

**CONVOLUTION WEIGHTED ORLICZ ALGEBRAS  
IN CONTEXT OF  $\sigma$ -COMPACT GROUPS**

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**ABSTRACT.** In this paper, some sufficient condition for a weighted Orlicz space,  $L_w^\Phi(G)$ , to be a Banach algebra with convolution as multiplication in context of a  $\sigma$ -compact groups. We also for a class of Orlicz spaces, obtain an equivalent condition, such that a weighted Orlicz space to be a convolution Banach algebra. This resultes generalized some known results in Lebesgue spaces.

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**1. Introduction and Background**

The condition that the Lebesgue spaces become an algebra by convolution multiplication has always been considered. It is known that for each locally compact group  $G$ ,  $L^1(G)$  is a convolution Banach algebra, and for each  $p > 1$ ,  $L^p(G)$  is a convolution algebra if and only if  $G$  is compact ([4], [15]). Kuznetsova in [8] proved that  $L_w^1(G)$  is convolution Banach algebra if and only if  $w$  is submultiplicative, and in the case of  $G$  is abelian for a  $p > 1$  and a weight  $w$   $L_w^p(G)$  is a convolution algebra if and only if  $G$  is  $\sigma$ -compact.

Given that Orlicz spaces are a very important generalization of Lebesgue spaces, the natural question is when Orlicz spaces or weighted Orlicz spaces are convolution Banach algebra. For a locally compact abelian group  $G$  and a  $\Delta_2$ -regular Young function  $\Phi$ , H. Hudzik, A. Kamiska and J. Musielak in [6] prove that the Orlicz space  $L^\Phi(G)$  is a convolution Banach algebra if and only if  $L^\Phi(G) \subseteq L^1(G)$ , and this holds if and only if  $G$  is compact or  $\lim_{x \rightarrow 0^+} \frac{\Phi(x)}{x} > 0$ . In [1, 13, 14] another necessary and sufficient condition for an Orlicz space  $L^\Phi(G)$ , and its weighted version  $L_w^\Phi(G)$ , to be a convolution Banach algebra is given. Also to observe similar results with the context of hypergroups see [2, 12].

In this paper,  $G$  is a locally compact group and  $\lambda$  is the left Haar measure on  $G$ . Let recall some properties of Orlicz spaces. For more details see three books [7, 10, 11] which are basic references for this subject.

A *Young function* is a convex even mapping  $\Phi : \mathbb{R} \rightarrow [0, \infty]$  such that  $\Phi(0) = \lim_{x \rightarrow 0} \Phi(x) = 0$  and  $\lim_{x \rightarrow \infty} \Phi(x) = \infty$ . A Young function  $\Phi : \mathbb{R} \rightarrow [0, \infty)$  is called an *N-function* if it is continuous,  $\Phi(x) = 0$  implies  $x = 0$ ,  $\lim_{x \rightarrow 0} \frac{\Phi(x)}{x} = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Phi(x)}{x} = \infty$ .

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The *complementary* of a Young function  $\Phi$  is the function  $\Psi$  defined by

$$\Psi(x) := \sup\{y|x| - \Phi(y) : y \geq 0\}, \quad (x \in \mathbb{R}).$$

If  $\Psi$  is the complementary of a Young function  $\Phi$ , then  $(\Phi, \Psi)$  is called a *Young pair*. If we denote the set of all Borel measurable complex-valued functions on  $G$  by  $\mathcal{B}(G)$ , then the Orlicz space  $L^\Phi(G)$  define by

$$L^\Phi(G) = \{f \in \mathcal{B}(G) : \exists \alpha > 0, \int_G \Phi(\alpha|f(x)|) d\lambda(x) < \infty\}.$$

For each Young pair  $(\Phi, \Psi)$  we denote the set of all  $f \in \mathcal{B}(G)$  with  $\int_G \Phi(|f(x)|) d\lambda(x) \leq 1$  by  $B_\Phi$ . Then we can define for each  $f \in L^\Phi(G)$  the *Orlicz norm*  $\|f\|_\Phi$  and *Luxemburg norm*  $\|f\|_{[\Phi]}$  respectively by

$$\|f\|_\Phi := \sup \left\{ \int_G |f(x)g(x)| d\lambda(x) : g \in B_\Psi \right\}$$

$$\|f\|_{[\Phi]} := \inf \left\{ \alpha > 0 : \frac{1}{\alpha} f \in B_\Phi \right\}.$$

If almost everywhere equal functions in  $L^\Phi(G)$  consider the same, then  $\|\cdot\|_\Phi$  and  $\|\cdot\|_{[\Phi]}$  are equivalent norms on  $L^\Phi(G)$  with

$$(1.1) \quad \|f\|_{[\Phi]} \leq \|f\|_\Phi \leq 2\|f\|_{[\Phi]}.$$

For each  $f \in L^\Phi(G)$  and  $g \in L^\Psi(G)$ , we have the *Hölder's inequality* for Orlicz spaces (see [10, Proposition 1, and the Remark after it, page 58]):

$$(1.2) \quad \int_G |f(x)g(x)| d\lambda(x) \leq 2\|f\|_{[\Phi]} \|g\|_{[\Psi]}.$$

A continuous positive function  $w$  on  $G$  called a *weight*. Similar to other function spaces, the *weighted Orlicz space*,  $L_w^\Phi(G)$ , is defined by

$$L_w^\Phi(G) = \{f \in \mathcal{B}(G) : wf \in L^\Phi(G)\}.$$

$L_w^\Phi(G)$  with the norm  $\|f\|_{\Phi,w} := \|wf\|_\Phi$  is a Banach space.

## 2. Main Results

Our motivation for giving the main result of this paper is to provide a generalization for [8, Theorem 1.1]. In that theorem it has been proved that a locally compact group  $G$  is  $\sigma$ -compact if and only if for any  $p > 1$  there exists a weight  $w$  such that  $w^{-q} * w^{-q} < w^{-q}$ , if and only if for any  $p > 1$  there exists a weight  $w$  such that  $L_w^\Phi(G)$  is a convolution Banach algebra. Recall that for any measurable functions  $f$  and  $g$  on  $G$ , the convolution product  $f * g$  is defined by

$$(f * g)(x) := \int_G f(y)g(y^{-1}x) d\lambda(y)$$

for all  $x \in G$ , while this integral exists. Also the weighted Orlicz space  $L_w^\Phi(G)$  is called a *convolution Banach algebra* if there exists a constant  $D > 0$  such that

$$\|f * g\|_{\Phi,w} \leq D \|f\|_{\Phi,w} \|g\|_{\Phi,w},$$

for all  $f, g \in L_w^\Phi(G)$ .

In sequel, for each weight function  $w$  on  $G$ , we define  $\Omega_w : G \times G \rightarrow (0, \infty)$  by

$$\Omega_w(x, y) = \frac{w(yx)}{w(x)w(y)}.$$

Also, we say that a Young function  $\Psi$  is satisfied the property  $(\mathcal{A})$  whenever, for each  $\sigma$ -compact group  $G$  there exists a weight  $w$  on  $G$  and there exists a real number  $C > 0$  such that for all  $v \in B_\Psi$ , the function  $H_v(y) := \|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_\Psi$ , ( $y \in G$ ) belongs to  $L^\Psi(G)$  and  $\|H_v\|_\Psi \leq C$  where for each function  $f$  on  $G$  and each  $y \in G$ , the function  $L_y f$  on  $G$  define by  $L_y f(x) = f(y^{-1}x)$ .

**Remark 2.1.** By [8, Theorem 1.1] easily one can see that for each  $1 < q < \infty$ , the Young function  $\Psi_q := |\cdot|^q$  satisfies the property  $(\mathcal{A})$ .

Indeed for each locally compact group  $G$  by [8, Theorem 1.1] always there exists a weight  $w$  such that  $w^{-q} * w^{-q} < w^{-q}$ . So for each  $v \in B_{\Psi_q}$  we have

$$\begin{aligned} \|H_v\|_{\Psi_q} &= \left( \int_G (H_v(y))^q dy \right)^{\frac{1}{q}} \\ &= \left( \int_G (\|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_{\Psi_q})^q dy \right)^{\frac{1}{q}} \\ &= \left( \int_G \int_G |\Omega_w(x, y)L_{y^{-1}}v(x)|^q dx dy \right)^{\frac{1}{q}} \\ &= \left( \int_G \int_G \frac{w(yx)^q}{w(x)^q w(y)^q} |v(yx)|^q dx dy \right)^{\frac{1}{q}} \\ &= \left( \int_G w(x)^q |v(x)|^q \left( \int_G \frac{1}{w(y)^q w(y^{-1}x)^q} dy \right) dx \right)^{\frac{1}{q}} \\ &= \left( \int_G w(x)^q |v(x)|^q (w^{-q} * w^{-q})(x) dx \right)^{\frac{1}{q}} \\ &\leq \left( \int_G |v(x)|^q dx \right)^{\frac{1}{q}} \\ &= \|v\|_{\Psi_q} \leq 1. \end{aligned}$$

Therefore  $\Psi_q$  satisfies the property  $(\mathcal{A})$  with  $C = 1$ .

Before stating the main thesis of this article we also recall that for a locally compact abelian group  $G$ , a weight function  $w$  and a complementary pair of Young functions  $(\Phi, \Psi)$  a function  $\xi : G \rightarrow \mathbb{C} \setminus \{0\}$  is called a *generalized character* if  $\xi(xy) = \xi(x)\xi(y)$  for all  $x, y \in G$ , and  $\frac{\xi}{w} \in L^\Psi(G)$ . The set of all generalized characters of  $G$  is denoted by  $\widehat{G_\Psi}(w)$ .

**Theorem 2.2.** *Let  $G$  be a locally compact group. Then, the following conditions are equivalent:*

- (1)  $G$  is  $\sigma$ -compact.
- (2) There are a complementary pair  $(\Phi, \Psi)$  of  $N$ -functions in which for some weight function  $w$ , we have  $\frac{1}{w} \in L^\Psi(G)$  and

$$\Psi\left(\frac{1}{w}\right) * \Psi\left(\frac{1}{w}\right) \leq \Psi\left(\frac{1}{w}\right).$$

- (3) For all complementary pair  $(\Phi, \Psi)$  of  $N$ -functions there exists some weight function  $w$  such that  $\frac{1}{w} \in L^\Psi(G)$  and

$$\Psi\left(\frac{1}{w}\right) * \Psi\left(\frac{1}{w}\right) \leq \Psi\left(\frac{1}{w}\right).$$

If  $G$  is abelian, then the above conditions are equivalent to the following conditions:

- (4) There are a complementary pair  $(\Phi, \Psi)$  of  $N$ -functions in which  $\Psi$  satisfies the property  $(\mathcal{A})$  for some weight  $w$  on  $G$ , and  $L_w^\Phi(G)$  is a convolution Banach algebra.  
(5) For each complementary pair  $(\Phi, \Psi)$  of  $N$ -functions in which  $\Psi$  satisfies the property  $(\mathcal{A})$  for some weight  $w$  on  $G$ ,  $L_w^\Phi(G)$  is a convolution Banach algebra.

*Proof.* (2)  $\Rightarrow$  (1): Let  $(\Phi, \Psi)$  be a complementary pair of  $N$ -functions and  $w$  be a weight on  $G$  such that  $\frac{1}{w} \in L^\Psi(G)$ . Then for some  $\alpha > 0$ , we have  $\Psi\left(\frac{\alpha}{w}\right) \in L^1(G)$ . Since  $\Psi(t) = 0$  implies that  $t = 0$ , and  $\Psi\left(\frac{\alpha}{w}\right) > 0$  on  $G$ , we have

$$G = \{x \in G : \Psi\left(\frac{\alpha}{w}\right) > 0\},$$

and so thanks to [3, Proposition 2.20] and [4, Theorem 1.40],  $G$  is  $\sigma$ -compact.

(1)  $\Rightarrow$  (3): Let  $G$  be a  $\sigma$ -compact group. By the proof of [8, Theorem 1.1], there is an integrable function  $u > 0$  such that  $u * u \leq u$ . Now for each  $N$ -function  $\Psi$  if we put  $w := \frac{1}{\Psi^{-1}(u)}$ , then we have  $\frac{1}{w} \in L^\Psi(G)$  and  $\Psi\left(\frac{1}{w}\right) * \Psi\left(\frac{1}{w}\right) \leq \Psi\left(\frac{1}{w}\right)$ .

Since (3)  $\Rightarrow$  (2) is trivial, the conditions 1 – 3 are equivalent.

(1)  $\Rightarrow$  (5): Let  $G$  be a  $\sigma$ -compact group. Then for all pairs  $(\Phi, \Psi)$  of  $N$ -functions such that  $\Psi$  satisfies the property  $(\mathcal{A})$ , there exists a weight  $w$  on  $G$  that satisfies the property  $(\mathcal{A})$ . Suppose that  $f, g \in L_w^\Phi(G)$ . Then for each  $v \in B_\Psi$ , since for almost every elements  $y \in G$  we have  $\|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_\Psi < \infty$ , thanks to Hölder's inequality (1.2) and inequality (1.1) we have

$$\begin{aligned} & \int_G |(f * g)(x)| w(x) |v(x)| d\lambda(x) \\ & \leq \int_G |f(y)| w(y) \int_G |g(x)| w(x) \frac{w(yx)}{w(x)w(y)} |v(yx)| d\lambda(x) d\lambda(y) \\ & \leq 2 \int_G |f(y)| w(y) \|fw\|_\Phi \|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_\Psi d\lambda(x) d\lambda(y) \\ & \leq 4 \|f\|_{\Phi, w} \|g\|_{\Phi, w} \|H\|_\Psi \\ & \leq 4C \|g\|_{\Phi, w} \|f\|_{\Phi, w}. \end{aligned}$$

So we have  $\|f * g\|_{\Phi, w} \leq 4C \|f\|_{\Phi, w} \|g\|_{\Phi, w}$  and  $L_w^\Phi(G)$  is a convolution algebra.

Now, let  $G$  be abelian.

(4)  $\Rightarrow$  (1): Let  $L_w^\Phi(G)$  be a convolution algebra. Since  $G$  is abelian, by [9, Proposition. 5.7]  $L_w^\Phi(G)$  is not radical. This implies that there exists a non-zero homomorphism  $T$  from  $L_w^\Phi(G)$  into  $\mathbb{C}$ , and so by [9, Theorem 5.2], there exists an element  $\eta \in \widehat{G_\Psi}(w)$  such that

$$T(f) = \int_G f \eta d\lambda, \quad (f \in L_w^\Phi(G)).$$

Hence we have  $\frac{\eta}{w} \in L^\Psi(G)$ . In other words, there exists an  $\alpha > 0$  such that  $\Psi\left(\frac{|\eta|}{\alpha w}\right) \in L^1(G)$ . Therefore  $G$  is  $\sigma$ -compact.  $\square$

By [5, Proposition 5] and [9, Lamma 4.3], for any  $N$ -function  $\Psi$ ,  $L^\Psi(G)$  is a Banach algebra with respect to pointwise multiplication if and only if  $L^\Psi(G) \subseteq L^\infty(G)$  if and only if  $G$  is discrete. Using these facts, we can express the following conclusion.

**Corollary 2.3.** *Let  $G$  be a discrete group. Then for any weight  $w$  on  $G$  such that*

- (1) *for each  $y \in G$ ,  $\Omega_w(\cdot, y) \in B_\Psi$  and,*
- (2) *if  $H(y) = \|\Omega_w(\cdot, y)\|_\Psi$ , ( $y \in G$ ), then  $H \in L^\Psi(G)$ ,*

*$\ell_w^\Phi(G)$  is a convolution Banach algebras.*

*Proof.* Since  $G$  is discrete, by [5, Proposition 5] and [9, Lamma 4.3],  $\ell^\Psi(G)$  is Banach algebra with respect to pointwise multiplication. So there exists a real number  $B$  such that for each  $f, g \in \ell^\Psi(G)$  we have  $\|fg\|_\Psi \leq B\|f\|_\Psi\|g\|_\Psi$ . By substituting  $\frac{1}{B}\Psi$  for  $\Psi$ , conditions (1) and (2) imply that if  $v \in B_\Psi$ , then

- (1) for each  $y \in G$ ,

$$\|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_\Psi \leq \|\Omega_w(\cdot, y)\|_\Psi\|L_{y^{-1}}|v|\|_\Psi \leq \|\Omega_w(\cdot, y)\|_\Psi, \text{ and}$$

- (2) the function  $H(y) = \|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_\Psi$  ( $y \in G$ ), belongs to  $B_\Psi$ ,

since the mapping  $x \mapsto v(yx)$  belongs to  $B_\Psi$ . So the Young function  $\Psi$  is satisfied the property (A) and by Theorem 2.2,  $\ell_w^\Phi(G)$  is a convolution Banach algebra.  $\square$

Before stating the next corollary, we remind that a Young function  $\Phi$  satisfies  $\Delta_2$ -condition (and write  $\Phi \in \Delta_2$ ) if there are  $c > 0$  and  $x_0 \geq 0$  such that

$$\Phi(2x) \leq c\Phi(x), \quad (x \geq x_0).$$

**Corollary 2.4.** *If  $\Psi$  is globally  $\Delta_2$ -regular and  $w$  is a weight on  $\mathbb{Z}$  such that*

$$(2.1) \quad \exists \beta > 0, \forall m, n \in \mathbb{Z}, \forall v \in B_\Psi, \Omega_w(m, n)|v(m-n)| \leq \frac{\beta}{(|m|+1)(|n|+1)}$$

*then  $\ell_w^\Phi(\mathbb{Z})$  is a convolution Banach algebras.*

*Proof.* Since  $\Psi \in \Delta_2$  globally by [10, Chapter II, Corollary 5] there exist a real numbers  $M > 0$  and  $\alpha > 1$  such that for all  $x \in \mathbb{R}$ ,  $\Psi(x) \leq M|x|^\alpha$ . So by (2.1) for each  $n \in \mathbb{N}$  we have

$$\begin{aligned} \int_{\mathbb{Z}} \Psi(\Omega_w(\cdot, n)L_{-n}|v|) &= \sum_{m=-\infty}^{+\infty} \Psi(\Omega_w(m, n)L_{-n}|v|) \\ &\leq \sum_{m=-\infty}^{+\infty} M|\Omega_w(m, n)L_{-n}|v|^\alpha \\ &\leq \sum_{m=-\infty}^{+\infty} M \left( \frac{\beta}{(|m|+1)(|n|+1)} \right)^\alpha \\ &= \frac{M\beta^\alpha}{(1+|n|)^\alpha} \sum_{m=-\infty}^{+\infty} \frac{1}{(1+|m|)^\alpha} \\ &\leq M\beta^\alpha \sum_{m=-\infty}^{+\infty} \frac{1}{(1+|m|)^\alpha} \\ &= M\beta^\alpha(2\zeta(\alpha) - 1) < \infty, \end{aligned}$$

where  $\zeta$  is the Riemann zeta function.

On the other hand if  $f \in B_\Phi$  we have  $\sum_{m=-\infty}^{+\infty} |\Phi(f(m))| \leq 1$ . So that for each  $m \in \mathbb{Z}$  we have  $|\Phi(f(m))| \leq 1$ . Therefore for each  $n \in \mathbb{Z}$  we see that

$$\begin{aligned} \|\Omega_\omega(\cdot, n)L_{-n}|v\|_\Psi &= \sup\left\{ \sum_{m=-\infty}^{+\infty} |\Omega_\omega(m, n)v(m-n)f(m)| : f \in B_\Phi \right\} \\ &\leq \Phi^{-1}(1) \sum_{m=-\infty}^{+\infty} |\Omega_\omega(m, n)v(m-n)| \end{aligned}$$

So for the function  $H : y \mapsto \|\Omega_\omega(\cdot, n)L_{-n}|v\|_\Psi$  we have

$$\begin{aligned} \int_{\mathbb{Z}} \Psi(H) &= \sum_{n=-\infty}^{+\infty} \Psi(H)(n) \\ &\leq \sum_{n=-\infty}^{+\infty} \Psi\left(\Phi^{-1}(1) \sum_{m=-\infty}^{+\infty} |\Omega_\omega(m, n)v(m-n)|\right) \\ &\leq \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} M |\Phi^{-1}(1)\Omega_\omega(m, n)v(m-n)|^\alpha \\ &\leq \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} M \Phi^{-1}(1)^\alpha \left(\frac{\beta}{(|m|+1)(|n|+1)}\right)^\alpha \\ &= M \Phi^{-1}(1)^\alpha \beta^\alpha (2\zeta(\alpha) - 1)^2. \end{aligned}$$

If  $\gamma = \max\{M\beta^\alpha(2\zeta(\alpha) - 1), M\Phi^{-1}(1)^\alpha\beta^\alpha(2\zeta(\alpha) - 1)^2\}$  then considering  $\frac{\Psi}{\gamma}$  instead of  $\Psi$  it is observed that  $\Psi$  is true in the condition of of Theorem 2.2, so that  $\ell_w^\Phi(\mathbb{Z})$  is a convolution Banach algebras.  $\square$

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