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Research Paper

CHARACTER AMENABILITY AND CHARACTER PSEUDO-AMENABILITY OF CERTAIN BANACH ALGEBRAS

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Abstract. In this paper, we study character amenability of semi-group algebras $\ell^1(S)$ and weighted semi-group algebras $\ell^1(S,\omega)$, for a certain semi-groups such as right(left) zero semi-group, rectangular band semi-group, band semi-group and uniformly locally finite inverse semi-group. In particular, we show that for a right (left) zero semi-group or a rectangular band semi-group, character amenability, amenability, pseudo - amenability of $\ell^1(S,\omega)$, for each weight ω, are equivalent. We also show that for an Archimedean semi-group S, character pseudo - amenability, amenability, approximate amenability and pseudo-amenable of $\ell^1(S)$ are equivalent.

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Keywords: character amenability, character pseudo-amenable, rectangular band semi-group, Archimedean semi-group.

1. **Introduction**

The notions of amenability in Banach algebra was initiated by Jonson in [12]. Let $\mathfrak{A}$ be a Banach algebra and E be a Banach $\mathfrak{A}$- bimodule. We regards the dual space E^* as a Banach $\mathfrak{A}$- bimodule with the following module actions:
$(a.f)(x) = f(x.a)$, $(f.a)(x) = f(a.x)$ $(a \in \mathfrak{A}, f \in E^*, x \in E)$
For a Banach algebra $\mathfrak{A}$ the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule in a natural manner and the multiplication map $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}$ defined by $\pi(a \otimes b) = ab$ for $a, b \in \mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule homomorphism.

Amenability for Banach algebras introduced by B. E. Johnson [12]. Let $\mathfrak{A}$ be a Banach algebra and E be a Banach $\mathfrak{A}$-bimodule. A continuous linear operator $D : \mathfrak{A} \longrightarrow E$ is a *derivation* if it satisfies $D(ab) = D(a) . b + a . D(b)$ for all $a, b \in \mathfrak{A}$. Given $x \in E$, the *inner* derivation $ad_x : \mathfrak{A} \longrightarrow E$ is defined by $ad_x(a) = a . x - x . a$. A Banach algebra $\mathfrak{A}$ is *amenable* if for every Banach $\mathfrak{A}$-bimodule E, every derivation from $\mathfrak{A}$ into E^*, the dual of E, is inner.

An *approximate diagonal* for a Banach algebra $\mathfrak{A}$ is a net $(m_i)_i$ in $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $a . m_i - m_i . a \longrightarrow 0$ and $a\pi(m_i) \longrightarrow a$, for each $a \in \mathfrak{A}$. The concept of pseudo-amenability introduced by F. Ghahramani and Y. Zhang in [9]. A Banach algebra $\mathfrak{A}$ is *pseudo-amenable* if it has an approximate diagonal. It is well-known that amenability of $\mathfrak{A}$ is equivalent to the existence of a *bounded* approximate diagonal. One may see [15, 16, 18] for more details and related notions.

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The notions of biprojectivity and biflatness of Banach algebras introduced by Helemskiĭ in [10]. A Banach algebra $\mathfrak{A}$ is *biprojective* if there is a bounded $\mathfrak{A}$-bimodule homomorphism $\rho : \mathfrak{A} \longrightarrow \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $\pi o \rho = I_{\mathfrak{A}}$, where $I_{\mathfrak{A}}$ is the identity map on $\mathfrak{A}$. We say that $\mathfrak{A}$ is *biflat* if there is a bounded $\mathfrak{A}$-bimodule homomorphism $\rho : \mathfrak{A} \longrightarrow (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ such that $\pi^{**} o \rho = k_{\mathfrak{A}}$, where $k_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{A}^{**}$ is the natural embedding of $\mathfrak{A}$ into its second dual.

Kaniuth, Lau and Pym have introduced and studied in [14] and [13], the notion of φ-amenability for Banach algebras, where $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ is a character. In [20], and also M. S. Monfared in [19] introduced the notion of character amenability for Banach algebras. Let $\mathfrak{A}$ be a Banach algebra over $\mathbb{C}$ and $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ be a character on $\mathfrak{A}$, that is, an algebra homomorphism from $\mathfrak{A}$ in to $\mathbb{C}$, and let $\Phi_{\mathfrak{A}}$ denote the character space of $\mathfrak{A}$ (that is, the set of all character on $\mathfrak{A}$).In [19], see also [20], Monfared introduced the notion of character amenable Banach algebras. He called $\mathfrak{A}$ character amenable if it has a bounded approximate identity and it is $\varphi-$ amenable for all nonzero character φ on $\mathfrak{A}$.

Character amenability for Banach algebras was introduced by Aghababa, Shi and Wu in [1].These notions have been studied for various classes of Banach algebras. For more details see, [19, 14, 13, 20] and [4]. As such character amenability is weaker than the classical amenability introduced by Johnson in [12], so all amenable Banach algebras are character amenable.

It is show in [15], that the character amenability of semi-group algebra $\ell^1(S)$ implies that the semi-group S is amenable , and the authors focus on certain semi-groups such as inverse semi-group, Rees semi-group, Clifford semi-group and Brandt semi-group and study the character amenability of $\ell^1(S)$ in relation to the semi-group S.

Nasr-Isfahani and Nemati, in [22] introduced and studied a notion of *character amenability* based on the existence of a $\varphi-$ approximate digonal that is not necessarily bounded. They study character pseudo- amenability of certain Banach algebras.

Let S be a semi-group and

$$\ell^1(S) = \{f : S \to \mathbb{C}, \|f\|_1 = \sum_{s\in S} \mid f(s) \mid < \infty\}.$$

We define the convolution of two elements $f, g \in \ell^1(S)$ by $(f * g)(s) = \sum_{uv=s} f(u)g(v)$, where $\sum_{uv=s} f(u)g(v) = 0$, when there are no elements $u, v \in S$ with $uv = s$. Then $(\ell^1(S), *, \|.\|_1)$ becomes a Banach algebra, that is called the *semi-group algebra of S*.

Let S be a semi-group. A continuous function $\omega : S \longrightarrow (0, \infty)$ is a *weight* on S if $\omega(st) \leq \omega(s)\omega(t)$, for all $s, t \in S$. Then it is standard that

$$\ell^1(S, \omega) = \left\{ f = \sum_{s\in S} f(s)\delta_s : \|f\|_\omega = \sum_{s\in S} |f(s)|\omega(s) < \infty \right\}$$

is a Banach algebra with the convolution product $\delta_s * \delta_t = \delta_{st}$. These algebras are called *Weighted convolution algebras*.

In [17], the authors introduced the *character pseudo- amenability* of semi-group algebras. No much work has been done to date on the character amenability version for weighted semi-group algebra $\ell^1(S, \omega)$ on a semi-group S, as in the other notions for amenability.

It will be good to study this and see how the character amenability of $\ell^1(S, \omega)$ affects the structure of S. Thus, in this work, we study the character pseudo - amenability and character amenability of semi-group algebras and weighted convolution algebras on certain semi-groups.

2. Preliminaries

We let $M_{\varphi_r}^{\mathfrak{A}}$ denote the class of Banach $\mathfrak{A}$- bimodule X for which the right module action of $\mathfrak{A}$ on X is given by

$$x.a = \varphi(a)x \ (a \in \mathfrak{A} \ , \ x \in X \ , \ \varphi \in \Phi_{\mathfrak{A}}),$$

and $M_{\varphi_l}^{\mathfrak{A}}$ denote the class of Banach $\mathfrak{A}$-bimodule X for which the left module action of $\mathfrak{A}$ on X is given by

$$a.x = \varphi(a)x \ (a \in \mathfrak{A} \ , \ x \in X \ , \ \varphi \in \Phi_{\mathfrak{A}}).$$

It is easy to see that the left module action of $\mathfrak{A}$ on the dual module X^* is given by

$$a.f = \varphi(a)f \ (a \in \mathfrak{A} \ , \ f \in X^* \ , \ \varphi \in \Phi_{\mathfrak{A}}).$$

Thus, we note that $X \in M_{\varphi_r}^{\mathfrak{A}}$ (resp $.X \in M_{\varphi_l}^{\mathfrak{A}}$) if and only if $X^* \in M_{\varphi_l}^{\mathfrak{A}}$ (resp $.X^* \in M_{\varphi_r}^{\mathfrak{A}}$).
Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Phi_{\mathfrak{A}}$, we recall from [20] and [19] that
$i)$ $\mathfrak{A}$ is left φ-amenable if every continuous derivation $D : \mathfrak{A} \longrightarrow X^*$ is inner for every $X \in M_{\varphi_r}^{\mathfrak{A}}$;
$ii)$ $\mathfrak{A}$ is right φ-amenable if every continuous derivation $D : \mathfrak{A} \longrightarrow X^*$ is inner for every $X \in M_{\varphi_l}^{\mathfrak{A}}$;
$iii)$ $\mathfrak{A}$ is left character amenable if it is left φ-amenable for every $\varphi \in \Phi_{\mathfrak{A}}$;
$iv)$ $\mathfrak{A}$ is right character amenable if it is right φ-amenable for every $\varphi \in \Phi_{\mathfrak{A}}$;
$v)$ $\mathfrak{A}$ is character amenable if it is both left and right character amenable.
We also recall the following defintions from that, for $\varphi \in \Phi_{\mathfrak{A}}$, a left (right) φ-approximate digonal for $\mathfrak{A}$ is a net (m_α) in $\mathfrak{A} \otimes \mathfrak{A}$ such that

$$(i) \ \|m_\alpha.a - \varphi(a)m_\alpha\| \to 0 \ (\|a.m_\alpha - \varphi(a)m_\alpha\| \to 0) \ (a \in \mathfrak{A});$$

$$(ii) \ \langle \varphi \otimes \varphi, m_\alpha \rangle = \varphi(\pi(m_\alpha)) \to 1,$$

where $\pi : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ defined by $\pi(a \otimes b) = ab \ (a, b \in \mathfrak{A})$ is the product map. The notion of φ-approximate digonal was introduced and studied by Hu, Monfared and Traynor [20]. Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Phi_{\mathfrak{A}}$. We recall from [21] that
(i) $\mathfrak{A}$ is φ-pseudo-amenable if there is a φ-approximate digonal for $\mathfrak{A}$;
(ii) $\mathfrak{A}$ is character pseudo - amenable if $\mathfrak{A}$ has a right approximate identity and it is φ-pseudo-amenable for all $\varphi \in \Phi_{\mathfrak{A}}$.

3. Character amenability of semi-group algebras

In the section, we prove some general results for semi-group algebras. It is clear that every biprojective Banach algebra is biflat. Also we recall that a Banach algebra $\mathfrak{A}$ is amenable if and only if it is biflat and has a bounded approximate identity. By [19, Theorem 2.6], If $\mathfrak{A}$ is character amenable, then $\mathfrak{A}$ has a bounded approximate identity. So we can have the following results.

A semi-group S is a *left zero semi-group* if $st = s$, and it is a *right zero semi-group* if $st = t$ for each $s, t \in S$.

Proposition 3.1. *Suppose that S is a right (left) zero semi-group. Then $\ell^1(S)$ is character amenable if and only if it is amenable.*

proof: From [6, proposition 3.1], $\ell^1(S)$ is biflat. Since $\ell^1(S)$ is character amenable, thus $\ell^1(S)$ has a bounded approximate identity. By the above argument, $\ell^1(S)$ is amenable. □

Let S be a semi-group and let $E(S) = \{p \in S : p^2 = p\}$. We say that S is a *band semi-group* if $S = E(S)$. A band semi-group S satisfying $sts = s$, for each $s, t \in S$ is called a *rectangular band semi-group.*

Corollary 3.2. *Let S be a rectangular band semi-group. Then $\ell^1(S)$ is character amenable if and only if it is amenable.*

proof For a rectangular band semi-group S, it is known that $S \simeq L \times R$, where L and R are left and right zero semi-groups, respectively [11, Theorem 1.1.3]. So,

$$\ell^1(S) \cong \ell^1(L \times R) \cong \ell^1(L) \otimes \ell^1(R).$$

Since $\ell^1(S)$ is character amenable, now by [1, proposition 6.3], $\ell^1(L)$ and $\ell^1(R)$ are character amenable. From proposition 3.1, it follows that $\ell^1(L)$ and $\ell^1(R)$ are amenable, so $\ell^1(S)$ is amenable. □

Corollary 3.3. *Let S be a rectangular band semi-group. Then the following are equivalent:*
(i) $\ell^1(S)$ is character amenable.
(ii) $\ell^1(S)$ is amenable.
(iii) S is singleton.

proof From corollary 3.2 , (i) and (ii) is equivalent and also by [6, theorem 3.3], (ii) and (iii) are equivalent. □

Let S be a band semi-group. Then by [11, Theorem 4.4.1], S is a semilattice of rectangular band semi-groups. Indeed, $S = \cup_{\alpha \in Y} S_\alpha$ where $Y = \frac{S}{\tau}$ and for each $\alpha = [s] \in Y$, $S_\alpha = [s]$.

Theorem 3.4. *Suppose that S be a band semi-group. If $\ell^1(S)$ is character amenable, then S is a finite semilattice.*

proof: By the above argument, let $S = \cup_{\alpha \in Y} S_\alpha$ is a semilattice of rectangular band semi-groups and $\ell^1(S)$ is character amenable. Indeed, we have $S_\alpha.S_\beta \subseteq S_{\alpha\beta}$ for each $\alpha, \beta \in Y$. It follows that $\ell^1(S)$ is ℓ^1-graded of $(\ell^1(S_\alpha))$'s over the semilattice Y. Indeed, we have

$$\ell^1(S) \cong \bigoplus_{\alpha \in Y} \ell^1(S_\alpha).$$

By [1, proposition 6.3], $\ell^1(S_\alpha)$ is character amenable and Y is finite. Since each S_α is rectangular band semi-group, so by corollary 3.3, S_α is singleton for each $\alpha \in Y$. So S is isomorphic to Y. Thus, S is a semilattice . □

Corollary 3.5. *Let S be a uniformly locally finite band semi-group. Then, the following are equivalent:*
(i) $\ell^1(S)$ is character amenable.
(ii) S is a finite semilattice.
(iii) $\ell^1(S)$ is approximately amenable.
(iv) $\ell^1(S)$ is amenable.

proof: $(i) \to (ii)$, by theorem 3.4 and $(ii) \longleftrightarrow (iii) \longleftrightarrow (iv)$ from [26, corollary 4.9] and also $(iv) \longleftrightarrow (i)$, is clear. □

3.1. **Character amenability of weighted semi-group algebras.** In this section, we extend the results for $\ell^1(S)$ to the weighted case $\ell^1(S,\omega)$. For $f,g \in \ell^1(S,\omega)$, it is obvious that $f*g = \varphi_S(f)g$ if S is a right zero semi-group, and $f*g = \varphi_S(g)f$ if S is a left zero semi-group, where φ_S is the *augmentation character* on $\ell^1(S,\omega)$.

Proposition 3.6. *Suppose that S is a right (left) zero semi-group and ω be a weight on S. Then $\ell^1(S,\omega)$ is character amenable if and only if S is singleton.*

proof : From [23, Proposition 2.1], $\ell^1(S,\omega)$ is biflat. Since $\ell^1(S,\omega)$ is character amenable so it has a bounded approximate identity and thus is amenable. Now it is immediate by [23, Proposition 2.5]. □

Corollary 3.7. *Let S be a right (left) zero semi-group and ω be a weight on S. Then the following are equivalent:*
(i) $\ell^1(S,\omega)$ is character amenable.
(ii) S is singleton.
(iii) $\ell^1(S,\omega)$ is pseudo-amenable.
(iv) $\ell^1(S,\omega)$ is amenable.

Proof. By proposition 3.6 and [23, corollary 2.8] is clear. □

Theorem 3.8. *Let S be a rectangular band semi-group and ω be a weight on S. Then $\ell^1(S,\omega)$ is character amenable if and only if S singleton.*

Proof: suppose that $\ell^1(S,\omega)$ be character amenable, by [18, Theorem 3.4], has a bounded approximate identity and by [23, proposition 2.3] is biflat. So, [23, Theorem 2.4] completes the proof. □

The following is a combination of Theorems 3.8 and [23, corollary 2.9].

Corollary 3.9. *Let S be a rectangular band semi-group, and let ω be a separable weight on S. Then the following are equivalent:*
(i) $\ell^1(S,\omega)$ is character amenable.
(ii) S is singleton.
(iii) $\ell^1(S,\omega)$ is pseudo-amenable.
(iv) $\ell^1(S,\omega)$ is amenable.

Let $(P,\leq)$ is a partially ordered set. Then $(P,\leq)$ is *locally finite* if $(x] = \{y \in S : y \leq x\}$ is finite for every $x \in S$, and it is *uniformly locally finite* if $\sup\{|(x]| : s \in S\} < \infty$.

We recall that a semi-group S is an *inverse semi-group* if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$.

Let S be an inverse semi-group. We define an equivalence relation D on S by sDt if and only if there exists $x \in S$ such that $s^*s = xx^*$ and $t^*t = x^*x$. Let $\{D_\lambda : \lambda \in \Lambda\}$ be the collection of all D_- classes of S. For each $p_\lambda \in E(D_\lambda)$, the maximal subgroup of S at p_λ is denoted by G_{p_λ}. It is easily verified that $G_{p_\lambda} = \{s \in S : ss^* = s^*s = p_\lambda\}$.
The following result is Theorem 2.3 and Corollary 2.5 of [7].

Theorem 3.10. *Let S be a semi-group and ω be a weight on S.*
(i) If $\omega \geq 1$ and $\ell^1(S,\omega)$ is character amenable, then $\ell^1(S)$ is character amenable.
(ii) If $\omega \leq 1$ and $\ell^1(S)$ is character amenable, then $\ell^1(S,\omega)$ is character amenable

Corollary 3.11. *Let $S = M^0(G, I)$ be the Brandt semi-group and ω be a weight on S. Then the following are equivalent:*
(i) $\ell^1(S, \omega)$ is character amenable.
(ii) $\ell^1(S)$ is character amenable.
(iii) I is finite and in the case where $|I| = 1$ then G is amenable.

We now consider character amenability of $\ell^1(S, \omega)$, where S is a uniformly locally finite inverse semi-group and $\omega \geq 1$.

Theorem 3.12. *Let S be a uniformly locally finite inverse semi-group and $\omega \geq 1$ be a weight on S. Then $\ell^1(S, \omega)$ is character amenable,if and only if $E(S)$ is finite and G_{p_λ} is amenable for each $\lambda \in \Lambda$ with $|E(D_\lambda)| = 1$.*

proof: Let $\ell^1(S, \omega)$ be character amenable , By theorem 3.10, $\ell^1(S)$ is character amenable, so by [5, theorem 2.6] is clear. Conversely, since $E(S)$ is finite and S is inverse, then S has a principal series

$$S = S_1 \supset S_2 \supset S_3 \supset .... \supset S_{m-1} \supset S_m = K(S)$$

of ideals of S, where $K(S)$ is the minimum ideal, see [19, theorem 3.12]. $\frac{S_i}{S_{i+1}}$ is a simple inverse semi-group with a finite number of idempotents, and so is a group. Also, for $i = 1, 2, ..., n-1$, $\frac{S_i}{S_{i+1}}$ is 0-simple with a finite number of idempotents, and so is a completely 0-simple inverse semi-group, that is a Brandt semi-group. By corollary 3.11, $\ell^1(S, \omega)$ is character amenable if and only if $\ell^1(S)$ is character amenable and by proof of [15, proposition 3.1] , $\ell^1(S)$ is character amenable if and only if $\ell^1(\frac{S_i}{S_{i+1}})$ is character amenable for $i = 1, 2, ..., n-1$. For $i = 1, 2, ..., n-1$, let G_i be the group of the Brandt semigrop $\frac{S_i}{S_{i+1}}$ and $\ell^1(\frac{S_i}{S_{i+1}})$ is amenable if G_i is amenable for $i = 1, 2, ..., n-1$. So $\ell^1(S, \omega)$ is character amenable if G_i is amenable and the groups G_i are maximal subgroups of S. □

Corollary 3.13. *Let S be a uniformly locally finite semilattice and $\omega \geq 1$ be a weight on S. Then $\ell^1(S, \omega)$ is character amenable,if and only if S is finite.*

proof: Suppose that $\ell^1(S, \omega)$ is character amenable, then by theorem 3.10 and [5, theorem 2.6] is clear.
Conversely, since S is finite, $\ell^1(S) \cong \ell^1(S, \omega)$ and $\ell^1(S)$ is finite-dimensional. Then by [5, corollary 2.8], $\ell^1(S)$ is character amenable so $\ell^1(S, \omega)$ is character amenable. □

4. Character pseudo - amenability of semi-group algebras

In this section, we would like to present a class of commutative semi-groups S which character pseudo-amenability and approximate amenability over $\ell^1(S)$ are equivalent. Recall that a semi-group S is *Archimedean* if S is commutative and for each $s, t \in S$ there exists $n \in \mathbb{N}$ such that

$$s^n \in tS = \{tu \ : \ u \in S\}.$$

Theorem 4.1. *Let S be an Archimedean semi-group. If $\ell^1(S)$ is character pseudo-amenable, then for each $s, t \in S$, $Ss = St$ and S has an idempotent element.*

Proof. Suppose that $\ell^1(S)$ is character pseudo-amenable. By definition of character pseudo - amenability, $\ell^1(S)$ has a right approximate identity and so $\overline{\ell^1(S)^2} = \ell^1(S)$. We know that $\ell^1(St)$ is a complemented and so $\ell^1(St)$ is weakly complemented ideal of $\ell^1(S)$.It follows that

$\ell^1(St)$ has a right approximate identity. Now, we conclude that $\overline{\ell^1(St)^2} = \ell^1(St)$ and so $StSt = St$. This implies that $St^m = St$ for each $t \in S$ and $m \in \mathbb{N}$. Fix two element $s, t \in S$. Since S is Archimedean, there is $n \in \mathbb{N}$ such that $s^n \in St$. Thus, we have $Ss \subseteq St$ and so we have $Ss = St$.
To prove the second part, we use the result of the previous part. So we conclude $S = S^2 = \cup_{t\in S} St$ and $S = St$ for all $t \in S$. Fix an element $t \in S$. There exist $u, v \in S$ such that $t = ut$ and $u = vt$. Thus,

$$u^2 = vtvt = vut = vt = u$$

and the proof is complete. □

In the sequel, we show that if S is Archimedean semi-group, then character pseudo-amenability, pseudo- amenability, amenability and approximate amenability for $\ell^1(S)$ are equivalent.

Theorem 4.2. *Let S be an Archimedean semi-group. Then the following are equivalent:*
(i) $\ell^1(S)$ is character pseudo-amenable.
(ii) S is an amenable group.
(iii) $\ell^1(S)$ is pseudo-amenable.
(iv) $\ell^1(S)$ is amenable.
(v) $\ell^1(S)$ is approximately amenable.

Proof. $(i) \to (ii)$ Let $\ell^1(S)$ is character pseudo-amenable. By theorem 4.1, for $u \in E(S)$, we have $S = Su$. It follows that u is the identity element of S. Now, if $ts = ks$ then exists $v \in S$ such that $u = sv$. Thus,

$$t = tu = tsv = ksv = ku = k.$$

Therefore S is cancellative and by [17, corollary 4.6], S is amenable group.
$(ii) \longleftrightarrow (iii) \longleftrightarrow (iv) \longleftrightarrow (v)$ follows from [27, theorem 2.3].
$(v) \to (i)$ follows from [22, corollary 2.6]. □

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