



A COMPLEX LIMIT CYCLE NOT INTERSECTING THE REAL PLANE

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ABSTRACT. We give a precise example of a polynomial vector field on \mathbb{R}^2 whose corresponding singular holomorphic foliation of \mathbb{C}^2 possesses a complex limit cycle which does not intersect the real plane \mathbb{R}^2 .

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1. Introduction

Every polynomial vector field $\begin{cases} \frac{dz}{dt} = P(z, w) \\ \frac{dw}{dt} = Q(z, w) \end{cases}$ defines a singular foliation by holomorphic

curves on \mathbb{C}^2 . The qualitative study of the behavior of these curves plays a crucial role in the area of holomorphic dynamical system. Every non-singular curve of the complex equation is considered as a leaf of the corresponding singular foliation. In the real dynamical system a limit cycle is a closed orbit whose Poincare return map is not equal to the identity map. For introduction on limit cycle phenomena see [2, page 250]. The number of limit cycles of polynomial vector fields is the main object of the second part of the Hilbert 16th Problem. For details on Hilbert 16th problem see [4]. However, in the complex setting a complex limit cycle is a complex leaf which contains a closed curve γ whose holonomy map is not equal to the identity map. The concept of holonomy is an immediate complex generalization of the Poincare return map. Historically, the idea of consideration of such kind of foliations arising from algebraic vector fields goes back to the innovative idea by I. Petrovskii and E. Landis who suggested considering a real polynomial vector field as a complex vector field. So, according to this idea we consider a real limit cycle as the intersection of a complex limit cycle with the real plane. For the seminal paper of I. Petrovskii and E. Landis see [7]. For a fundamental and deep contribution to this innovative idea of Petrovski-Landis, see the influential paper by Yu. Ilyashenko in [3].

In this paper, we consider the following question:

Question. Is there a polynomial vector field on \mathbb{R}^2 whose corresponding singular holomorphic foliation on \mathbb{C}^2 possess a complex limit cycle L which does not intersect the real plane \mathbb{R}^2 ?

By real plane we mean $\mathbb{R}^2 \subset \mathbb{C}^2$. That is

$$\{(z, w) \in \mathbb{C}^2 \mid \Im(z) = \Im(w) = 0\}.$$

A motivation for this question, is the following: Every real limit cycle is the intersection of a complex limit cycle with the real plane. But in the above question we somehow search for a precise example of an invisible limit cycle, namely a complex limit cycle which does not intersect the real plane.

We shall prove that the answer to the above question is affirmative as it is indicated in the following theorem. In this theorem by z', w' we mean $dz/dt, dw/dt$ respectively where $t \in \mathbb{C}$ is a complex time-parameter.

2. MAIN RESULTS

Main Theorem. The polynomial vector field

$$(2.1) \quad \begin{cases} z' = w + z(z^2 + w^2 + 1) \\ w' = -z + w(z^2 + w^2 + 1) \end{cases}$$

has $z^2 + w^2 + 1 = 0$ as a complex limit cycle which does not intersect the real plane \mathbb{R}^2 .

PRELIMINARIES AND PROOFS

A k dimensional foliation of a complex manifold M is a decomposition of $M = \bigcup L_\alpha$ where each L_α is an immersed k dimensional holomorphic submanifold of M . Moreover for every $p \in M$ there is a neighborhood U and a biholomorphism $\phi : U \rightarrow V \times W \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$, where V and W are open poly discs in \mathbb{C}^k and \mathbb{C}^{n-k} respectively. Moreover, the biholomorphism ϕ maps each connected component of $L_\alpha \cap U$ to a plaque $V \times \{c\}$ for a fixed $c \in W$. The triple $(U, V \times W, \phi)$ is called a foliation chart. An important example of a holomorphic foliation of \mathbb{C}^2 arises from a non vanishing holomorphic vector field. More precisely, to a vector field

$$(2.2) \quad P(z, w)\partial/\partial z + Q(z, w)\partial/\partial w$$

on \mathbb{C}^2 one associate a differential 1-form

$$(2.3) \quad \omega = -Q(z, w)dz + P(z, w)dw.$$

Then leaves of the foliation defined by $\omega = 0$ are the integral curve of the initial vector field. For more details on real and holomorphic foliations see [6] and [8].

Assume that L is a leaf of a holomorphic foliation. Let $\gamma : [0, 1] \rightarrow L$ be a continuous curve in L . At points $\gamma(0)$ and $\gamma(1)$ we choose two $n - k$ dimensional transversal sections S_0 and S_1 , respectively. They are $n - k$ dimensional complex local sections which are transversal to the leaf L . These transversal sections play the role of Poincare sections in the real dynamical system. Then holonomy map along γ is a holomorphic function $h : S_0 \rightarrow S_1$ with the property that $z \in S_0$ and $h(z) \in S_1$ lie on the same leaf of the foliation. This concept is the complex analogy of the Poincare map in the real context. In the real case, we have a regular orbit γ with two transversal sections S_0, S_1 . Then the Poincare map is a well defined map from S_0 to S_1 which is naturally defined via the flow of the vector field. In the Holomorphic case we simulate the same situation but via foliation charts moving along γ since the flow has no a well behaved definition in the holomorphic case. For details on definition and geometric description of the holonomy map of a foliation see [8]. For a leaf L of foliation, we fix a base point p . For every loop $\gamma \subset L$ based at p one may consider the holonomy map h_γ . This gives an obvious group structure on the set $\{h_\gamma \mid \gamma \in \pi_1(L, p)\}$, where $\pi_1(L, p)$ is the fundamental group of the leaf L associated to loops based at point p . This group is called the monodromy

group of L . We denote it by $G(L)$. So there is a natural group homomorphism from $\pi_1(L)$ into $G(L)$. A complex limit cycle is a leaf L whose monodromy group $G(L)$ is a non trivial group.

Every differential 1-form $\omega = -Q(z, w)dz + P(z, w)dw$ defines a 1 dimensional singular holomorphic foliation of \mathbb{C}^2 . The leaves of these foliations are the integral curves of

$$(2.4) \quad \begin{cases} z' = P(z, w) \\ w' = Q(z, w) \end{cases}$$

A natural question is that in this special case that the foliation is given by a global differential 1-form, how can we compute the holonomy of a typical leaf L of the foliation along a closed curve γ lying in L . The following beautiful theorem by Khanehdani-Suwa gives us a formula for the first variation of the holonomy map. Before we state the theorem Khanehdani-Suwa we give a very short introduction based on [5, Khanehdani-Suwa]:

Suppose that the foliation is defined via equation

$$\omega = -Q(z, w)dz + P(z, w)dw = 0.$$

Assume that L is a leaf of the foliation and γ is an arbitrary closed curve on L . Then we can write

$$d\omega = \alpha \wedge \omega,$$

where α is a multi-valued 1-form in a neighbourhood of γ and the restriction of α to each leaf is single-valued. Assume that h_γ is the holonomy of L associated to γ defined in a neighbourhood of origin in \mathbb{C} which models a local section transversal to the leaf. Then we have:

Theorem[Khanehdani-Suwa][5]. The first variation of the holonomy is given by $h'_\gamma(0) = \exp \int_\gamma \alpha$.

The proof of our main theorem is based on usage of the above theorem:

Proof of the Main Theorem. First we show that the algebraic curve $L : z^2 + w^2 + 1 = 0$ is an integral curve of the equation (2.1). We compute the derivation $z^2 + w^2 + 1$ along the vector field (2.1):

$$\begin{aligned} (z^2 + w^2 + 1)' &= 2zz' + 2ww' = \\ &= 2z(w + z(z^2 + w^2 + 1)) + 2w(-z + w(z^2 + w^2 + 1)) = \\ &= (2z^2 + 2w^2)(z^2 + w^2 + 1). \end{aligned}$$

This obviously shows that the algebraic curve $z^2 + w^2 + 1 = 0$ is an integral curve of equation (2.1). It is a non-singular solution which does not intersect the real plane $\mathbb{R}^2 \subset \mathbb{C}^2$. Now we prove that L is a complex limit cycle. To prove this we give a closed curve γ on L such that the holonomy map h_γ is different from the identity map. For a differential 1-form $\omega = -Qdz + Pdw$ we have $d\omega = \alpha \wedge \omega$, where

$$(2.5) \quad \alpha = \left(\frac{P_z + Q_w}{P^2 + Q^2} \right) (Pdz + Qdw).$$

We apply this formula to

$$(2.6) \quad \begin{cases} P(z, w) = w + z(z^2 + w^2 + 1) \\ Q(z, w) = -z + w(z^2 + w^2 + 1) \end{cases}$$

We shall introduce a closed curve $\gamma \subset L$ for which $\int_{\gamma} \alpha \neq 2k\pi i$, $\forall k \in \mathbb{Z}$. This would show that the holonomy map h_{γ} is different from the identity map since $h'_{\gamma}(0) \neq 1$. Notice that the restriction of α in (2.5) to L is equal to $2(wdz - zdw)$.

Because $P_z + Q_w = 2(z^2 + w^2 + 1) + 2z^2 + 2w^2 \equiv -2$ on L . Similarly $P^2 + Q^2 \equiv -1$ on L . So α restricts to $wdz - zdw$ on curve L . To compute $\int_{\gamma} \alpha$ we give a global parametrization of $L : z^2 + w^2 + 1 = 0$ then we apply the change of variable formula in integration:

$$(2.7) \quad \phi : \mathbb{C} \setminus \{0\} \rightarrow L \quad \phi(t) = (z(t), w(t)),$$

where

$$z(t) = \frac{i}{2}\left(t + \frac{1}{t}\right), w(t) = \frac{1}{2}\left(t - \frac{1}{t}\right).$$

Note that the global parametrization ϕ of L gives us an embedding of $\mathbb{C} \setminus \{0\}$ into \mathbb{C}^2 whose image is whole L . Moreover we have

$$\begin{aligned} \phi^*(wdz - zdw) &= w(t)d(z(t)) - z(t)d(w(t)) = \\ &= \frac{i}{4} \left(\left(t - \frac{1}{t}\right)\left(1 - \frac{1}{t^2}\right) - \left(t + \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right) \right). \end{aligned}$$

The closed curve γ in L whose holonomy map h_{γ} is nontrivial is the curve

$$\phi(e^{i\theta}), \quad \theta \in [0, 2\pi].$$

Put $\beta(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$, the unit circle in $\mathbb{C} \setminus \{0\}$. So we have $\phi(\beta) = \gamma$. For this closed curve γ we have $\int_{\gamma} \alpha = \int_{\beta} \phi^*(\alpha)$. This is a consequence of change of variables in integration. As we computed in the above lines, the restriction of α to L is $\alpha = 2(wdz - zdw)$ so we can write:

$$\begin{aligned} \int_{\gamma} \alpha &= \int_{\beta} \phi^*(\alpha) = \\ &= \frac{i}{4} \int_{\beta} \left(\left(t - \frac{1}{t}\right)\left(1 - \frac{1}{t^2}\right) - \left(t + \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right) \right) dt = \\ &= \frac{i}{4} \int_{\beta} \left(\frac{4}{t} + \frac{2}{t^3} \right) dt = -2\pi, \end{aligned}$$

by residue formula in complex function theory, see [1]

So according to theorem by Khanehdani-Suwa we see that the first variation of the holonomy is $e^{-2\pi}$ which is different from 1. So the holonomy map is not the identity map. Thus $L : z^2 + w^2 + 1 = 0$ is a complex limit cycle which obviously does not intersect the real plane \mathbb{R}^2 . This completes the proof of the main theorem.

DISCUSSIONS AND FURTHER RESEARCHES

In this paper, inspired by the relation between real and complex limit cycle theory, we introduced a precise example of a polynomial vector field on \mathbb{R}^2 whose corresponding complex foliation possesses a complex limit cycle not intersecting the real plane. In fact we observed

that the algebraic curve $L : z^2 + w^2 + 1 = 0$ is a complex limit cycle for the real coefficient algebraic vector field

$$\begin{cases} z' = w + z(z^2 + w^2 + 1) \\ w' = -z + w(z^2 + w^2 + 1) \end{cases}$$

The algebraic property of L in our example is a motivation to ask the following question:

Question 1. Is there an algebraic vector field on \mathbb{R}^2 whose corresponding singular foliation of \mathbb{C}^2 possess a complex limit cycle L which does not intersect the real plane \mathbb{R}^2 and L is not an algebraic curve?

Moreover the construction of a complex limit cycle not intersecting the real plane naturally arises the following question:

Question 2. Is there a polynomial vector field X on \mathbb{R}^2 such that every complex limit cycle of the corresponding foliation of \mathbb{C}^2 must necessarily intersect the real plane \mathbb{R}^2 ? In this question we do not require this intersection would be in form of a closed curve but we merely require the intersection would be a non empty set.

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