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Research Paper

ON THE GLOBAL STABILITY, EXISTENCE AND NONEXISTENCE OF LIMIT CYCLES IN A PREDATOR-PREY SYSTEM

TOHID KASBI AND VAHID ROOMI*

Abstract. The existence and number of limit cycles is an important problem in the study of ordinary differential equations and dynamical systems. In this work, we consider 2-dimensional predator-prey system and, using Poincaré-Bendixson theorem and LaSalle's invariance principle, present some new necessary and some new sufficient conditions for the existence and nonexistence of limit cycles of the system. These results extend and improve the previous results in this subject. Local or global stability of the rest points of a system is also an important issue in the study of the systems. In this paper, a sufficient condition about global stability of a critical point of the system will also be presented. Our results are sharp and are applicable for predator-prey systems with the functional response which is the function of prey and predator. At the end of the manuscript, some examples of well-known predator-prey systems are provided to illustrate our results.

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1. Introduction

Consider the following autonomous planar system

$$
\begin{aligned}
&\frac{dx}{dt} = rx\big(1-\frac{x}{k}\big) - yh(x,y) = P(x,y)\\
&\frac{dy}{dt} = y(\mu h(x,y) - D) = Q(x,y),\\
&x(0) > 0,\ y(0) > 0,
\end{aligned}
\tag{1.1}
$$

where x and y are the prey and predator population, respectively; r, k, μ and D are positive constants and h is a given functional response which satisfies in the following conditions.

($\mathbf{A}_1$) $h(0,y) = 0 \quad \text{for} \quad y > 0,$

($\mathbf{A}_2$) $\dfrac{\partial h(x,y)}{\partial x} > 0, \quad \text{for} \quad x, y > 0,$

($\mathbf{A}_3$) $\dfrac{\partial h(x,y)}{\partial y} < 0, \quad \text{for} \quad x, y > 0,$

($\mathbf{A}_4$) $\lim_{(x,y)\to(+\infty,0)} h(x,y) = C < \infty.$

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*Corresponding author.

Limit cycles have been used to model the behavior of a great many real-world oscillatory systems. The study of limit cycles was initiated by Henri Poincaré (1854-1912). The existence and number of limit cycles for 2-dimensional systems are related to Hilbert's 16th problem and oscillatory problems in mathematical models (see [1-6, 9-14] and reference cited therein).

First, note that $(0,0)$ and $(k,0)$ are two critical points of system (1.1). If $D/\mu \in Range\ h$ and $D/\mu < h(k,0)$, then this system has a third critical point $E_* = (x_*, y_*)$ where

$$h(x_*, y_*) = \frac{D}{\mu} \quad and \quad y_* = \frac{rx_*(1-\frac{x_*}{k})}{h(x_*, y_*)}.$$

This critical point is located in the first quadrant $\{(x,y)|x>0,\ y>0\}$ if $0<x<k$. When assumption $0 < D/\mu < h(k,0)$ fails, system (1.1) has no critical point in the first quadrant and hence no limit cycles of (1.1) exist.

Many authors have studied the nonexistence problem for limit cycles and some results are provided in [11, 12]. Recently, Aghajani and Moradifam in [1] considered system (1.1) with $h(x,y) = \phi(x)$ and presented the following theorem about the nonexistence of limit cycles of system (1.1) with functional response $\phi(x)$.

Theorem 1.1. *Suppose that* ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$) *and* ($\boldsymbol{A}_4$) *hold,* $\dfrac{D}{\mu} \in Range\ \phi$, *and* $\dfrac{D}{\mu} < \phi(k)$. *Moreover,*

$$\phi'(0) \le \frac{r}{y_*}.$$

Let

$$\psi(x) = \frac{\phi(x)}{x}, \quad F(x) = \frac{rx(1-\frac{x}{k})}{\phi(x)}.$$

If one of the following conditions holds:

(i) $\psi''(x)$ *has no zero in* $(0,k)$,
(ii) $F'(0) > 0$ *and* $\psi''(x)$ *has at most one zero in* $(0,k)$,

then system (1.1) *with* $h(x,y) = \phi(x)$ *has no limit cycles.*

In [13], Moghadas gave the following theorem for the nonexistence of limit cycles of system (1.1) with $h(x,y) = \phi(x)$.

Theorem 1.2. *Assume* ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$) *and* ($\boldsymbol{A}_4$) *hold. Also assume that* $\phi''(x) > 0$ *for* $x > 0$, $\frac{D}{\mu} \in Range\ \phi$ *and* $\frac{D}{\mu} < \phi(k)$. *Furthermore, there is a unique constant* $0 < \alpha < 1$ *such that* $\phi'''(x) < 0$ *for* $0 < x < \alpha$, $\phi'''(\alpha) = 0$ *and* $\phi'''(\alpha) > 0$ *for* $x > \alpha$. *If* $2\phi'(0) + \phi''(0) < 0$ *and* $\dfrac{r}{\phi'(0)} - y_* \ge 0$, *then*

$$2x_* + x_*(1-x_*)\frac{\phi'(x_*)}{\phi(x_*)} > 1, \tag{1.2}$$

is a necessary and sufficient condition for the nonexistence of limit cycles of system (1.1) *with* $h(x,y) = \phi(x)$.

The following theorem in [5] guarantees the boundedness of the solutions of system (1.1).

Theorem 1.3. *Suppose that ($\boldsymbol{A}_1$)-($\boldsymbol{A}_4$) hold. Let*

$$\Gamma = \left\{(x,y) \in \mathbb{R}_+^2 \mid 0 \le x \le k,\ 0 \le x + \frac{y}{\mu} \le \frac{1}{k} + \frac{M}{\mu}\right\},$$

where $M = \max\{rx(1-\frac{x}{k}) \ :\ x \in [0,k]\}$. *Then,*
1- Γ *is positively invariant,*
2- for all $(x_0, y_0) \in \mathbb{R}_+^2$,

$$(x(t), y(t)) \to \Gamma \quad as \quad t \to \infty.$$

In the next section, our main results will be presented. The results are simpler and more explicit than the results in [1, 13] and can be applied for the predator-prey systems with the functional response of two variables.

2. Main Results

In this section, using Poincaré-Bendixson theorem and LaSalle's invariance principle, a necessary condition and some sufficient conditions will be given about the existence and nonexistence of limit cycles for system (1.1). Moreover, the global stability of positive equilibrium of system (1.1) will be presented.

The following theorem gives a sufficient condition for system (1.1) to have limit cycle.

Theorem 2.1. *Assume that ($\boldsymbol{A}_1$)-($\boldsymbol{A}_4$) hold. System (1.1) has at least one limit cycle if*

$$\begin{aligned} &\mu y_* \frac{\partial h}{\partial y}(x_*, y_*) - y_* \frac{\partial h}{\partial x}(x_*, y_*) + r(1 - \frac{2x_*}{k}) > 0, \\ &h(x_*, y_*)\frac{\partial h}{\partial x}(x_*, y_*) + r(1 - \frac{2x_*}{k})\frac{\partial h}{\partial y}(x_*, y_*) > 0. \end{aligned} \tag{2.1}$$

Proof. The Jacobian matrix of the linearized system at E_* is as follows.

$$\begin{bmatrix} r(1 - \frac{2x_*}{k}) - y_* \frac{\partial h}{\partial x}(x_*, y_*) & -h(x_*, y_*) - y_* \frac{\partial h}{\partial y}(x_*, y_*) \\ \mu y_* \frac{\partial h}{\partial x}(x_*, y_*) & \mu y_* \frac{\partial h}{\partial y}(x_*, y_*) \end{bmatrix}.$$

Therefore, the characteristic polynomial of this matrix is as

$$\begin{aligned} P(z) = z^2 &+ \left(\mu y_* \frac{\partial h}{\partial y}(x_*, y_*) - y_* \frac{\partial h}{\partial x}(x_*, y_*) + r(1 - \frac{2x_*}{k})\right) z \\ &+ \mu y_* \left(\frac{\partial h}{\partial x}(x_*, y_*) h(x_*, y_*) + \frac{\partial h}{\partial y}(x_*, y_*) r(1 - \frac{2x_*}{k})\right). \end{aligned}$$

The roots of characteristic polynomial have positive real parts if and only if (2.1) holds. Thus, E_* is unstable if (2.1) holds. It is easy to check that the stable and unstable manifolds at $(0,0)$ are on the y-axis and x-axis, respectively. Also, the unstable manifold at $(k,0)$ is in the first quadrant. By Theorem 1.3, the positive solutions of (1.1) are eventually uniformly bounded. Since E_* is unstable, from Poincaré-Bendixson theorem, the ω-limit set of each orbit initiating at a point in the first quadrant is a limit cycle and the proof is complete. □

In the following, a necessary condition for the nonexistence of limit cycles for system (1.1) will be given.

Theorem 2.2. *Let*

$$F(x,y) = \frac{rx(1-\frac{x}{k})}{h(x,y)}.$$

Suppose that ($\boldsymbol{A}_1$)-($\boldsymbol{A}_4$) *hold. If system* (1.1) *has no limit cycles, then*

$$\frac{\partial F}{\partial x}(x_*, y_*) - \mu \frac{\partial F}{\partial y}(x_*, y_*) \leq 0. \tag{2.2}$$

Proof. By calculating the following derivatives

$$\frac{\partial F}{\partial x}(x_*, y_*) - \mu \frac{\partial F}{\partial y}(x_*, y_*) = \frac{\mu y_* \frac{\partial h}{\partial y}(x_*, y_*) - y_* \frac{\partial h}{\partial x}(x_*, y_*) + r(1 - \frac{2x_*}{k})}{h(x_*, y_*)}, \tag{2.3}$$

and using Theorem 2.1, it can be concluded that

$$\frac{\partial F}{\partial x}(x_*, y_*) - \mu \frac{\partial F}{\partial y}(x_*, y_*) \leq 0,$$

if system (1.1) has no limit cycles. This completes the proof. □

As mentioned, condition (2.2) is a necessary condition for system (1.1) about the nonexistence of limit cycles. In the following, some sufficient conditions are provided for system (1.1) about the nonexistence of limit cycles. To state our results, consider functions $\psi(x,y)$ and $F(x,y)$ as follows:

$$\psi(x,y) = \frac{h(x,y)}{x} \ , \qquad F(x,y) = \frac{rx(1-\frac{x}{k})}{h(x,y)}.$$

Hereafter assume that the conditions of Theorem 2.1 hold. In the following, a sufficient condition for the global stability of E_* for system (1.1) will be given. The condition says that if the horizontal line $y = y_*$ divides the prey isocline $y = F(x,y)$ into two parts, then E_* is globally asymptotically stable in positive cone.

Theorem 2.3. *Suppose that* ($\boldsymbol{A}_1$)-($\boldsymbol{A}_4$) *hold. If*

$$(x - x_*)(F(x,y) - y_*) < 0 \quad for \quad 0 < x < k, \quad x \neq x_*, \tag{2.4}$$

then the solutions of (1.1) *satisfy*

$$\lim_{t\to\infty} x(t) = x_* \qquad and \qquad \lim_{t\to\infty} y(t) = y_*. \tag{2.5}$$

Proof. Define the Lyapunov function

$$V(x,y) = \int_{x_*}^{x} \frac{(\mu h(\eta,y) - D)}{h(\eta,y)} \eta + \int_{y_*}^{y} \frac{\psi - y_*}{\psi} d\psi.$$

The time derivative of $V(x,y)$ along the positive solutions of system (1.1) can be written as

$$\begin{aligned}\dot{V} &= \dot{x}\left(\frac{\mu h(x,y) - D}{h(x,y)}\right) + \dot{y}\left(\frac{y - y_*}{y}\right) \\ &= \left(rx(1 - \frac{x}{k}) - yh(x,y)\right)\left(\frac{\mu h(x,y) - D}{h(x,y)}\right) + \left(y(\mu h(x,y) - D)\right)\left(\frac{y - y_*}{y}\right).\end{aligned}$$

Since

$$(x - x_*)(\mu h(x,y) - D) > 0 \quad for \quad x \neq x_*,$$

then

$$\dot{V} = (\mu h(x,y) - D)\left(\frac{rx(1-\frac{x}{k})}{h(x,y)} - y_*\right) = (\mu h(x,y) - D)(F(x,y) - y_*) \leq 0.$$

Hence, the equilibrium E_* is stable. On the other hand, $\dot{V} = 0$ if and only if $x = x_*, y = y_*$. Let M be the largest invariant set in

$$D = \{(x,y) \mid \dot{V}(x,y) = 0\} = \{E_*\}.$$

We have that $M = \{E_*\}$. The global stability of E_* or (2.5) follows from Theorem 1.3 and LaSalle's invariance principle [7, 8]. □

Remark 2.4. If the prey isocline $y = F(x,y)$ is nonincreasing on $0 < x < k$, then (2.4) holds. Thus E_* is globally asymptotically stable or system (1.1) has no limit cycles.

In the following, sufficient conditions for the nonexistence of limit cycles for system (1.1) will be given by using Dulac-Bendixson Theorem.

Theorem 2.5. *Suppose that (**A**$_1$)-(**A**$_4$) hold. Also, assume that*

$$\frac{\partial \psi}{\partial x} - \mu \frac{\partial \psi}{\partial y} \geq 0, \tag{2.6}$$

on $\Omega = \{(x,y)|\ 0 < x < k,\ 0 < y < \infty\}$. *Then, system* (1.1) *has no limit cycles.*

Proof. Choosing Dulac function $D(x,y) = \dfrac{1}{xy}$ for system (1.1) in Ω we have

$$\frac{\partial(DP)}{\partial x} + \frac{\partial(DQ)}{\partial y} = -\frac{r}{ky} - \left(\frac{\partial \psi}{\partial x} - \mu \frac{\partial \psi}{\partial y}\right) < 0 \quad (\neq 0).$$

Thus, by Dulac-Bendixson theorem, there is no closed orbit in Ω. This implies that the interior critical point $E_*(x_*, y_*)$ is globally asymptotically stable or system (1.1) has no limit cycles. □

Theorem 2.6. *Suppose that (**A**$_1$)-(**A**$_4$) hold. Also, assume that*

$$\frac{\partial F}{\partial x} \leq 0 \tag{2.7}$$

on $\Omega = \{(x,y)|\ 0 < x < k,\ 0 < y < \infty\}$. *Then, system* (1.1) *has no limit cycles.*

Proof. For system (1.1), choosing Dulac function $D(x,y) = \dfrac{1}{yh(x,y)}$, we have

$$\frac{\partial(DP)}{\partial x} + \frac{\partial(DQ)}{\partial y} = \frac{1}{y}\frac{\partial F}{\partial x} + D\frac{\frac{\partial h}{\partial y}}{h(x,y)^2} < 0 \quad (\neq 0)$$

in Ω. By Dulac-Bendixson theorem, there is no closed orbit in Ω. This implies that the interior critical point $E_*(x_*, y_*)$ is globally asymptotically stable or system (1.1) has no limit cycles. □

The following corollaries are applicable for system (1.1) with functional response of one variable. Let $h(x,y) = \phi(x)$. In this case $F(x) = \frac{rx(1-\frac{x}{k})}{\phi(x)}$ and $\psi(x) = \frac{\phi(x)}{x}$.

Corollary 2.7. *Suppose that*

$$\psi'(x) \geq 0 \tag{2.8}$$

on $\Omega = \big\{(x, y) |\ 0 < x < k,\ 0 < y < \infty\big\}$*. Then, system* (1.1) *with* $h(x, y) = \phi(x)$ *has no limit cycles.*

Corollary 2.8. *Suppose that*

$$F'(x) < 0 \tag{2.9}$$

on $\Omega = \big\{(x, y) |\ 0 < x < k,\ 0 < y < \infty\big\}$*. Then, system* (1.1) *with* $h(x, y) = \phi(x)$ *has no limit cycles.*

Remark 2.9. Notice that in the case of $h(x, y) = \phi(x)$, Theorems 2.2, 2.3 and Corollary 2.8 are reduced to Theorems 2.1, 2.2 and Corollary 2.1 in [1], respectively.

3. **Examples**

In this section, some examples are provided to illustrate our results. These examples show that how our results are simply applicable to some well-known predator-prey systems.

Example 3.1. Consider system (1.1) with

$$h(x, y) = \frac{\alpha x}{1 + ax + by}, \quad \alpha, a, b > 0.$$

This system is said to have functional response of Beddington-DeAngelis type. For this system

$$\psi(x, y) = \frac{\alpha}{1 + ax + by}.$$

If $\mu \geq \frac{a}{b}$, then

$$\frac{\partial \psi}{\partial x} - \mu \frac{\partial \psi}{\partial y} = \frac{\alpha(\mu b - a)}{(1 + ax + by)^2} \geq 0.$$

Thus, by Theorem 2.5 this system has no limit cycles.

Example 3.2. Consider system (1.1) with the functional response of Crowley-Martin type,

$$h(x, y) = \frac{\alpha x}{1 + ax + by + abxy}, \quad \alpha, a, b > 0,$$

and assume that $a \leq \frac{1}{k}$. Then,

$$F(x, y) = \frac{r}{\alpha}\big(1 - \frac{x}{k}\big)(1 + ax + by + abxy).$$

Thus,

$$\frac{\partial F}{\partial x} = \frac{r}{\alpha}\bigg(\big(a - \frac{1}{k}\big) + by\big(a - \frac{1}{k}\big) - 2\frac{a}{k}x(1 + y)\bigg).$$

If $a \leq \frac{1}{k}$, then

$$\frac{\partial F}{\partial x}(x, y) < 0,$$

in the first quadrant. Therefore, Theorem 2.6 implies that this system has no limit cycles.

Example 3.3. Consider system (1.1) with the functional response of one variable,

$$\phi(x) = \frac{x^2}{(x+a)(x+b)}, \quad a, b > 0.$$

Assume that $k \leq \sqrt{ab}$. Then,

$$\psi(x) = \frac{x}{(x+a)(x+b)}.$$

Thus,

$$\psi'(x) = \frac{ab - x^2}{\big((x+a)(x+b)\big)^2} \geq 0.$$

If $k \leq \sqrt{ab}$, then

$$\psi'(x) \geq 0$$

on $\Omega = \big\{(x,y)|\ 0 < x < k,\ 0 < y < \infty\big\}$. Therefore, Corollary 2.7 implies that this system has no limit cycles.

Remark 3.4. By comparing Example 3.3 in our work and Example 3.2 in [1], which both represent the same systems, it can be seen that with much simpler conditions than in [1], the nonexistence of limit cycles of system (1.1) with $h(x,y) = \phi(x)$ has been obtained.

Example 3.5. Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x\big(1-x\big) - \frac{7}{2}\frac{xy}{1+x+y} = P(x,y) \\ \frac{dy}{dt} &= y(\frac{7}{2}\frac{x}{1+x+y} - 1) = Q(x,y). \end{aligned} \tag{3.1}$$

It is easy to check that $(\frac{1}{2}, \frac{1}{4})$ is an equilibrium point of (3.1) in the region $\{(x,y) : 0 < x < 1,\ y > 0\}$. We have

$$F(x,y) = \frac{2}{7}(1 + y - x^2 - xy).$$

By using Theorem 2.1, it can be concluded that

$$\frac{\partial F}{\partial x}(\frac{1}{2}, \frac{1}{4}) - \frac{\partial F}{\partial y}(\frac{1}{2}, \frac{1}{4}) = -\frac{1}{2} < 0.$$

Thus, we have the necessary condition for the nonexistence of a limit. On the other hand, the variational matrix of system (3.1) at $(\frac{1}{2}, \frac{1}{4})$ takes the form

$$\begin{bmatrix} -\frac{1}{4}\frac{\partial h}{\partial x}(\frac{1}{2}, \frac{1}{4}) & -h(\frac{1}{2}, \frac{1}{4}) - \frac{1}{4}\frac{\partial h}{\partial y}(\frac{1}{2}, \frac{1}{4}) \\ \frac{1}{4}\frac{\partial h}{\partial x}(\frac{1}{2}, \frac{1}{4}) & \frac{1}{4}\frac{\partial h}{\partial y}(\frac{1}{2}, \frac{1}{4}) \end{bmatrix} = \begin{bmatrix} \frac{-5}{14} & \frac{-6}{7} \\ \frac{5}{14} & \frac{-1}{7} \end{bmatrix},$$

where $h(x,y) = \frac{7}{2}\frac{x}{1+x+y}$. From the above matrix, we concluded that point $(\frac{1}{2}, \frac{1}{4})$ is locally asymptotic stable. By using Dulac-Bendixson theorem with $D(x,y) = \frac{1}{xy}$, we have

$$\frac{\partial(DP)}{\partial x} + \frac{\partial(DQ)}{\partial y} = -\frac{1}{y} < 0.$$

This implies that the interior critical point $(\frac{1}{2}, \frac{1}{4})$ is globally asymptotically stable or system (3.1) has no limit cycles (Fig. 1).

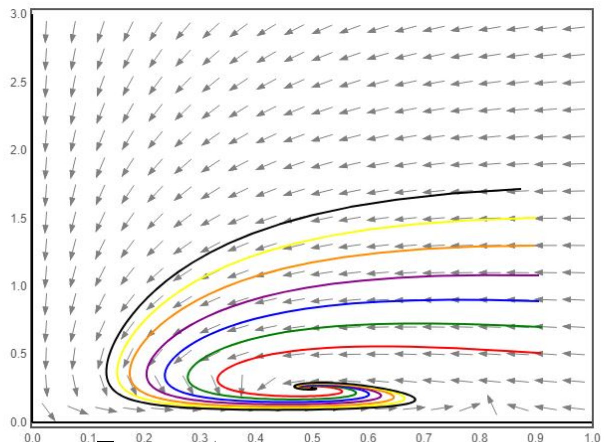

FIGURE 1. Phase portrait of system (3.1)

Example 3.6. Consider system

$$\frac{dx}{dt} = rx\big(1 - \frac{x}{k}\big) - \frac{\beta xy}{x+y}$$
$$\frac{dy}{dt} = y\big(\mu\frac{\beta xy}{x+y} - D\big), \tag{3.2}$$

with $\beta < r$ and let $B = \dfrac{\beta k}{r}$. For this system, the prey isocline is

$$y = \frac{kx - x^2}{B - k + x}.$$

Thus,

$$y' = \frac{-x^2 + 2(k-B)x + k(B-k)}{(B-k+x)^2)}.$$

The prey isocline is nonincreasing on $0 < x < k$ if

$$-x^2 + 2(k-B)x + k(B-k) < 0. \tag{3.3}$$

The relation (3.3) holds if

$$\Delta = 4B(B-k) < 0,$$

or $\beta < r$. Thus, $E_* = (x_*, y_*)$ is globally asymptotically stable or system (3.2) has no limit cycles.

Remark 3.7. Theorems 1.1 and 1.2 are just applicable for the systems with functional responses of one variable. Therefore, they are not applicable to Examples 3.1, 3.2 and 3.5 since the functional response of the systems in these examples are defined in terms of two variables.

The authors in [1] and [13], under many complicated conditions including some inequalities and other conditions on the derivatives of the functional response, gave necessary and sufficient conditions for the nonexistence of limit cycles of system (1.1). In this work, much simpler explicit conditions have been given for the nonexistence of limit cycles of the system. Moreover, the predator-prey system with the general functional response with two variables has been studied here and some results of [1] have been extended.

4. **Conclusion**

In this work, using Poincaré-Bendixson theorem and LaSalle's invariance principle, we have presented some new necessary and some new sufficient conditions for the existence and nonexistence of limit cycles of the 2-dimensional predator-prey system. A sufficient condition about the global stability of the critical point of the system has also been presented. The obtained results are applicable for predator-prey systems with functional response which is the function of both prey and predator. Some examples of well-known predator-prey systems have been provided to illustrate the results.

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(Tohid Kasbi) Faculty of Mathematical Science, University of Tabriz, Tabriz, Iran.
Email address: tohid.kasbi@gmail.com

(Vahid Roomi) Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.
Email address: roomi@azaruniv.ac.ir