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Research Paper

GENERALIZED COMMON FIXED POINT RESULTS IN CONE METRIC SPACES

GURUSAMY SIVA

Abstract. Common fixed point theorems for three self mappings satisfying generalized contractive conditions in cone metric spaces are derived. Also, some common fixed point results for two self mappings are deduced. Moreover, these results generalize some important familiar results. Given example to illustrate our main result. Furthermore, an existence theorem for the common solution of the two Urysohn integral equations obtained by using our main result.

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1. **Introduction**

Some common fixed point(or, CFP) theorems are derived for commuting maps in [9]. The concept of cone metric spaces(or, CMSs) was introduced by L. G. Huang et al in [7]. Every metric space is a cone metric space(or, CMS) with respect to the natural cone $P = [0, \infty)$ of the real line Banach space $\mathbb{R}$. After introduction of the concept of CMSs for fixed point theory, CFP results for CMSs were derived in more articles. See, for example [1, 2, 3, 4, 5, 6, 8, 11, 13, 14, 15].

The notion of weakly compatible maps was introduced by Jungck in [10]. M. Abbas and G. Jungck [2], proved some CFP theorems for two non commuting weakly compatible mappings without continuity in CMSs. In [15] P. Vetro proved CFP result for two self mappings, which is generalisation of result of M. Abbas and G. Jungck. M. Abbas and B. E. Rhoades proved CFP theorems for two self mappings satisfying generalized contractive condition in CMSs in [1]. Also, M. Arshad et al proved some CFP results for three self maps on CMSs in [3].

In this paper, we use the technique of M. Arshad et al to derive some CFP theorems for three mappings without continuity satisfying certain generalized contraction conditions in CMSs. We deduce some CFP results for two self mappings in CMSs. Also, we deduce some more results for self maps in CMSs. Moreover, we generalize some important familiar results of [1, 2, 3, 7, 15]. Furthermore, we write an example to illustrate our main result, and we prove an existence theorem for the unique common solution(or, UCS) of the two Urysohn integral equations(UIEs) by using our main result.

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2. **Cone metric spaces**

This section include some definitions and known results in this section. See [7] for more details.

Let E always be a real Banach space and $P \subset E$. The set P is called a cone if and only if

(I) P is closed, $P \neq \emptyset$, and $P \neq \{0\}$;
(II) $a, b \in \mathbb{R}$, $a, b \geq 0$, $\nu, \eta \in P \Rightarrow a\nu + b\eta \in P$; and
(III) $\nu \in P$ and $-\nu \in P \Rightarrow \nu = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with regard to P by $\nu \leq \eta$ iff $\eta - \nu \in P$. We shall write $\nu < \eta$ to mention that $\nu \leq \eta$ but $\nu \neq \eta$, and $\nu \ll \eta$ iff $\eta - \nu \in intP$, where $intP$ represent the interior of P.

The cone P is called normal if there is a number $K > 0$ such that $\forall\ \nu, \eta \in E$ satisfying $0 \leq \nu \leq \eta$ we have $\| \nu \| \leq K \| \eta \|$. The smallest positive number K is called the normal constant of P.

In this paper, E is a Banach space, P is a cone in E with $intP \neq \emptyset$, and $\leq$ is the partial ordering induced by P.

Definition 2.1. [7] Let $W \neq \emptyset$ be a set. The function $d : W \times W \to E$ is said to be a cone metric if

(i) $0 \leq d(\nu, \eta), \forall \nu, \eta \in W$ and $d(\nu, \eta) = 0$ iff $\nu = \eta$;
(ii) $d(\nu, \eta) = d(\eta, \nu)$, $\forall \nu, \eta \in W$; and
(iii) $d(\nu, \eta) \leq d(\nu, \mu) + d(\mu, \eta), \forall \nu, \eta, \mu \in W$.

Then (W, d) is called a cone metric space(CMS).

Definition 2.2. [7] Let (W, d) be a CMS, and $\{\nu_n\}$ be a sequence in W.

(i) $\{\nu_n\}$ is said to be order convergent if for every $c \in E$ with $0 \ll c$ there exists $N \in \mathbb{N}$ such that $\forall\ n > N$, $d(\nu_n, \nu) \ll c$, for some $\nu \in W$.
(ii) $\{\nu_n\}$ is called an order Cauchy sequence(or, OCS) if for any $c \in E$ with $0 \ll c$, there exists $N \in \mathbb{N}$ such that $\forall\ n, m > N$, $d(\nu_n, \nu_m) \ll c$.
(iii) (W, d) is called an order complete, if every OCS is order convergent in W.

For the main results of this paper we need the following lemma and definitions.

Lemma 2.3. [7] *Let (W, d) be a CMS, and the corresponding P be a normal cone with normal constant(or, NCWNC) K. Let $\{\nu_n\}$ and $\{\eta_n\}$ be two sequences in W. Then the following conditions hold.*

(i) *The sequence $\{\nu_n\}$ order converges to ν in W iff $d(\nu_n, \nu) \to 0$ as $n \to \infty$.*
(ii) *If $\nu_n \to \nu$, $\nu_n \to \eta$ as $n \to \infty$, then $\nu = \eta$.*
(iii) *The sequence $\{\nu_n\}$ is an OCS iff $d(\nu_n, \nu_m) \to 0 (n, m \to \infty)$.*
(iv) *If $\nu_n \to \nu$, $\eta_n \to \eta$ as $n \to \infty$, then $d(\nu_n, \eta_n) \to d(\nu, \eta)$ as $n \to \infty$.*

Definition 2.4. [2] Let $h, f, g : W \to W$ be three maps. If $\eta = h\nu = f\nu = g\nu$ for some ν in W, then ν is called a coincidence point of h, f and g, and η is called a point of coincidence(or, POC) of h, f and g.

Definition 2.5. [10] A pair (h, f) of self maps on a set $W \neq \emptyset$ is called weakly compatible(or, WC) if the condition $h\nu = f\nu$ implies that $hf\nu = fh\nu$.

3. main results

In this section, we establish the result on points of coincidence and common fixed points for three self mappings and then show that our main result generalizes some important results of fixed point.

Definition 3.1. Let h, f and g be three self maps on a CMS (W, d) and $f(W) \cup g(W) \subseteq h(W)$. Let $\nu_0 \in W$. Choose points ν_1 and ν_2 in W such that $f\nu_0 = h\nu_1$ and $g\nu_1 = h\nu_2$. Continuing this process we get $\nu_1, \nu_2, ..., \nu_{2i}$. Now we choose ν_{2i+1} and ν_{2i+2} in W such that $f(\nu_{2i}) = h(\nu_{2i+1})$, $g(\nu_{2i+1}) = h(\nu_{2i+2})$, $i = 0, 1, 2, ...$ Then the sequence $\{h\nu_n\}$ is called a $f - g$ sequence.

Proposition 3.2. *Let (W, d) be a CMS, and the corresponding P be a NCWNC K. Let h, f and g be three self maps on a set W and $g(W) \cup f(W) \subseteq h(W)$. If the following conditions hold*

(i) $d(f\nu, g\eta) \leq \alpha d(h\eta, h\nu) + \beta[d(h\nu, f\nu) + d(g\eta, h\eta)] + \gamma[d(h\nu, g\eta) + d(f\nu, h\eta)]$, $\forall \nu, \eta \in W$, *with* $\nu \neq \eta$, *where* $\alpha, \gamma, \beta \geq 0$, *with* $\alpha + 2\beta + 2\gamma < 1$;
(ii) $d(f\nu, g\nu) < d(h\nu, f\nu) + d(h\nu, g\nu)$, $\forall \nu \in W$, *where* $f\nu \neq g\nu$,

then every $f - g$ sequence is an OCS.

Proof. Fix $\nu_0 \in W$, and let $\{h\nu_n\}$ be a $f - g$ sequence.
Case(I): Suppose $h\nu_n \neq h\nu_{n+1}$, for all $n \in \mathbb{N}$. So $\nu_n \neq \nu_{n+1}$, for all n, and hence

$$d(h\nu_{2i+1}, h\nu_{2i+2}) \quad = \quad d(f\nu_{2i}, g\nu_{2i+1})$$

$$\leq \alpha d(h\nu_{2i}, h\nu_{2i+1}) + \beta[d(h\nu_{2i}, f\nu_{2i}) + d(h\nu_{2i+1}, g\nu_{2i+1})] + \gamma[d(h\nu_{2i}, g\nu_{2i+1}) + d(h\nu_{2i+1}, f\nu_{2i})]$$

$$\leq \alpha d(h\nu_{2i}, h\nu_{2i+1}) + \beta[d(h\nu_{2i}, h\nu_{2i+1}) + d(h\nu_{2i+1}, h\nu_{2i+2})] + \gamma[d(h\nu_{2i}, h\nu_{2i+2}) + d(h\nu_{2i+1}, h\nu_{2i+1})]$$

$$d(h\nu_{2i+1}, h\nu_{2i+2}) \quad \leq \quad (\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma})d(h\nu_{2i}, h\nu_{2i+1}).$$

Similarly,

$$d(h\nu_{2i+2}, h\nu_{2i+3}) \quad \leq \quad (\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma})d(h\nu_{2i+1}, h\nu_{2i+2}).$$

By putting $\lambda = (\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}) < 1$, we have

$$d(h\nu_{2i+1}, h\nu_{2i+2}) \quad \leq \quad \lambda d(h\nu_{2i}, h\nu_{2i+1}), \text{ and}$$

$$d(h\nu_{2i+2}, h\nu_{2i+3}) \quad \leq \quad \lambda d(h\nu_{2i+1}, h\nu_{2i+2}).$$

Now, by using the induction, for each i=0,1,2,...., we get

$$d(h\nu_{2i+1}, h\nu_{2i+2}) \quad \leq \quad \lambda^{2i+1} d(h\nu_0, h\nu_1).$$

Similarly,

$$d(h\nu_{2i+2}, h\nu_{2i+3}) \quad \leq \quad \lambda^{2i+2} d(h\nu_0, h\nu_1).$$

Now, for $p < q$, we get

$$\begin{aligned} d(h\nu_{2p+1}, h\nu_{2q+1}) &\leq d(h\nu_{2p+1}, h\nu_{2p+2}) + d(h\nu_{2p+2}, h\nu_{2p+3}) + ... \\ &\quad + d(h\nu_{2q}, h\nu_{2q+1}) \\ &\leq \Big(\sum_{i=2p+1}^{2q} \lambda^i\Big) d(h\nu_0, h\nu_1). \end{aligned}$$

Hence,

$$||d(h\nu_{2p+1}, h\nu_{2q+1})|| \leq \Big(\sum_{i=2p+1}^{2q+1} \lambda^i\Big) K||d(h\nu_0, h\nu_1)||.$$

So, $d(h\nu_{2p+1}, h\nu_{2q+1}) \to 0$ as $p, q \to \infty$.
Also we get $d(h\nu_{2p}, h\nu_{2q+1}) \to 0$ as $p, q \to \infty$, $d(h\nu_{2p}, h\nu_{2q}) \to 0$ as $p, q \to \infty$, and $d(h\nu_{2p+1}, h\nu_{2q}) \to 0$ as $p, q \to \infty$.
Therefore, $d(h\nu_m, h\nu_n) \to 0$ as $m, n \to \infty$ for $m < n$. Hence $\{h\nu_n\}$ is an OCS.
Case (II): Suppose $h\nu_m = h\nu_{m+1}$, for some $m \in \mathbb{N}$.
If $\nu_m = \nu_{m+1}$, and $m = 2i$, then by (ii), we have,

$$\begin{aligned} d(h\nu_{2i+1}, h\nu_{2i+2}) &= d(f\nu_{2i}, g\nu_{2i+1}) \\ &< d(h\nu_{2i}, f\nu_{2i}) + d(h\nu_{2i+1}, g\nu_{2i+1}) \\ &= d(h\nu_{2i+1}, h\nu_{2i+2}). \end{aligned}$$

Therefore, $h\nu_{2i+1} = h\nu_{2i+2}$.
If $\nu_m \neq \nu_{m+1}$, then we have

$$d(h\nu_{2i+1}, h\nu_{2i+2}) = d(f\nu_{2i}, g\nu_{2i+1})$$

$$\leq \alpha d(h\nu_{2i}, h\nu_{2i+1}) + \beta[d(h\nu_{2i}, f\nu_{2i}) + d(h\nu_{2i+1}, g\nu_{2i+1})] + \gamma[d(h\nu_{2i}, g\nu_{2i+1}) + d(h\nu_{2i+1}, f\nu_{2i})]$$

$$= \alpha d(h\nu_{2i}, h\nu_{2i+1}) + \beta[d(h\nu_{2i}, h\nu_{2i+1}) + d(h\nu_{2i+1}, h\nu_{2i+2})] + \gamma[d(h\nu_{2i}, h\nu_{2i+2}) + d(h\nu_{2i+1}, h\nu_{2i+1})]$$

$$= (\beta + \gamma) d(h\nu_{2i+1}, h\nu_{2i+2}).$$

Then $d(h\nu_{2i+1}, h\nu_{2i+2}) \leq (\beta + \gamma)^k d(h\nu_{2i+1}, h\nu_{2i+2})$, for every $k = 1, 2, 3, ...$, and $||d(h\nu_{2i+1}, h\nu_{2i+2})|| \leq (\beta+\gamma)^k K||d(h\nu_{2i+1}, h\nu_{2i+2})||$, for every $k = 1, 2, 3, ...$. Since $(\beta+\gamma)^k \to 0$ as $k \to \infty$, we get $d(h\nu_{2i+1}, h\nu_{2i+2}) = 0$. Therefore, $h\nu_{2i+1} = h\nu_{2i+2}$.
Similarly we conclude that, $h\nu_{2i+2} = h\nu_{2i+3}$. So, $h\nu_n = h\nu_m$, for all $n \geq m$. Hence $\{h\nu_n\}$ is an OCS, and this complete the proof. □

We are now ready to state the main result.

Theorem 3.3. *Let (W, d) be a CMS, and the corresponding P be a NCWNC K. Let h, f and g be three self maps on a set W such that $h(W)$ be an order complete subspace of W and $g(W) \cup f(W) \subseteq h(W)$. If*

(i) $d(f\nu, g\eta) \leq \alpha d(h\eta, h\nu) + \beta[d(h\nu, f\nu) + d(g\eta, h\eta)] + \gamma[d(h\nu, g\eta) + d(f\nu, h\eta)]$, $\forall \nu, \eta \in X$, *with $\nu \neq \eta$, where $\alpha, \gamma, \beta \geq 0$, with $\alpha + 2\beta + 2\gamma < 1$; and*
(ii) $d(f\nu, g\nu) < d(h\nu, f\nu) + d(h\nu, g\nu)$, *for all $\nu \in W$, where $f\nu \neq g\nu$,*

then h, f and g have a unique POC. Moreover, if (f,h) and (g,h) are WC, then h, f and g have unique CFP.

Proof. Fix $\nu_0 \in W$. By using the above proposition, we conclude that the $f-g$ sequence $\{h\nu_n\}$ is an OCS. By the order completeness of $h(W)$, $h\nu_n \to v$, for some v in W. Also there exists u in W such that $hu = v$. Now for every $n \in N$:

$$\begin{aligned} d(h\nu_n, fu) &\leq d(h\nu_n, h\nu_{2n}) + d(h\nu_{2n}, fu) \\ &= d(h\nu_n, h\nu_{2n}) + d(g\nu_{2n-1}, fu) \end{aligned}$$

$$\begin{aligned} &\leq d(h\nu_n, h\nu_{2n}) + \alpha d(hu, h\nu_{2n-1}) + \beta[d(hu, fu) + d(h\nu_{2n-1}, g\nu_{2n-1})] \\ &\qquad + \gamma[d(hu, g\nu_{2n-1}) + d(h\nu_{2n-1}, fu)] \\ &= d(h\nu_n, h\nu_{2n}) + \alpha d(v, h\nu_{2n-1}) + \beta[d(v, fu) + d(h\nu_{2n-1}, h\nu_{2n})] \\ &\qquad + \gamma[d(v, h\nu_{2n}) + d(h\nu_{2n-1}, fu)]. \end{aligned}$$

$$\begin{aligned} d(h\nu_n, fu) \leq \frac{1}{1-\beta-\gamma}&[(1+\beta)d(h\nu_n, v) + (1+\beta+2\gamma)d(h\nu_{2n}, v) \\ &+ (1+\beta+\gamma)d(v, h\nu_{2n-1})] \end{aligned}$$

and hence,

$$\begin{aligned} ||d(h\nu_n, fu)|| \leq \frac{1}{1-\beta-\gamma}K||&[(1+\beta)d(h\nu_n, v) + (1+\beta+2\gamma)d(h\nu_{2n}, v) \\ &+ (1+\beta+\gamma)d(v, h\nu_{2n-1})]||. \end{aligned}$$

Therefore, $d(h\nu_n, fu) \to 0$ as $n \to \infty$. Hence $h\nu_n \to fu$. Also we have $h\nu_n \to hu$, and by using Lemma 2.3, we get $fu = hu$.
Similarly from $d(h\nu_n, gu) \leq d(h\nu_n, h\nu_{2n+1}) + d(h\nu_{2n+1}, gu)$ we have $hu = gu$. Therefore $hu = fu = gu = v$, where v is POC.
For the proof of uniqueness of v, let v^* be an arbitrary POC such that $hu^* = fu^* = gu^* = v^*$. Then

$$\begin{aligned} d(v, v^*) &= d(fu, gu^*) \\ &\leq \alpha d(hu^*, hu) + \beta[d(hu, fu) + d(hu^*, gu^*)] + \gamma[d(gu^*, hu) \\ &\quad + d(hu^*, fu)] \\ &\leq \alpha d(v^*, v) + \beta[d(v, v) + d(v^*, v^*)] + \gamma[d(v, v^*) + d(v^*, v^*)] \\ &\leq (\alpha + 2\gamma)d(v, v^*). \end{aligned}$$

$d(v, v^*) \leq (\alpha+2\gamma)^m d(v, v^*)$, for every $m = 1, 2, 3, ...$, and
$||d(v, v^*)|| \leq (\alpha+2\gamma)^m K||d(v, v^*)||$. Since $(\alpha+2\gamma)^m \to 0$ as $m \to \infty$, we have $d(v, v^*) = 0$. Therefore $v = v^*$. Thus the POC is unique.
Also, if (g,h) and (f,h) are WC pairs, then

$$fv = fhu = hfu = hv, \text{ and } gv = ghu = hgu = hv,$$

which implies that $fv = gv = hv = z$. Therefore z is a POC of h,f and g. Since the POC is unique, we conclude that $v = z$, and so h, f and g have unique CFP v. □

By putting $g = f$ in Theorem 3.3, we conclude the following result.

Theorem 3.4. *Let (W, d) be a CMS, and the corresponding P be a NCWNC K. Let $h, g : W \to W$, $h(W)$ be an order complete subspace of W and $g(W) \subseteq h(W)$. If*

$$d(g\nu, g\eta) \leq \alpha d(h\nu, h\eta) + \beta[d(h\nu, g\nu) + d(h\eta, g\eta)] + \gamma[d(g\eta, h\nu) + d(h\eta, g\nu)],$$
$$\text{for all } \nu, \eta \in W, \text{ where } \alpha, \gamma, \beta \in [0, 1), \text{ with } \alpha + 2\beta + 2\gamma < 1,$$

then h and g have a unique POC. Moreover, if a pair (h, g) is WC, then h and g have unique CFP.

The following corollary, is a direct result of Theorem 3.4.

Corollary 3.5. *Let (W, d) be a CMS, and the corresponding P be a NCWNC K. Let $h, g : W \to W$ be two maps such that $h(W)$ be an order complete subspace of W and $g(W) \subseteq h(W)$. If*

$$d(g\nu, g\eta) \leq t_1 d(h\nu, h\eta) + t_2 d(h\nu, g\nu) + t_3 d(h\eta, g\eta) + t_4 d(h\nu, g\eta) + t_5 d(h\eta, g\nu),$$
$$\text{for all } \nu, \eta \in W, \text{ where } t_i \in [0, 1), \forall i \in \{1, 2, ...5\}, \text{ with } \sum_{i=1}^{5} t_i < 1,$$

then h and g have a unique POC. Moreover, if a pair (h, g) is WC, then h and g have unique CFP.

Proof. Adding the new inequality to contractive condition, by interchanging roles of ν and η in contractive condition with $\alpha = t_1$, $\beta = \frac{t_1+t_2}{2}$, and $\gamma = \frac{t_3+t_4}{2}$, and by Theorem 3.4, we proved this result. □

The above Corollary is a generalization of Theorem 1 of [15], Theorem 2.3, Theorem 2.1, and Theorem 2.4 of [2].
By putting $h = I$ in Theorem 3.4, we conclude the Corollary 2.3 of [1] as follow.

Corollary 3.6. *Let (W, d) be an order complete CMS, and the corresponding P be a NCWNC K. Let $g : W \to W$ be a map such that*

$$d(g\nu, g\eta) \leq \alpha d(\eta, \nu) + \beta[d(\nu, g\nu) + d(\eta, g\eta)] + \gamma[d(g\eta, \nu) + d(\eta, g\nu)],$$
$$\text{for all } \nu, \eta \in W, \text{ where } \alpha, \gamma, \beta \in [0, 1), \text{ with } \alpha + 2\beta + 2\gamma < 1.$$

Then g has unique fixed point(or, UFP).

The above Corollary is Corollary 2.3 of [1].

Corollary 3.7. *Let (W, d) be an order complete CMS, and the corresponding P be a NCWNC K. Let $g : W \to W$ be a map such that*

$$d(g^r\nu, g^r\eta) \leq \alpha d(\nu, \eta) + \beta[d(\nu, g^r\nu) + d(\eta, g^r\eta)] + \gamma[d(\nu, g^r\eta) + d(\eta, g^r\nu)],$$
$$\text{for all } \nu, \eta \in W, r \in \mathbb{N}, \text{ where } \alpha, \gamma, \beta \in [0, 1), \text{ with } \alpha + 2\beta + 2\gamma < 1.$$

Then g has UFP.

Proof. The mapping g^r has UFP ν^*, because of previous Corollary 3.6. But $g(g^r\nu^*) = g^r(g\nu^*) = g\nu^*$, so $g\nu^*$ is also a fixed point of g^r. Hence $g\nu^* = \nu^*$, and ν^* is a fixed point of g. Therefore g has a UFP, because a fixed point of g is also a fixed point of g^r. □

Remark 3.8. The Corollary 3.7 is a generalization of Corollary 1 of [15], Theorem 1, Theorem 3, Theorem 4, and Corollary 2 of [7].

Next we give an example to find a unique common fixed point of three self maps, and which is illustrate our main result.

Example 3.9. Let $P = \{(\nu,\eta) \in E : \nu,\eta \geq 0\}$ be a subset of the real Banach space $E = \mathbb{R}^2$, (the Euclidean plane). Then P is a NCWNC with $K = 1$. Let $W = \{a, b, c\}$ and $d : W \times W \to E$ be defined by $d(\nu,\eta) = 0$ if $\nu = \eta$, and suppose $\nu \neq \eta$,

$$d(\nu,\eta) = \begin{cases} (\frac{2}{5}, 2), & \text{if } \nu,\eta \in \{a,c\} \\ (1,5), & \text{if } \nu,\eta \in \{a,b\} \\ (\frac{3}{5}, 3), & \text{if } \nu,\eta \in \{b,c\}. \end{cases}$$

Then (W, d) is a order complete CMS.
Define $h, f, g : W \to W$ by $h(\nu) = \nu, f(\nu) = c$, and

$$g\nu = \begin{cases} a, & \text{if } \nu = b \\ c, & \text{if } \nu \neq b. \end{cases}$$

For $\alpha = 0, \gamma = 0$, and $\beta = \frac{2}{5}$, we have

$$d(fx, gy) = \begin{cases} (\frac{2}{5}, 2), & \text{if } \eta = b, \\ (0,0), & \text{if } \eta \neq b, \end{cases}$$

$$\beta[d(h\nu, f\nu) + d(h\eta, g\eta)] = \begin{cases} (\frac{2^2}{5^2} + \frac{2}{5}, \frac{14}{5}), & \text{if } \nu = a \text{ and } \eta = b \\ (\frac{6}{5^2} + \frac{2}{5}, \frac{16}{5}), & \text{if } \nu = b \text{ and } \eta = b. \end{cases}$$

Also, (g, h) and (f, h) are WC pairs on W, because $hf(c) = fh(c)$ and $hg(c) = gh(c)$. Then, all conditions of Theorem 3.3 are satisfied. Moreover, the unique CFP of h, f and g is c.

4. Applications

We show that an existence theorem for the unique common solution(UCS) of the two Urysohn integral equations(UIEs). In the following, assume $W = C([a, b], \mathbb{R}^n)$, $P = \{(k, l) \in E : k, l \geq 0\} \subseteq E = \mathbb{R}^2$, and $d(\nu,\eta) = (||\nu - \eta||_\infty, w||\nu - \eta||_\infty)$, $\forall \nu,\eta \in W$, $w \geq 0$, so (W, d) is an order complete CMS.

Theorem 4.1. *Let consider two UIEs*

$$\nu(r) = \int_a^b L_1(r, t, \nu(t))dt + p(r), \tag{4.1}$$

$$\nu(r) = \int_a^b L_2(r, t, \nu(t))dt + q(r), \tag{4.2}$$

where $r \in [a, b] \subset \mathbb{R}$, *and* $\nu, p, q \in W$. *Let* L_1 *and* L_2 *be two maps from* $[a, b] \times [a, b] \times \mathbb{R}^n$ *to* $\mathbb{R}^n$ *such that*
(i) $A_\nu, B_\nu \in W$, *for every* $\nu \in W$, *where*

$$A_\nu(r) = \int_a^b L_1(r, t, \nu(t))dt, B_\nu(r) = \int_a^b L_2(r, t, \nu(t))dt, \forall r \in [a, b].$$

(ii) There exists $\alpha, \gamma, \beta \geq 0$, such that

$$\begin{aligned}(|A_\nu(r) - B_\eta(r) + p(r) - q(r)|, w|A_\nu(r) - B_\eta(r) + p(r) - q(r)|) \\ \leq \alpha(|\nu(r) - \eta(r)|, w|\nu(r) - \eta(r)|) \\ +\beta[(|A_\nu(r) + p(r) - \nu(r)|, w|A_\nu(r) + p(r) - \nu(r)|) \\ +(|B_\eta(r) + q(r) - \eta(r)|, w|B_\eta(r) + q(r) - \eta(r)|)] \\ +\gamma[(|B_\eta(r) + q(r) - \nu(r)|, w|B_\eta(r) + q(r) - \nu(r)|) \\ +(|A_\nu(r) + p(r) - \eta(r)|, w|A_\nu(r) + p(r) - \eta(r)|)],\end{aligned}$$

where $\alpha + 2\beta + 2\gamma < 1$, $\forall \nu, \eta \in W$ with $\nu \neq \eta$, and $r \in [a, b]$.
(iii) Whenever $A_\nu + p \neq B_\nu + q$,

$$\begin{aligned}\sup_{r\in[a,b]} (|A_\nu(r) - B_\nu(r) + p(r) - q(r)|, w|A_\nu(r) - B_y(r) + p(r) - q(r)|) \\ < \sup_{r\in[a,b]} (|A_\nu(r) + p(r) - \nu(r)|, w|A_\nu(r) + p(r) - \nu(r)|) \\ + \sup_{r\in[a,b]} (|B_\nu(r) + q(r) - \nu(r)|, w|B_\nu(r) + q(r) - \nu(r)|), \ \forall \nu \in W.\end{aligned}$$

Then UIEs (4.1) and (4.2) have a UCS.

Proof. Let $g, f : W \to W$ by $f(\nu) = A_\nu + p$, $g(\nu) = B_\nu + q$, then

$$\begin{aligned}(||f - g||_\infty, w||f - g||_\infty) \leq \alpha(||\nu - \eta||_\infty, w||\nu - \eta||_\infty) \\ +\beta[(||f(\nu) - \nu||_\infty, w||f(\nu) - \nu||_\infty) + (||g(\eta) - \eta||_\infty, w||g(\eta) - \eta||_\infty)] \\ +\gamma[(||g(\eta) - \nu||_\infty, w||g(\eta) - \nu||_\infty) + (||f(\nu) - \eta||_\infty, w||f(\nu) - \eta||_\infty)],\end{aligned}$$

$\forall \nu, \eta \in W$, with $\nu \neq \eta$, and if $f(\nu) \neq g(\nu)$

$$\begin{aligned}(||f - g||_\infty, w||f - g||_\infty) < (||f(\nu) - \nu||_\infty, w||f(\nu) - \nu||_\infty) \\ +(||g(\nu) - \nu||_\infty, w||g(\nu) - \nu||_\infty),\end{aligned}$$

$\forall \nu \in W$. If $h = I$ on W, UIEs (4.1) and (4.2) have a UCS, by Theorem 3.3. □

Conclusion

Fixed point iteration methods are applied to solve Urysohn integral equations. So, we have to find all possible fixed point results in connection with fixed point iteration methods.

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(Gurusamy Siva) Department of Mathematics, Alagappa University, Karaikudi-630 003, India
Email address: gsivamaths2012@gmail.com