Mathematical Analysis & Convex Optimization

Vol. 2 (2021), No. 1, 1–10
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.2.1.1

Research Paper

SOME MULTIPLICATIVE INEQUALITIES FOR HEINZ OPERATOR MEAN

SILVESTRU SEVER DRAGOMIR

ABSTRACT. In this paper we obtain some new multiplicative inequalities for Heinz operator mean..

MSC(2010): 47A63, 47A30, 15A60, 26D15, 26D10.
Keywords: Young's Inequality, Real functions, Arithmetic mean-Geometric mean inequality, Heinz means.

1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators and $\nu \in [0, 1]$

$$A\nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean,* and

$$A\sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean* [14]. When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

Define the *Heinz operator mean* by

$$H_\nu (A, B) := \frac{1}{2} (A\sharp_\nu B + A\sharp_{1-\nu} B).$$

The following interpolatory inequality is obvious

$$A\sharp B \leq H_\nu (A, B) \leq A\nabla B \tag{1.1}$$

for any $\nu \in [0, 1]$.

We recall that *Specht's ratio* is defined by [16]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln \left(h^{\frac{1}{h-1}} \right)} \text{ if } h \in (0,1) \cup (1, \infty), \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

Date: Received: October 8, 2020, Accepted: January 11, 2021.

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following result provides an upper and lower bound for the Heinz mean in terms of the operator geometric mean $A\sharp B$:

Theorem 1.1 (Dragomir, 2015 [6]). *Assume that A, B are positive invertible operators and the constants $M > m > 0$ are such that*

$$mA \leq B \leq MA. \tag{1.3}$$

Then we have

$$\omega_\nu(m,M)\, A\sharp B \leq H_\nu(A,B) \leq \Omega_\nu(m,M)\, A\sharp B, \tag{1.4}$$

where

$$\Omega_\nu(m,M) := \begin{cases} S\left(m^{|2\nu-1|}\right) \text{ if } M < 1, \\ \\ \max\left\{S\left(m^{|2\nu-1|}\right), S\left(M^{|2\nu-1|}\right)\right\} \text{ if } m \leq 1 \leq M, \\ \\ S\left(M^{|2\nu-1|}\right) \text{ if } 1 < m \end{cases} \tag{1.5}$$

and

$$\omega_\nu(m,M) := \begin{cases} S\left(M^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } M < 1, \\ \\ 1 \text{ if } m \leq 1 \leq M, \\ \\ S\left(m^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } 1 < m, \end{cases} \tag{1.6}$$

where $\nu \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.7}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

We have:

Theorem 1.2 (Dragomir, 2015 [7]). *Assume that A, B are positive invertible operators and the constants $M > m > 0$ are such that the condition (1.3) is valid. Then for any $\nu \in [0,1]$ we have*

$$(A\sharp B \leq)\, H_\nu(A,B) \leq \exp\left[\Theta_\nu(m,M) - 1\right] A\sharp B \tag{1.8}$$

where

$$\Theta_\nu(m,M) := \begin{cases} K\left(m^{|2\nu-1|}\right) \text{ if } M < 1, \\ \\ \max\left\{K\left(m^{|2\nu-1|}\right), K\left(M^{|2\nu-1|}\right)\right\} \text{ if } m \leq 1 \leq M, \\ \\ K\left(M^{|2\nu-1|}\right) \text{ if } 1 < m \end{cases} \tag{1.9}$$

and

$$(0 \leq) H_{\nu}(A,B) - A\sharp B \leq \frac{1}{4m^{1-\nu}} \max_{x\in[m,M]} D\left(x^{2\nu-1}\right) A, \tag{1.10}$$

where the function $D:(0,\infty)\to[0,\infty)$ *is defined by* $D(x)=(x-1)\ln x$.

The following bounds for the Heinz mean $H_{\nu}(A,B)$ in terms of $A\nabla B$ are also valid:

Theorem 1.3 (Dragomir, 2015 [7])**.** *With the assumptions of Theorem 2.2 we have*

$$(0 \leq) A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)\Upsilon(m,M) A, \tag{1.11}$$

where

$$\Upsilon(m,M) := \begin{cases} (m-1)\ln m \text{ if } M<1, \\ \\ \max\{(m-1)\ln m, (M-1)\ln M\} \text{ if } m\leq 1\leq M, \\ \\ (M-1)\ln M \text{ if } 1<m \end{cases} \tag{1.12}$$

and

$$A\nabla B \exp\left[-4\nu(1-\nu)\left(F(m,M)-1\right)\right] \leq H_{\nu}(A,B) \,(\leq A\nabla B) \tag{1.13}$$

where

$$F(m,M) := \begin{cases} K(m) \text{ if } M<1, \\ \\ \max\{K(m), K(M)\} \text{ if } m\leq 1\leq M, \\ \\ K(M) \text{ if } 1<m. \end{cases} \tag{1.14}$$

For other recent results on operator geometric mean inequalities, see [2]-[13], [15] and [17]-[18].

Motivated by the above results, we establish in this paper some multiplicative inequalities providing bounds for $H_{\nu}(A,B)$ in terms of $A\sharp B$ and $A\nabla B$ under various assumptions for positive invertible operators A, B.

2. Bounds for $H_{\nu}(A,B)$ in Terms of $A\sharp B$

For $\nu\in(0,1)\setminus\left\{\frac{1}{2}\right\}$ we consider the following function $d_{\nu}:(0,\infty)\to[1,\infty)$ defined by

$$d_{\nu}(x) = \frac{x^{\nu}+x^{1-\nu}}{2\sqrt{x}}. \tag{2.1}$$

The properties of this function are collected in the following lemma.

Lemma 2.1. *For any* $\nu\in(0,1)\setminus\left\{\frac{1}{2}\right\}$ *we have that* $\lim_{x\to 0+} d_{\nu}(x) = \lim_{x\to\infty} d_{\nu}(x) = \infty$, *the function is decreasing on* $(0,1)$, *increasing on* $(1,\infty)$, $d_{\nu}(1)=1$ *and* $d_{\nu}\left(\frac{1}{x}\right) = d_{\nu}(x)$ *for any* $x\in(0,\infty)$.

Proof. We have

$$d_{\nu}(x)=\frac{x^{\nu}+x^{1-\nu}}{2\sqrt{x}}=\frac{1}{2}\left(x^{\nu-\frac{1}{2}}+x^{\frac{1}{2}-\nu}\right)$$

for any $x\in(0,\infty)$ and any $\nu\in(0,1)\setminus\left\{\frac{1}{2}\right\}$.

By taking the derivative we have

$$\begin{aligned}d_{\nu}'(x)&=\frac{1}{2}\left(\left(\nu-\frac{1}{2}\right)x^{\nu-\frac{3}{2}}+\left(\frac{1}{2}-\nu\right)x^{-\nu-\frac{1}{2}}\right)\\&=\frac{1}{2}\left(\nu-\frac{1}{2}\right)\left(x^{\nu-\frac{3}{2}}-x^{-\nu-\frac{1}{2}}\right)\\&=\frac{1}{2}\left(\nu-\frac{1}{2}\right)x^{-\nu-\frac{1}{2}}\left(x^{2\nu-1}-1\right)\end{aligned}$$

for any $x\in(0,\infty)$ and any $\nu\in(0,1)\setminus\left\{\frac{1}{2}\right\}$.

If $\nu>\frac{1}{2}$ then $x^{2\nu-1}-1$ is negative for $x\in(0,1)$ and positive for $x\in(1,\infty)$ giving that $d_{\nu}'(x)$ is negative for $x\in(0,1)$ and positive for $x\in(1,\infty)$.

If $\nu<\frac{1}{2}$ then $x^{2\nu-1}-1$ is positive for $x\in(0,1)$ and negative for $x\in(1,\infty)$ giving that $d_{\nu}'(x)$ is negative for $x\in(0,1)$ and positive for $x\in(1,\infty)$.

These imply that d_{ν} is decreasing on $(0,1)$ and increasing on $(1,\infty)$. The rest is obvious. $\square$

Theorem 2.2. *Let A, B be positive invertible operators and the constants $M>m>0$ such that*

$$mA\leq B\leq MA. \tag{2.2}$$

If for $\nu\in(0,1)\setminus\left\{\frac{1}{2}\right\}$ we define

$$\Lambda_{\nu}(m,M):=\begin{cases}\frac{m^{\nu}+m^{1-\nu}}{2\sqrt{m}} \text{ if } M<1,\\ \max\left\{\frac{m^{\nu}+m^{1-\nu}}{2\sqrt{m}},\frac{M^{\nu}+M^{1-\nu}}{2\sqrt{M}}\right\} \text{ if } m\leq 1\leq M,\\ \frac{M^{\nu}+M^{1-\nu}}{2\sqrt{M}} \text{ if } 1<m\end{cases} \tag{2.3}$$

and

$$\lambda_{\nu}(m,M):=\begin{cases}\frac{M^{\nu}+M^{1-\nu}}{2\sqrt{M}} \text{ if } M<1,\\ 1 \text{ if } m\leq 1\leq M,\\ \frac{m^{\nu}+m^{1-\nu}}{2\sqrt{m}} \text{ if } 1<m,\end{cases} \tag{2.4}$$

then we have the double inequality

$$\lambda_{\nu}(m,M)\,A\sharp B\leq H_{\nu}(A,B)\leq\Lambda_{\nu}(m,M)\,A\sharp B, \tag{2.5}$$

for $\nu\in(0,1)\setminus\left\{\frac{1}{2}\right\}$.

Proof. By the properties of function d_ν we have

$$\begin{cases} d_\nu(M) \text{ if } M<1, \\ 1 \text{ if } m \le 1 \le M, \\ d_\nu(m) \text{ if } 1<m, \end{cases} \le \frac{x^\nu + x^{1-\nu}}{2\sqrt{x}}$$

$$\le \begin{cases} d_\nu(m) \text{ if } M<1, \\ \max\{d_\nu(m), d_\nu(M)\} \text{ if } m \le 1 \le M, \\ d_\nu(M) \text{ if } 1<m \end{cases}$$

for any $x \in [m, M]$ and any $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

This is equivalent to

$$\lambda_\nu(m, M)\sqrt{x} \le \frac{x^\nu + x^{1-\nu}}{2} \le \Lambda_\nu(m, M)\sqrt{x} \tag{2.6}$$

for any $x \in [m, M]$ and any $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

Using the continuous functional calculus, we have for any operator X with $mI \le X \le MI$ that

$$\lambda_\nu(m, M) X^{1/2} \le \frac{X^\nu + X^{1-\nu}}{2} \le \Lambda_\nu(m, M) X^{1/2} \tag{2.7}$$

for any $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

Now, if we multiply both sides of (2.2) by $A^{-1/2}$ we have $mI \le A^{-1/2}BA^{-1/2} \le MI$ and by writing the inequality (2.7) for $X = A^{-1/2}BA^{-1/2}$ we get

$$\begin{aligned} \lambda_\nu(m, M)\left(A^{-1/2}BA^{-1/2}\right)^{1/2} &\le \frac{\left(A^{-1/2}BA^{-1/2}\right)^\nu + \left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}}{2} \\ &\le \Lambda_\nu(m, M)\left(A^{-1/2}BA^{-1/2}\right)^{1/2} \end{aligned} \tag{2.8}$$

for any $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

Finally, if we multiply both sides of (2.8) by $A^{1/2}$, then we get the desired result (2.5). □

Corollary 2.3. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. If either of the following conditions*

(i) If $0 < mI \le A \le m'I < M'I \le B \le MI$,

(ii) If $0 < mI \le B \le m'I < M'I \le A \le MI$,

hold, then

$$\frac{(h')^\nu + (h')^{1-\nu}}{2\sqrt{h'}} A\sharp B \le H_\nu(A, B) \le \frac{h^\nu + h^{1-\nu}}{2\sqrt{h}} A\sharp B. \tag{2.9}$$

Proof. If the condition (i) is valid, then we have

$$I < h'I = \frac{M'}{m'}I \leq A^{-1/2}BA^{-1/2} \leq \frac{M}{m}I = hI,$$

which implies, by (2.5) that

$$d_\nu\left(h'\right) A\sharp B \leq H_\nu\left(A,B\right) \leq d_\nu\left(h\right) A\sharp B$$

and the inequality (2.9) is proved.

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h}I \leq A^{-1/2}BA^{-1/2} \leq \frac{1}{h'}I < I,$$

which, by (2.5) gives

$$d_\nu\left(\frac{1}{h'}\right) A\sharp B \leq H_\nu\left(A,B\right) \leq d_\nu\left(\frac{1}{h}\right) A\sharp B.$$

Since

$$d_\nu\left(\frac{1}{h'}\right) = d_\nu\left(h'\right) \text{ and } d_\nu\left(\frac{1}{h}\right) = d_\nu\left(h\right),$$

then the inequality (2.9) is also valid. □

3. Bounds for $H_\nu\left(A,B\right)$ in Terms of $A\nabla B$

We introduce the function $c_\nu : (0,\infty) \to [1,\infty)$ defined by

$$c_\nu\left(x\right) = \frac{x+1}{x^\nu + x^{1-\nu}}, \tag{3.1}$$

where $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

The properties of this function are as follows:

Lemma 3.1. *For any $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$ we have that $\lim_{x\to 0+} c_\nu\left(x\right) = \lim_{x\to\infty} c_\nu\left(x\right) = \infty$, the function is decreasing on $(0,1)$, increasing on $(1,\infty)$, $c_\nu\left(1\right) = 1$ and $c_\nu\left(\frac{1}{x}\right) = c_\nu\left(x\right)$ for any $x \in (0,\infty)$.*

Proof. Taking the derivative of c_ν, we have

$$\begin{aligned}
c_\nu'\left(x\right) &= \frac{\left(x+1\right)'\left(x^\nu + x^{1-\nu}\right) - \left(x+1\right)\left(x^\nu + x^{1-\nu}\right)'}{\left(x^\nu + x^{1-\nu}\right)^2} \\
&= \frac{x^\nu + x^{1-\nu} - \left(x+1\right)\left(\nu x^{\nu-1} + \left(1-\nu\right)x^{-\nu}\right)}{\left(x^\nu + x^{1-\nu}\right)^2} \\
&= \frac{x^\nu + x^{1-\nu} - \nu x^\nu - \left(1-\nu\right)x^{1-\nu} - \nu x^{\nu-1} - \left(1-\nu\right)x^{-\nu}}{\left(x^\nu + x^{1-\nu}\right)^2} \\
&= \frac{\left(1-\nu\right)x^\nu + \nu x^{1-\nu} - \nu x^{\nu-1} - \left(1-\nu\right)x^{-\nu}}{\left(x^\nu + x^{1-\nu}\right)^2} \\
&= \frac{\left(1-\nu\right)\left(x^\nu - x^{-\nu}\right) + \nu\left(x^{1-\nu} - x^{\nu-1}\right)}{\left(x^\nu + x^{1-\nu}\right)^2}
\end{aligned}$$

for any $x \in (0, \infty)$ and $\nu \in (0,1) \setminus \left\{ \frac{1}{2} \right\}$.

Consider the function $\ell_\nu : (0, \infty) \to \mathbb{R}$ defined by

$$\begin{aligned}\ell_\nu (x) &:= (1-\nu)\left(x^\nu - x^{-\nu}\right) + \nu \left(x^{1-\nu} - x^{\nu-1}\right) \\ &= (1-\nu)\left(x^\nu - \frac{1}{x^\nu}\right) + \nu \left(x^{1-\nu} - \frac{1}{x^{1-\nu}}\right) \\ &= (1-\nu)\left(\frac{x^{2\nu}-1}{x^\nu}\right) + \nu \left(\frac{x^{2(1-\nu)}-1}{x^{1-\nu}}\right).\end{aligned}$$

We also have

$$\begin{aligned}\ell_\nu' (x) &= (1-\nu)\left(\nu x^{\nu-1} + \nu x^{-\nu-1}\right) + \nu \left((1-\nu) x^{-\nu} + (1-\nu) x^{\nu-2}\right) \\ &= (1-\nu)\nu \left(x^{\nu-1} + x^{-\nu-1} + x^{-\nu} + x^{\nu-2}\right)\end{aligned}$$

for any $x \in (0, \infty)$ and $\nu \in (0,1) \setminus \left\{ \frac{1}{2} \right\}$.

Since $\ell_\nu' (x) > 0$ for any $x \in (0, \infty)$ and $\nu \in (0,1) \setminus \left\{ \frac{1}{2} \right\}$ it follows that the equation $\ell_\nu (x) = 0$ has a unique solution on $(0, \infty)$, namely $x = 1$ and $\ell_\nu' (x) < 0$ for $x \in (0,1)$ and $\ell_\nu' (x) > 0$ for $x \in (1, \infty)$.

These show that the function c_ν is decreasing on $(0,1)$ and increasing on $(1, \infty)$.

The rest of properties are obvious. □

We have:

Theorem 3.2. *Let A, B be positive invertible operators and the constants $M > m > 0$ such that the condition (2.2) holds. If for $\nu \in (0,1) \setminus \left\{ \frac{1}{2} \right\}$ we define*

$$\Phi_\nu (m, M) := \begin{cases} \frac{M^\nu + M^{1-\nu}}{M+1} \text{ if } M < 1, \\ 1 \text{ if } m \le 1 \le M, \\ \frac{m^\nu + m^{1-\nu}}{m+1} \text{ if } 1 < m, \end{cases} \tag{3.2}$$

and

$$\phi_\nu (m, M) := \begin{cases} \frac{m^\nu + m^{1-\nu}}{m+1} \text{ if } M < 1, \\ \min\left\{ \frac{m^\nu + m^{1-\nu}}{m+1}, \frac{M^\nu + M^{1-\nu}}{M+1} \right\} \text{ if } m \le 1 \le M, \\ \frac{M^\nu + M^{1-\nu}}{M+1} \text{ if } 1 < m, \end{cases} \tag{3.3}$$

then we have the double inequality

$$\phi_\nu (m, M) A \nabla B \le H_\nu (A, B) \le \Phi_\nu (m, M) A \nabla B, \tag{3.4}$$

for $\nu \in (0,1) \setminus \left\{ \frac{1}{2} \right\}$.

Proof. From Lemma 3.1 we have

$$\begin{cases} \frac{M+1}{M^{\nu}+M^{1-\nu}} \text{ if } M<1, \\ \\ 1 \text{ if } m \leq 1 \leq M, \\ \\ \frac{m+1}{m^{\nu}+m^{1-\nu}} \text{ if } 1<m, \end{cases}$$
$$\leq \frac{x+1}{x^{\nu}+x^{1-\nu}}$$
$$\leq \begin{cases} \frac{m+1}{m^{\nu}+m^{1-\nu}} \text{ if } M<1, \\ \\ \max\left\{\frac{m+1}{m^{\nu}+m^{1-\nu}}, \frac{M+1}{M^{\nu}+M^{1-\nu}}\right\} \text{ if } m \leq 1 \leq M, \\ \\ \frac{M+1}{M^{\nu}+M^{1-\nu}} \text{ if } 1<m, \end{cases}$$

which implies that

$$\frac{x+1}{2} \times \begin{cases} \frac{m^{\nu}+m^{1-\nu}}{m+1} \text{ if } M<1, \\ \\ \min\left\{\frac{m^{\nu}+m^{1-\nu}}{m+1}, \frac{M^{\nu}+M^{1-\nu}}{M+1}\right\} \text{ if } m \leq 1 \leq M, \\ \\ \frac{M^{\nu}+M^{1-\nu}}{M+1} \text{ if } 1<m \end{cases}$$
$$\leq \frac{x^{\nu}+x^{1-\nu}}{2}$$
$$\leq \frac{x+1}{2} \times \begin{cases} \frac{M^{\nu}+M^{1-\nu}}{M+1} \text{ if } M<1, \\ \\ 1 \text{ if } m \leq 1 \leq M, \\ \\ \frac{m^{\nu}+m^{1-\nu}}{m+1} \text{ if } 1<m, \end{cases}$$

namely

$$\phi_{\nu}(m, M) \frac{x+1}{2} \leq \frac{x^{\nu}+x^{1-\nu}}{2} \leq \Phi_{\nu}(m, M) \frac{x+1}{2}$$

for any $x \in (0, \infty)$ and $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

Using the continuous functional calculus, we have for any operator X with $mI \leq X \leq MI$ that

$$\phi_{\nu}(m, M) \frac{X+I}{2} \leq \frac{X^{\nu}+X^{1-\nu}}{2} \leq \Phi_{\nu}(m, M) \frac{X+I}{2} \tag{3.5}$$

for any $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

Now, if we multiply both sides of (2.2) by $A^{-1/2}$ we have $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by writing the inequality (3.5) for $X = A^{-1/2}BA^{-1/2}$ we get

$$\phi_\nu(m,M)\frac{A^{-1/2}BA^{-1/2}+I}{2} \leq \frac{\left(A^{-1/2}BA^{-1/2}\right)^\nu + \left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}}{2} \leq \Phi_\nu(m,M)\frac{A^{-1/2}BA^{-1/2}+I}{2} \tag{3.6}$$

for any $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

Finally, if we multiply both sides of (3.6) with $A^{1/2}$, then we get the desired result (3.4). □

Finally, we have:

Corollary 3.3. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. If either of the following conditions*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$,

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$,

hold, then

$$\frac{h^\nu + h^{1-\nu}}{h+1}A\nabla B \leq H_\nu(A,B) \leq \frac{(h')^\nu + (h')^{1-\nu}}{h'+1}A\nabla B. \tag{3.7}$$

References

[1] S. S. Dragomir, Some new reverses of Young's operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18**: Art. 130, 2015. [http://rgmia.org/papers/v18/v18a130.pdf].

[2] S. S. Dragomir, On new refinements and reverses of Young's operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18**: Art. 135, 2015. [http://rgmia.org/papers/v18/v18a135.pdf].

[3] S. S. Dragomir, Some inequalities for operator weighted geometric mean, Preprint *RGMIA Res. Rep. Coll.* **18**: Art. 139, 2015. [http://rgmia.org/papers/v18/v18a139.pdf].

[4] S. S. Dragomir, Refinements and reverses of Hölder-McCarthy operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18**: Art. 143, 2015. [http://rgmia.org/papers/v18/v18a143.pdf].

[5] S. S. Dragomir, Some reverses and a refinement of Hölderoperator inequality, Preprint *RGMIA Res. Rep. Coll.* **18**: Art. 147, 2015. [http://rgmia.org/papers/v18/v18a147.pdf].

[6] S. S. Dragomir, Some inequalities for Heinz operator mean, Preprint *RGMIA Res. Rep. Coll.* **18**: Art. 163, 2015. [Online http://rgmia.org/papers/v18/v18a163.pdf].

[7] S. S. Dragomir, Further inequalities for Heinz operator mean, Preprint *RGMIA Res. Rep. Coll.* **18**: Art. 167, 2015. [Online http://rgmia.org/papers/v18/v18a167.pdf].

[8] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5**: 21–31, 2011.

[9] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** : 46–49, 2012.

[10] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361**: 262-269, 2010.
[11] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Lin. Multilin. Alg.,* **59**: 1031–1037, 2011.
[12] F. Kittaneh, M. Krnić, N. Lovričević and J. Pečarić, Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators, *Publ. Math. Debrecen*, **80**(3-4): 465–478, 2012.
[13] M. Krnić and J. Pečarić, Improved Heinz inequalities via the Jensen functional, *Cent. Eur. J. Math.* **11** (9): 1698-1710, 2013.
[14] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **264**: 205–224, 1980.
[15] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19**(2): 467–479, 2015.
[16] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74**:91–98, 1960.
[17] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55**: 583–588, 2002.
[18] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5**: 551–556, 2011.

(Silvestru Sever Dragomir) Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
Email address: sever.dragomir@vu.edu.au