first order non-linear ordinary differential equations. Then, the obtained system is solved by SSP-RK43 method [42]. The accuracy and efficiency of the proposed method are demonstrated by some test examples. The numerical results are discussed in L_∞ errors and figures form. The obtained numerical solutions are very similar to the exact solutions.

2. The continuous problem

Let us consider the following generalized Black-Scholes equation

$$\mathcal{C}_\tau + \frac{1}{2}\sigma^2(S,\tau)S^2\mathcal{C}_{SS} + (r(S,\tau) - d(S,\tau))\, S\mathcal{C}_S - r(S,\tau)\mathcal{C} = 0,\ \ (S,\tau) \in \mathbb{R}^+ \times (0,T), \tag{2.1}$$

equipped with the terminal condition

$$\mathcal{C}(S,T) = \max(S - E, 0),\ \ S \in \mathbb{R}^+, \tag{2.2}$$

where $\mathcal{C}(S,\tau)$ is the value of the European call option at the asset price S and at time τ, E is the exercise price, T is the maturity date, $(S,\tau) > 0$ is the risk-free interest rate, $d(S,\tau)$ is the dividend, and $\sigma(S,\tau) > 0$ represents volatility function of underlying asset.

Here, we assume that $r(S,\tau)$, $d(S,\tau)$, and $\sigma(S,\tau)$ are sufficiently smooth and bounded on the domain. When r, d and σ are constant functions, it becomes the classical Black-Scholes equation.

The existence and uniqueness of a classical solution of (2.1)-(2.2) is well known (see, [14, 15, 16]). In fact, it can be transformed to a Cauchy problem for a uniformly parabolic operator.

Now, we see that the above model is described in an infinite domain $\mathbb{R}^+ \times (0,T)$, which makes difficulties in constructing numerical solutions. This motivates the consideration of the following model defined on a truncated domain $\Omega = (0, S_{\max}) \times (0, T)$, where $S_{\max}$ is suitably chosen positive number:

$$\begin{aligned} &V_\tau + \frac{1}{2}\sigma^2(S,\tau)S^2 V_{SS} + (r(S,\tau) - d(S,\tau))\, SV_S - r(S,\tau)V = 0,\ (S,\tau) \in \Omega, \\ &V(S,T) = \max(S-E, 0),\ S \in [0, S_{\max}], \\ &V(0,\tau) = 0,\ \tau \in [0,T]. \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} V(S_{\max}, \tau) = &S_{\max} \exp\left(-\int_\tau^T d(S_{\max}, s)ds\right) \\ &- E\exp\left(-\int_\tau^T r(S_{\max}, s)ds\right),\ \tau \in [0,T]. \end{aligned} \tag{2.4}$$

The existence and uniqueness of classical solution of (2.3)-(2.4) can be found in (see, [17,18,19]). Here the boundary conditions are chosen according to Vázquez [13].

It is proved in [14] that if $\mathcal{C}$ and V are solutions of (2.1)-(2.2) and (2.3)-(2.4) respectively, then at every point $(S,\tau) \in (0, S_{\max}) \times [0,T]$ satisfying $\ln(\frac{S_{\max}}{S}) \geq -D(T-\tau)$, we have

$$\begin{aligned} |\mathcal{C}(S,\tau) - V(S,\tau)| \leq &\|\mathcal{C} - V\|_{L^\infty(\Gamma\times(\tau,T))} \\ &\times \left[\exp\left(-\frac{\left(\ln\frac{S_{\max}}{S}\right)\left((T-\tau)\times\min\{0,D\} + \ln\frac{S_{\max}}{S}\right)}{2(T-\tau)[\min_{(S,\tau)\in[0,S_{\max}]\times[0,T]}\sigma^2(S,\tau)]}\right)\right], \end{aligned}$$

where $\Gamma = \{0, S_{\max}\}$ and

$$D = \inf\left\{\sigma^2(S,\tau) - 2r(S,\tau) + 2d(S,\tau) : (S,\tau) \in \Omega\right\}.$$

Since the final condition is not smooth, the resulting solution is not smooth enough for the convergence of numerical aproximations [17]. Replacing $\max(S-E,0)$ in the terminal condition by a smooth function $\phi(S)$, we obtain

$$W_\tau + \frac{1}{2}\sigma^2(S,\tau)S^2 W_{SS} + (r(S,\tau) - d(S,\tau))\, SW_S - r(S,\tau)W = 0,\ (S,\tau) \in \Omega, \tag{2.5}$$

with final condition

$$W(S,T) = \phi(S),\ S \in [0, S_{\max}], \tag{2.6}$$

and boundary conditions

$$\begin{aligned} &W(0,\tau) = 0,\ \tau \in [0,T] \\ &W(S_{\max},\tau) = S_{\max}\exp\left(-\int_\tau^T d(S_{\max}, s)ds\right) \\ &\qquad - E\exp\left(-\int_\tau^T r(S_{\max}, s)ds\right),\ \tau \in [0,T]. \end{aligned} \tag{2.7}$$

The existence and uniqueness of a classical solution of (2.5)-(11) can be found in [16], which also contains the proof of the following estimate:

There exists a positive constant Kindependent of $\phi(S)$ such that

$$|V(S,\tau) - W(S,\tau)| \leq K\|\phi - \max(S-E,0)\|_{L^\infty},\ (S,\tau) \in [0, S_{\max}] \times [0,T]. \tag{2.8}$$

It follows from (2.5) and (2.8) that we can make the solution of model (2.5)-(11) become close to that of the original model (2.1)-(2.2) by choosing sufficiently large $S_{\max}$ and sufficiently close approximation ϕ of final data. Further, we see that the partial differential equation given in (2.5)-(11) is degenerate and backward in time, so we transform this partial differential equation in to a non-degenerate and forward in time partial differential equation by using the transformations: $S = \exp(x) \Rightarrow x = \ln(S)$, and $\tau = T - t \Rightarrow t = T - \tau$ respectively. We have

$$U_t = \frac{1}{2}\hat{\sigma}^2(x,t)U_{xx} + \left(\hat{r}(x,t) - \hat{d}(x,t) - \frac{1}{2}\hat{\sigma}^2(x,t)\right)U_x \tag{2.9}$$

$$-\hat{r}(x,t)U, \ \ (x,t) \in (-\infty, x_{\max}) \times (0,T) \tag{2.10}$$

with final condition

$$U(x,0) = U_0(x), \ \ x \in (-\infty, x_{\max}] \tag{2.11}$$

and boundary conditions

$$U(x,t) = 0, \ \ x \to -\infty, \ \ t \in [0,T] \tag{2.12}$$

$$U(x_{\max}, t) = U_{\max}(t) \ t \in [0,T] \tag{2.13}$$

where

$$\begin{aligned} U(x,t) &= W(S,\tau) = W(\exp(x), T-t), \\ \hat{\sigma}(x,t) &= \sigma(\exp(x), T-t), \\ \hat{r}(x,t) &= r(\exp(x), T-t), \\ \hat{d}(x,t) &= d(\exp(x), T-t), \\ U_0(x) &= \phi(\exp(x)), \end{aligned}$$

and

$$U_{\max}(t) = \exp\left(x_{\max} - \int_0^t \hat{d}(x_{\max}, s)ds\right) - E\exp\left(-\int_0^t \hat{r}(x_{\max}, s)ds\right).$$

Now for computational purpose truncate the infinite interval $(-\infty, x_{\max})$ into the finite interval $(x_{\min}, x_{\max})$, where $x_{\min}$ is a sufficiently small negative real number.
Let us consider a general form of (2.9)-(16) on the truncated region $\Omega_1 = \Omega_x \times \Omega_t = (x_{\min}, x_{\max}) \times [0,T]$:

$$Lu(x,t) = f(x,t), \ \ (x,t) \in \Omega_1 \tag{2.14}$$

$$u(x,0) = \phi_0(x), \ \ x \in \bar{\Omega}_x \tag{2.15}$$

$$u(x_{\min}, t) = \phi_L(t), \ \ t \in \bar{\Omega}_t \tag{2.16}$$

$$u(x_{\max}, t) = \phi_R(t), \ \ t \in \bar{\Omega}_t \tag{2.17}$$

where $L : C(\bar{\Omega}_1) \cap C^{2,1}(\Omega_1) \to C(\Omega_1)$ is defined as

$$L \equiv \frac{\partial}{\partial t} - \alpha(x,t)\frac{\partial^2}{\partial x^2} - \beta(x,t)\frac{\partial}{\partial x} - \gamma(x,t)I,$$

with $\alpha(x,t) \geq \tilde{\alpha} > 0, \gamma(x,t) \leq -\tilde{\gamma} < 0$ on $\bar{\Omega}_1$ and $\alpha, \beta, \gamma, \phi_0, \phi_L, \phi_R, f$ are sufficiently smooth functions.

Here we assume that the problem satisfies sufficient regularity and compatibility conditions which guarantee the problem has a unique solution $u \in C(\bar{\Omega}_1) \cap C^{2,1}(\Omega_1)$ satisfying (see [16, 17])

$$\left|\frac{\partial^{m+n}u(x,t)}{\partial x^m \partial t^n}\right| \leq C \text{ on } \bar{\Omega}_1; \ 0 \leq n \leq 3 \text{ and } 0 \leq m+n \leq 5.$$

Note that C is a generic constant.

3. Polynomial-based differential quadrature method (PDQM)

Differential quadrature method is a numerical technique for solving linear or nonlinear differential equations. By this method, we approximate the spatial derivatives of unknown function at any grid points using weighted sum of all the functional values at certain points in the whole computational domain. Since the weighting coefficients are dependent only the spatial grid spacing, we assume N grid points on the real axis with the same step length. The differential quadrature discretizations of the first-and second-order spatial derivatives are given by, respectively:

$$u_x(x_i,t) = \sum_{j=1}^{N} a_{ij}u(x_j,t),\ i=1,2,\dots,N, \tag{3.1}$$

$$u_{xx}(x_i,t) = \sum_{j=1}^{N} b_{ij}u(x_j,t),\ i=1,2,\dots,N, \tag{3.2}$$

where a_{ij} and b_{ij} are the weighting coefficients of the first and second order partial derivatives respectively and N is the total number of grid points taken in the interval $[a,b]$.

There are many approaches to find these weighting coefficients such as Bellman's approaches [18], Quan and Chang's approach [19, 20] and Shu's approach [21]. Shu's approach is very general approach and in the recent years most of the differential quadrature methods using various test functions such as Legendre polynomials, Lagrange interpolation polynomials, spline functions, Lagrange interpolated cosine functions, are based on this approach. These days in the literature, the most frequently used differential quadrature methods are based on Lagrange interpolation polynomials and sine-cosine expansion. Korkmaz and Dağ [22, 23] proposed sine differential quadrature method and cosine expansion based differential quadrature method for many nonlinear partial differential equations while Jiwari et al. [25-28] have used polynomial based differential quadrature method for numerical solutions of some nonlinear partial differential equations.

In order to obtain the weighting coefficients a_{ij} and b_{ij} the following base functions are used

$$g_k(x) = \frac{L(x)}{(x-x_k)L^{(1)}(x_k)},\ k=1,2,\dots,N, \tag{3.3}$$

where $L(x)=(x-x_1)(x-x_2)\dots(x-x_N)$ and

$$L^{(1)}(x_i) = \prod_{k=1,k\neq i}^{N}(x_i-x_k),\ i=1,2,\dots,N. \tag{3.4}$$

Using the set of base functions given in Eq. (3.3), the weighting coefficients of the first order derivative are found as [18]

$$a_{ij} = \frac{L^{(1)}(x_i)}{(x_i-x_j)L^{(1)}(x_j)},\ i,j=1,2,\dots,N,\ i\neq j, \tag{3.5}$$

$$a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij},\ i=1,2,\dots,N, \tag{3.6}$$

and for weighting coefficients of the second order derivative, the formula is [21]

$$b_{ij} = 2a_{ij}\left(a_{ii} - \frac{1}{x_i - x_j}\right),\ i,j = 1,2,\ldots,N,\ i \neq j, \tag{3.7}$$

$$b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij},\ i = 1,2,\ldots,N. \tag{3.8}$$

3.1. **Selections of grid points and stability.** The accuracy, stability and rate of convergence of the numerical solutions depend on the choice of grid points selected. It is well known that uniformly grid points are not desirable (e.g. [28]). Therefore it is suggested that non-uniformly spaced grid points may give better solutions. The zeros of some orthogonal polynomials are commonly used as the grid points. In fact Bellman et al. [18] had proposed to use the zeros of the Legendre polynomials as the grid points in one of his papers. The stability of the DQM applied depends on the eigen values of differential quadrature discretization matrices. These eigen values in turn vary much depend on the distribution of grid points. It has been shown by Shu [21] in his book that the uniform grid point distribution does not give stable solution which we have also noticed in our numerical experiments. According to Shu the stable solution can be obtained when Chebyshev-Gauss-Lobatto grid points [35-41] are chosen. The Chebyshev-Gauss-Lobatto grid points are given by

$$x_i = a + \frac{1}{2}\left(1 - \cos\frac{(i-1)\pi}{N-1}\right)L_x,\ i = 1,2,\ldots,N, \tag{3.9}$$

where $L_x = b - a$ is the length of the interval $[a, b]$.

4. Implementation of method

On substituting the first and second order approximation of the spatial derivatives, obtained by using PDQM discussed in Section 3, the Generalized Black-Scholes equation (2.14) can be rewritten as

$$\begin{aligned}\frac{du(x_i,t)}{dt} &= \alpha(x_i,t)\sum_{k=1}^{N} b_{ik}u(x_k,t) + \beta(x_i,t)\sum_{k=1}^{N} a_{ik}u(x_k,t)\\ &\quad + \gamma(x_i,t)u(x_i,t) + f(x_i,t),\ x_i \in \Omega,\ i = 1,2,\ldots,N,\ t \geq 0,\end{aligned} \tag{4.1}$$

with initial condition

$$u(x_i, 0) = \phi_0(x_i),\ i = 1,2,\ldots,N, \tag{4.2}$$

and Dirichlet boundary conditions (19)-(20). Thus, Eq. (4.1) is reduced into a set of ordinary differential equations in time, that is, for $i = 1,\ldots,N$, we have

$$\frac{du_i}{dt} = \mathcal{L}(u_i) \tag{4.3}$$

where $\mathcal{L}$ is a spatial nonlinear differential operator. The time interval $[0, T]$ is divided into M small cells equally and let $k = \frac{T}{M}$ (time mesh size). There are various methods to solve this system of ODE. We preferred the optimal four-stage, order three strong stability-preserving time-stepping Runge-Kutta (SSP-RK43) scheme [42] to solve the system of ODE. In this scheme the Eq. (4.3) is integrated from time t_k (step k) to $t_k + \Delta t$ (step $k+1$) through the following operations

$$\begin{aligned} u^{(1)} &= u^k + \frac{1}{2}\Delta t \mathcal{L}(u^k) \\ u^{(2)} &= u^{(1)} + \frac{1}{2}\Delta t \mathcal{L}(u^{(1)}) \\ u^{(3)} &= \frac{2}{3}u^k + \frac{1}{3}u^{(2)} + \frac{1}{6}\Delta t \mathcal{L}(u^{(2)}) \\ u^{k+1} &= u^{(3)} + \frac{1}{2}\Delta t \mathcal{L}(u^{(3)}), \end{aligned}$$

and consequently the solution $u(x,t)$ at a particular time level is completely known.

5. Numerical experiments

In this section, to measure the accuracy of numerical solutions, difference between analytic and numerical solutions at some specified time is computed by using maximum error norm L_∞.

$$L_\infty = \|u^{exact} - u^{num}\| = \max_{1\le j\le N} |u_j^{exact} - u_j^{num}|.$$

The numerical rates of convergence (ROC) is calculated using the following formula

$$\text{ROC} \approx \frac{\log\left((E(N_{i-1})/E(N_i)\right)}{\log(N_i/N_{i-1})},$$

where $E(N_j)$ is the maximum error norm L_∞ when using N_j gride poinst.

Example 1. In this example we consider the general Black-Scholes equation (2.1) for European call option with $\sigma = 0.4,\ r = 0.06,\ d = 0.02,\ E = 1$ and $T = 1$. In this case the exact solution of the Black-Sholes equation with final condition $\mathcal{C}(S,T) = \max(S-E,0)$ and boundary conditions $\mathcal{C}(0,\tau) = 0$ and $\mathcal{C}(S,\tau) = S\exp(-d(T-\tau)) - E\exp(-r(T-\tau))$ as $S \to \infty$ is

$$\mathcal{C}(S,\tau) = SN(d_1)\exp(-d(T-\tau)) - EN(d_2)\exp(-r(T-\tau)), \tag{5.1}$$

where

$$\begin{aligned} N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{1}{2}y^2\right) dy, \\ d_1(S,\tau) &= \frac{\ln\left(\frac{S}{E}\right) + \left(r - d + \frac{1}{2}\sigma^2\right)(T-\tau)}{\sigma\sqrt{T-\tau}} \text{ and } d_2(S,\tau) = d_1(S,\tau) - \sigma\sqrt{T-\tau}, \end{aligned}$$

Now, we approximate the final condition '$\max(S-E,0)$' by sufficiently smooth function '$\phi(S)$' in the following manner.
The function '$\max(y,0)$' can be approximate by the smooth function

$$\psi(y) = \begin{cases} y, & y \ge \epsilon \\ c_0 + c_1 y + c_2 y^2 + \cdots + c_9 y^9, & -\epsilon < y < \epsilon \\ 0, & y \le \epsilon, \end{cases}$$

Table 1. Max absolute error (Max. err.) and the numerical rates of convergence (ROC) of the proposed method for Example 1, with time step length $\Delta t = 10^{-6}$ and different total gride numbers N.

$N \rightarrow$	10	20	40	80	160
For $\Delta t = 10^{-6}$					
Max. err.	$1.55E-002$	$4.27E-003$	$1.02E-003$	$2.48E-004$	$6.10E-005$
R.O.C	...	1.8543	2.0624	2.0441	2.0237

where for sufficiently small $\epsilon > 0$ (see[17])

$$\psi(-\epsilon) = \psi'(-\epsilon) = \psi''(-\epsilon) = \psi'''(-\epsilon) = \psi^{(4)}(-\epsilon) = 0,$$
$$\psi(\epsilon) = \epsilon, \ \psi'(\epsilon) = 1, \ \psi''(\epsilon) = \psi'''(\epsilon) = \psi^{(4)}(\epsilon) = 0,$$

by using these ten conditions one can easily find that

$$c_0 = \frac{35}{256}\epsilon, \ c_1 = \frac{1}{2}, \ c_2 = \frac{35}{64\epsilon}, \ c_4 = -\frac{35}{128\epsilon^3},$$
$$c_6 = \frac{7}{64\epsilon^5}, \ c_8 = -\frac{5}{256\epsilon^7}, \ c_3 = c_5 = c_7 = c_9 = 0,$$

this gives $\phi(S) = \psi(S - E)$.

Thus by using the above appproximation and the transformations $S = \exp(x)$ and $t = T - \tau$, the original Black-Scholes equation, which is degenerate and backward in time, transformed in to the following non-degenerate and forward in time partial differential equation with smooth initial condition

$$u_t = \alpha u_{xx} + \beta u_x + \gamma u, \ (x,t) \in (x_{\min}, x_{\max}) \times (0,T),$$
$$u(x,0) = \phi_0(x),$$
$$u(x_{\min}, t) = 0,$$
$$u(x_{\max}, t) = \exp(x_{\max} - dt) - E\exp(-rt),$$

where

$$u(x,t) = \mathcal{C}(\exp(x), T - \tau), \ \phi_0(x) = \phi(\exp(x)), \ \alpha = 0.08, \ \beta = -0.04, \ \text{and } \gamma = -0.06.$$

The transformed exact solution is

$$u(x,t) = N(d_1)\exp(x - dt) - EN(d_2)\exp(-rt), \tag{5.2}$$

where

$$d_2(x,t) = d_1(x,t) - \sigma\sqrt{t}.$$

For computational purpose here we assume that $x_{\min} = -2, \ x_{\max} = 2$ and $\epsilon = 10^{-6}$. The numerical results of Example 1 are shown in Table 1 and Fig. 1.

Example 2. As the second example we consider the following general Black-Scholes type equation

$$u_t = \alpha(x,t)u_{xx} + \beta(x,t)u_x + \gamma(x,t)u + f(x,t), \ (x,t) \in (-2,2) \times (0,1),$$

with

$$\alpha(x,t) = 0.08(2 + (1-t)\sin(\exp(x)))^2,$$
$$\beta(x,t) = 0.06(1 + t\exp(-\exp(x))) - 0.02\exp(-t - \exp(x)) - \alpha(x,t),$$
$$\gamma(x,t) = -0.06(1 + t\exp(-\exp(x))),$$

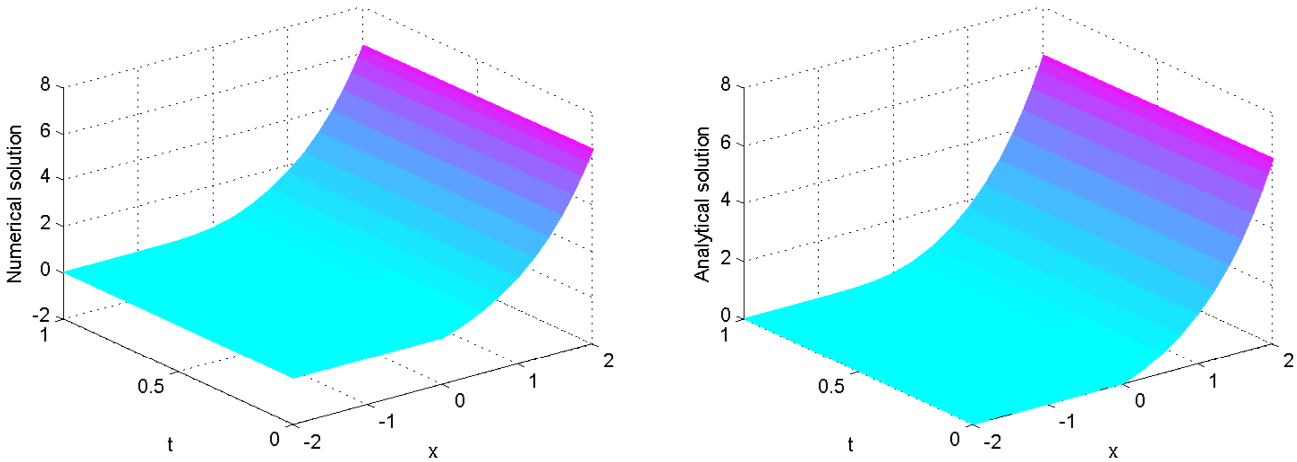

FIGURE 1. Numerical (Left) and analytical (Right) solutions of Example 1 with time step length $\Delta t = 10^{-4}$ and total gride number $N = 25$ up to time $t = 1$

TABLE 2. Max absolute error (Max. err.) and the numerical rates of convergence (ROC) of the proposed method for Example 2, with total number gride $N = 10$ and different step lengths Δt.

$\Delta t \rightarrow$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
For $N = 10$					
Max. err.	$7.85E-003$	$7.86E-004$	$7.86E-005$	$7.83E-006$	$7.88E-007$
R.O.C	...	1.8543	2.0624	2.0441	2.0237

where $f(x,t)$ is chosen in such a way that $u(x,t) = \exp(x-t)$. For computational purpose here we assume that $x_{\min} = -2$, $x_{\max} = 2$ and $\epsilon = 10^{-6}$. The numerical results of Example 2 are shown in Table 2. The solution profile is given in Fig. 2.

To obtain the rate of convergence in the spatial direction, we fix the time step size to be $\Delta t = 10^{-6}$, and increase the grid number in the S direction. As shown in Table 2, we find that the rate is approaching 2, which indicates that our method is indeed second order convergent in the spatial direction. Similarly, we fix the total number of grid points in the S direction and vary the number of time intervals from 10^2 to 10^6. From Table 2, it is quite clear that the rate is very close to 2. Therefore, a second order convergence is also achieved in the time direction.

6. Conclusion

In this paper, we have proposed polynomial differential quadrature method based on Lagrange interpolation to find the approximate solution of the generalized Black-Scholes equation, which is used for option pricing. Polynomial differential quadrature method has been used for discretizing the spasial derivatives and SSP-RK43 scheme for the time integration of resulting system of ordinary differential equations. Comparisons of the computed results with exact solutions showed that the method has the capability of solving the generalized Black-Scholes equation and is also capable of producing accurate solutions with minimal computational effort. The performance of the technique for the considered problems was measured by comparing with the exact solutions. It was seen that the combined technique approximates the exact solution very well. It is suggested that the Chebyshev-Gauss-Lobatto grid

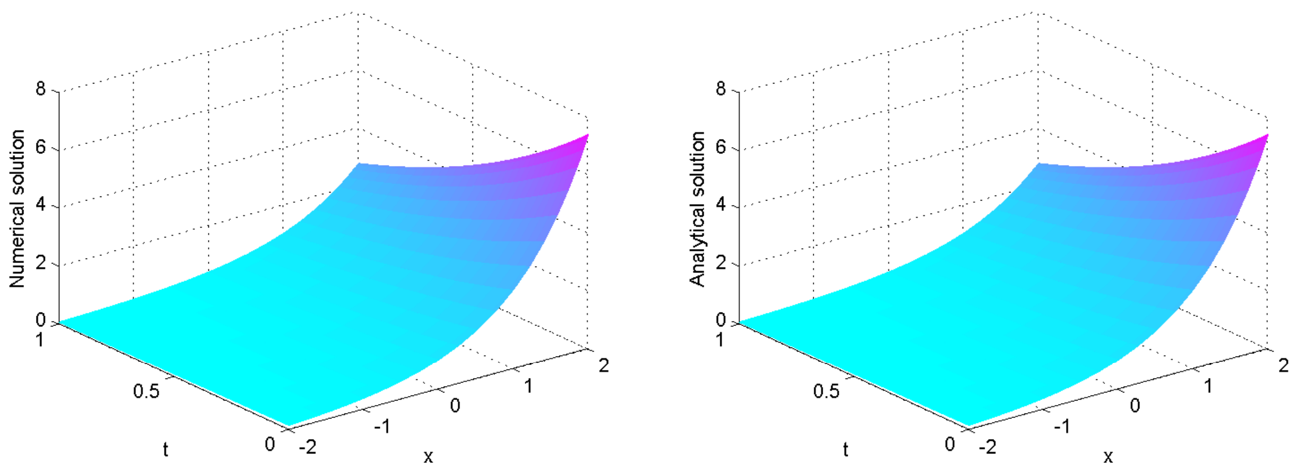

Figure 2. Numerical (Left) and analytical (Right) solutions of Example 2 with time step length $\Delta t = 10^{-4}$ and total gride number $N = 25$ up to time $t = 1$.

points produced accurate solutions. Finally, the authors conclude that the proposed methods give very accurate and similar results to the exact solutions by choosing less number of grid points. It is remained as a future study to prove the convergence of the presented method.

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