

**CONSTRUCTION OF NONNEGATIVE MATRIX FOR SPECIAL SPECTRUM**

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ABSTRACT. The construction of a nonnegative matrix for a given set of eigenvalues is one of the objectives of this paper. The generalization of the cases discussed in the previous papers as well as finding a recursive solution for the Suleimanova spectrum are other points that are studied in this paper.

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**1. Introduction**

A matrix  $A$  is called nonnegative if all its entries are nonnegative. The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a list  $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_n)$  of complex numbers in order that it be the spectrum of a nonnegative matrix. In this case, one says that  $\sigma$  is realizable and a nonnegative matrix  $A$  with spectrum  $\sigma$  is said to realize  $\sigma$  and it is referred to as a realizing matrix. There is a right and a left eigenvector associated with the Perron eigenvalue with nonnegative entries. The spectral radius of the nonnegative matrix  $A$  is denoted by  $\rho(A)$ . In addition  $s_k$  the  $k$ -th power sum of the eigenvalues  $\lambda_i$  and in the list  $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_n)$ ,  $\lambda_1$  is the Perron element. Some necessary conditions on the list of complex numbers  $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_n)$  to be the spectrum of a nonnegative matrix are listed below.

- (1) The Perron eigenvalue  $\max\{|\lambda_i|; \lambda_i \in \sigma\}$  belongs to  $\sigma$  (Perron -Frobenius Theorem).
- (2) The list  $\sigma$  is closed under complex conjugation.
- (3)  $s_k = \sum_{i=1}^n \lambda_i^k \geq 0$ .
- (4)  $s_k^m \leq n^{m-1} s_{km}$  for  $k, m = 1, 2, \dots$  (JLL inequality)[3,8].

A number of necessary conditions for realizability are known, as well as a number of sufficient conditions. In many cases, sufficiency is established by the direct construction of a realizing matrix [1-6].

In terms of  $n$ , complete solutions to the NIEP are available only for  $n \leq 4$ . Nazari and Sherafat in [14] tried to introduce a recursive method for solving (NIEP). They solved different cases for state  $n = 5$  and their recursive method can also be used for case  $n > 5$ . Although they found a nonnegative matrix for many cases of  $\sigma$ , we can say that complete solution for this problem when  $n \geq 5$  is an open problem.

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For the case of non-real spectra  $\sigma$  for  $n = 4$ , complete solutions are available through work of Laffey and Meehan [5] (see Meehan's 1998 doctoral thesis (National University of Ireland, Dublin [12])) and, independently, that of Torre-Mayo, Abril-Raymundo, Alarcia-Estevez, Marijuan and Pisanero by analyzing coefficients of the characteristic polynomial. EBL digraphs [11]. Oscar Rojo, Ricardo L. Soto found a necessary and sufficient condition for  $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  to be the spectrum of some circulant nonnegative matrix [13]. In [10] Helena Smigoc started with a realizable list of real numbers and obtained a realizable list that contains elements that are not real.

We start by Lemma 6 of paper from Helena Smigoc in [2] and in section 2 bring Theorem 2.1 from [14] that is similar to Smigoc's Lemma and find some new condition to solve (NIEP) for a given real list of  $\sigma$ . In section 3 we give some special sets of the spectrum and construct a nonnegative matrix corresponding to them.

**Lemma 1.1.** *Suppose  $B$  is an  $m \times m$  matrix with canonical form  $J(B)$  that contains at least one  $1 \times 1$  Jordan block corresponding to the eigenvalue  $c$ :*

$$J(B) = \begin{pmatrix} c & 0 \\ 0 & I(B) \end{pmatrix},$$

let  $t$  and  $s$ , respectively, be the left and right eigenvector of  $B$  associated with the  $1 \times 1$  Jordan block in the above canonical form. Furthermore, we normalize vectors  $t$  and  $s$  so that  $t^T s = 1$ . Let  $J(A)$  be a Jordan canonical form for an  $n \times n$  matrix

$$A = \begin{pmatrix} A_1 & a \\ b^T & c \end{pmatrix}$$

where  $A_1$  is an  $(n-1) \times (n-1)$  matrix and  $a$  and  $b$  are vectors in  $C^{n-1}$ . Then the matrix

$$C = \begin{pmatrix} A_1 & at^T \\ sb^T & B \end{pmatrix}$$

has Jordan canonical form

$$J(C) = \begin{pmatrix} J(A) & 0 \\ 0 & I(B) \end{pmatrix}.$$

## 2. Construction of nonnegative matrix with spectrum of two special nonnegative matrices

**Theorem 2.1.** *Let  $B$  be an  $m \times m$  nonnegative matrix and  $M_1 = \{\mu_1, \mu_2, \dots, \mu_m\}$  be its eigenvalues and  $\mu_1$  be Perron eigenvalue of  $B$ . Assume  $A$  be an  $n \times n$  nonnegative matrix in following form*

$$A = \begin{pmatrix} A_1 & a \\ b^T & \mu_1 \end{pmatrix},$$

where  $A_1$  is an  $(n-1) \times (n-1)$  matrix and  $a$  and  $b$  are arbitrary vectors in  $C^{n-1}$  and  $M_2 = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$  is the set of eigenvalues of  $A$ . Then there exist the  $(m+n-1) \times (m+n-1)$  nonnegative matrix such that  $M = \{\mu_2, \dots, \mu_m, \lambda_1, \lambda_2, \dots, \lambda_m\}$  is its eigenvalues.

*Proof.* Proof in [14]. □

We present a Corollary of the above Theorem, which can be used for problems that do not require the existence of Perron eigenvalue of the matrix  $B$  on the main diagonal of the matrix  $A$ .

**Corollary 2.2.** *Let the conditions of Theorem 2.1 satisfy but matrix  $A$  is in the following form:*

$$A = \begin{pmatrix} A_1 & a \\ b^T & \alpha\mu_1 \end{pmatrix},$$

then there exists a nonnegative matrix  $C$  with the order of  $(m+n-1) \times (m+n-1)$  as

$$C = \begin{pmatrix} A_1 & as^* \\ sb^T & \alpha B \end{pmatrix},$$

such that the elements of  $M = \{\lambda_1, \dots, \lambda_n, \alpha\mu_2, \dots, \alpha\mu_m\}$  are its eigenvalues.

*Proof.* Proof is very similar of the Theorem 2.1.  $\square$

**Remark 2.3.** We can use the above Theorem for both symmetric and nonsymmetric matrices. We illustrate this with two examples. It is easy to see that the nonnegative matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 250 & 55 & 2 \end{pmatrix} \text{ has eigenvalues } \begin{pmatrix} 10 \\ -4-3i \\ -4+3i \end{pmatrix} \text{ and the matrix } B = \begin{pmatrix} 0 & 28 \\ 1 & 12 \end{pmatrix} \text{ has}$$

eigenvalues  $\begin{pmatrix} 14 \\ -2 \end{pmatrix}$ , with normalized Perron eigenvector  $s = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ . In order to be able

to use the above theorem, we must create a nonnegative matrix whose its Perron eigenvalue is equal 2, because this number is on the main diagonal of matrix  $A$ . Now let  $a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and

$b^T = (250 \ 55)$  then we see that  $\frac{1}{7}B$  has eigenvalues  $\begin{pmatrix} 2 \\ -2/7 \end{pmatrix}$ . Then by above Theorem

the nonnegative matrix

$$C = \begin{pmatrix} A_1 & as^* \\ sb^T & \alpha B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/4\sqrt{2} & 1/4\sqrt{14} \\ \frac{125\sqrt{2}}{2} & \frac{55\sqrt{2}}{4} & 0 & 2/7\sqrt{7} \\ \frac{125\sqrt{14}}{2} & \frac{55\sqrt{14}}{4} & 2/7\sqrt{7} & \frac{12}{7} \end{pmatrix},$$

is nonnegative matrix with eigenvalues  $\begin{pmatrix} -2/7 \\ 10 \\ -4-3i \\ -4+3i \end{pmatrix}$ . It is easy to see that we can use

the above Theorem for symmetric matrices. The symmetric matrix  $A = \begin{pmatrix} 0 & \sqrt{15} \\ \sqrt{15} & 2 \end{pmatrix}$

has eigenvalues  $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$  and we choose the nonnegative  $2 \times 2$  symmetric matrix as  $B =$

$\begin{pmatrix} 0 & \sqrt{14} \\ \sqrt{14} & 12 \end{pmatrix}$  with eigenvalues  $\begin{pmatrix} 14 \\ -2 \end{pmatrix}$  and then the matrix  $\frac{1}{7}B$  has eigenvalues  $\begin{pmatrix} 2 \\ -2/7 \end{pmatrix}$

and the Perron eigenvalue of matrix  $\frac{1}{7}B$  lies in main diagonal of matrix  $A$  and since the Perron eigenvector of matrix  $\frac{1}{7}B$  is  $\begin{pmatrix} 1/4\sqrt{2} \\ 1/4\sqrt{14} \end{pmatrix}$ , then the following nonnegative symmetric matrix

$$\begin{pmatrix} A_1 & as^T \\ sa^T & \alpha B \end{pmatrix} = \begin{pmatrix} 0 & 1/4\sqrt{15}\sqrt{2} & 1/4\sqrt{15}\sqrt{14} \\ 1/4\sqrt{15}\sqrt{2} & 0 & 2/7\sqrt{7} \\ 1/4\sqrt{15}\sqrt{14} & 2/7\sqrt{7} & \frac{12}{7} \end{pmatrix},$$

has eigenvalues  $\begin{pmatrix} 5 \\ -3 \\ -2/7 \end{pmatrix}$ .

**Corollary 2.4.** *If we change the matrices  $A$  and  $C$  in corollary 2.2, in the following form:*

$$A = \begin{pmatrix} A_1 & a \\ b^T & \alpha + \mu_1 \end{pmatrix}, C = \begin{pmatrix} A_1 & as^* \\ sb^T & (\alpha I + B) \end{pmatrix},$$

then the set of  $M = \{(\mu_2 + \alpha), \dots, (\mu_m + \alpha), \lambda_1, \dots, \lambda_n\}$  is spectrum of nonnegative matrix  $C$ .

**Example 2.5.** Consider the matrix of Remark (2.3)  $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 250 & 55 & 1+1 \end{pmatrix}$  with eigen-

values  $\begin{pmatrix} 10 \\ -4-3i \\ -4+3i \end{pmatrix}$ , and by above Corollary  $\alpha = \mu_1 = 1$  and  $B = \begin{pmatrix} 1/2 & 1/14\sqrt{7} \\ 1/14\sqrt{7} & \frac{13}{14} \end{pmatrix}$

has eigenvalues  $\begin{pmatrix} 1 \\ 3/7 \end{pmatrix}$  then  $\alpha I + B = \begin{pmatrix} 3/2 & 1/14\sqrt{7} \\ 1/14\sqrt{7} & \frac{27}{14} \end{pmatrix}$  has eigenvalues  $\begin{pmatrix} 2 \\ 10/7 \end{pmatrix}$  and the Perron eigenvalue of matrix  $\alpha I + B$  lies in main diagonal of matrix  $A$  and then

$$C = \begin{pmatrix} A_1 & as^* \\ sb^T & (\alpha I + B) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/4\sqrt{2} & 1/4\sqrt{14} \\ \frac{125\sqrt{2}}{2} & \frac{55\sqrt{2}}{4} & 3/2 & 1/14\sqrt{7} \\ \frac{125\sqrt{14}}{2} & \frac{55\sqrt{14}}{4} & 1/14\sqrt{7} & \frac{27}{14} \end{pmatrix}$$

has eigenvalues

$$\begin{pmatrix} \frac{10}{7} \\ 10 \\ -4-3i \\ -4+3i \end{pmatrix}$$

**Theorem 2.6.** *Assume  $B$  is an  $m \times m$  nonnegative diagonal matrix and  $M_1 = \{\mu_1, \mu_2, \dots, \mu_m\}$  is set of its eigenvalues and  $\mu_i$  and  $\mu_j$  are two arbitrary elements of  $M$ , without loss of generalization of the problem, let  $i = 1$  and  $j = 2$ . Take  $A$  as an  $n \times n$  nonnegative matrix as*

follows

$$A = \begin{pmatrix} A_1 & a_1 & a_2 \\ b_1^T & 0 & 0 \\ b_2^T & 0 & 0 \end{pmatrix},$$

where  $A_1$  is an  $(n-2) \times (n-2)$  matrix and  $a_1, a_2, b_1$  and  $b_2$  are arbitrary vectors in  $C^{n-2}$ . If  $M_2 = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  be the set of eigenvalues of  $A$ , then there exists a  $(3m+n-4) \times (3m+n-4)$  nonnegative matrix, such that  $M = \{\mu_3, \dots, \mu_m, \mu_3, \dots, \mu_m, \lambda_1, \dots, \lambda_n, \underbrace{0, \dots, 0}_{m \text{ times}}\}$  is its spectrum.

*Proof.* Assume vectors  $s$  and  $t$  are orthonormal eigenvectors associated to eigenvalues  $\mu_1$  and  $\mu_2$ , respectively. By Schur decomposition theorem, there exists the unitary matrix  $Y$  such that  $Y^*BY = T_B = B$ . Now we partition the matrix  $Y$  and  $Y^*$  in the following form,

$$Y = \begin{pmatrix} s & t & T \end{pmatrix} \quad \text{and} \quad Y^* = \begin{pmatrix} s^* \\ t^* \\ T^* \end{pmatrix}$$

where  $T$  is  $m \times (m-2)$  matrix and  $s$  and  $t$  are  $m \times 1$  vectors and it is obvious that,

$$YY^* = ss^* + tt^* + TT^* = I_m,$$

$$Y^*Y = \begin{pmatrix} s^*s & s^*t & s^*T \\ t^*s & t^*t & t^*T \\ T^*s & T^*t & T^*T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix} \quad (2.5)$$

from the above relation we have,

$$Y^*BY = \begin{pmatrix} \mu_i & 0 & \star \\ 0 & \mu_j & \star \\ 0 & 0 & \hat{T}_B \end{pmatrix} = T_B \quad (2.5)$$

$\hat{T}_B$  is a diagonal matrix with set of  $\{\mu_3, \dots, \mu_m\}$  in its main diagonal. By Schur decomposition Theorem, there exist an unitary matrix  $X$ , such that  $X^*AX = T_A$ , is an upper triangular matrix with the elements  $M_2$  in its main diagonal. The matrices  $X$  and  $X^*$  are partitioned as below,

$$X = \begin{pmatrix} V \\ K \\ L \end{pmatrix} \quad \text{and} \quad X^* = \begin{pmatrix} V^* & K^* & L^* \end{pmatrix},$$

where the order of matrix  $V$  is  $(n-2) \times n$  and the order of  $K$  and  $L$  are both  $1 \times n$ , since  $X$  is a unitary matrix, we have,

$$XX^* = \begin{pmatrix} VV^* & VK^* & VL^* \\ KV^* & KK^* & KL^* \\ LV^* & LK^* & LL^* \end{pmatrix} = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.7)$$

$$X^*X = V^*V + K^*K + LL^* = I_n.$$

By (2.7) and  $X^*AX = T_A$ , we have,

$$T_A = V^*A_1V + K^*b^TV + L^*b_2^TV + V^*a_1K + V^*a_2L. \quad (2.8)$$

We consider matrices  $Z$  and  $Z^*$  and nonnegative matrix  $C$  with  $(3m + n - 4) \times (3m + n - 4)$  dimension in the following form,

$$Z = \begin{pmatrix} V & 0 & 0 & 0 \\ sK & ts^* & 0 & T \\ tL & st^* & T & 0 \\ 0 & T^* & 0 & T \end{pmatrix}, Z^* = \begin{pmatrix} V^* & K^*s^* & L^*t^* & 0 \\ 0 & st^* & ts^* & T \\ 0 & 0 & T^* & 0 \\ 0 & T^* & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} A_1 & a_1s^* & a_2t^* & 0 \\ sb_1^T & TT^*B & 0 & 0 \\ tb_2^T & 0 & TT^*B & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the relations (2.5) and (2.7), it is easy to show that  $Z$  is a unitary matrix. Now by the relations (2.5)-(2.8), we can calculate  $Z^*CZ$ ,

$$Z^*CZ = \begin{pmatrix} V^*A_1V + K^*b^TV + L^*b_2^TV + V^*a_1K + V^*a_2L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & T^*BT & 0 \\ 0 & 0 & 0 & T^*BT \end{pmatrix} = \begin{pmatrix} T_A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{T}_B & 0 \\ 0 & 0 & 0 & \hat{T}_B \end{pmatrix} = T_C,$$

$T_C$  is an upper triangular matrix and the elements of its main diagonal are the elements of set of  $M$ . On the other hand by the relation above  $C$  and  $T_C$  are similar, then  $C$  is the matrix, we were to find, and the proof is completes.  $\square$

**Corollary 2.7.** *Let the conditions of Theorem 2.5 satisfy, then there exist  $(3m + n - 4) \times (3m + n - 4)$  nonnegative matrix, that*

$$M = \{\alpha\mu_3 + \delta, \dots, \alpha\mu_m + \delta, \beta\mu_3 + \delta, \dots, \beta\mu_m + \delta, \gamma\lambda_1, \dots, \gamma\lambda_n, \underbrace{0, \dots, 0}_{m \text{ times}}\}$$

is its spectrum, where  $\alpha, \beta, \delta$  and  $\gamma$  are arbitrary nonnegative real numbers.

*Proof.* The nonnegative matrix  $C$  is as the following form:

$$C = \begin{pmatrix} \gamma A_1 & \gamma a_1 s^* & \gamma a_2 t^* & 0 \\ \gamma s b_1^T & \alpha T T^* B + \delta I & 0 & 0 \\ \gamma t b_2^T & 0 & \beta T^* T B + \delta I & 0 \\ 0 & 0 & 0 & \delta T^* T \end{pmatrix},$$

and it is a solution of the problem. Note that in this Corollary, we construct a unitary matrix for the matrices  $A, B$  and  $C$  which is the same as Theorem 2.5.  $\square$

### 3. Special cases of NIEP of order $n$

**Theorem 3.1.** *Assume  $\sigma = \{\lambda_1, \dots, \lambda_n\}$ , such that the elements of  $\sigma$  are real numbers and  $\sigma$  has only one real positive number  $\lambda_1$ . Let  $\sigma$  satisfies in the following conditions,*

$$\lambda_1 + \lambda_2 + \dots + \lambda_n \geq 0. \quad (3.1)$$

*Then there exist the nonnegative matrix of order  $n$ , such that  $\sigma$  is its spectrum.*

*Proof.* Although Suleimanova solved in [7] this problem in 1949 with a companion matrix, we want to find another solution here. We provide proof by induction.. Let  $n = 2$ , in this case the nonnegative matrix

$$A = \begin{pmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{pmatrix} \quad (3.2)$$

is solution of the problem.

Assume  $n = 3$ , put  $\sigma_1 = \{\lambda_1, \lambda_2\}$ , it is clear that  $\sigma_1$  satisfies in the conditions of theorem, so that  $\sigma_1$  is spectrum of the matrix (3.2). Let  $\sigma_2 = \{\lambda_1 + \lambda_2, \lambda_3\}$ , by the relation (3.1), we have,

$$(\lambda_1 + \lambda_2 > 0, \lambda_3 \leq 0) \quad \text{or} \quad (\lambda_1 + \lambda_2 = 0, \lambda_3 = 0), \quad (3.3)$$

then

$$\lambda_1 + \lambda_2 \geq |\lambda_3|. \quad (3.4)$$

The relations above show that  $\sigma_2$  satisfies in the conditions of theorem and by the case of  $n = 2$ ,  $\sigma_2$  is the spectrum of  $2 \times 2$  nonnegative matrix,

$$B = \begin{pmatrix} 0 & -(\lambda_1 + \lambda_2)\lambda_3 \\ 1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix}.$$

Consequently by (3.4),  $\lambda_1 + \lambda_2$  is Perron eigenvalue of nonnegative matrix  $B$ . It is easy to show that the right orthonormal eigenvector associated with the Perron eigenvalue of  $B$  is

$$s = \begin{pmatrix} \frac{-\lambda_3}{\sqrt{1+\lambda_3^2}} \\ \frac{1}{\sqrt{1+\lambda_3^2}} \end{pmatrix},$$

because of the Perron eigenvalue of  $B$  is placed on the main diagonal of nonnegative matrix  $A$ , then matrices  $A$  and  $B$  satisfy in theorem 2.1, therefore the nonnegative matrix

$$C = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\sqrt{1+\lambda_3^2}} & \frac{-\lambda_1\lambda_2}{\sqrt{1+\lambda_3^2}} \\ \frac{-\lambda_3}{\sqrt{1+\lambda_3^2}} & 0 & -(\lambda_1 + \lambda_2)\lambda_3 \\ \frac{1}{\sqrt{1+\lambda_3^2}} & 1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix} \quad (3.5)$$

has spectrum of  $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ .

Now assume problem holds for  $n - 1$ , in order to construct a  $n \times n$  nonnegative matrix with the set of eigenvalues  $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ , we have to do the following process.

Let  $\sigma_1 = \{\lambda_1, \lambda_2, \dots, \lambda_{n-1}\}$  and  $\lambda = \lambda_1 + \lambda_2 + \dots, \lambda_{n-2}$  then by relation (3.1) we have

$$(\lambda > 0, \lambda_{n-1} \leq 0) \quad \text{or} \quad (\lambda = 0, \lambda_{n-1} = 0), \text{ and then we have} \\ \lambda \geq |\lambda_{n-1}|.$$

So that,  $\sigma_1$  satisfies in the conditions of our theorem, by the hypothesis of induction we can construct the  $(n - 1) \times (n - 1)$  nonnegative matrix  $A$  with spectrum of  $\sigma_1$ . By (3.2) and (3.5), the nonnegative matrix  $A$  is as the following form

$$A = \begin{pmatrix} A_1 & a \\ b^T & \lambda + \lambda_{n-1} \end{pmatrix},$$

where  $A_1$  is  $(n - 2) \times (n - 2)$  matrix and  $a$  and  $b$  are the vectors with dimension of  $(n - 1) \times 1$ . Let  $\lambda' = \lambda + \lambda_{n-1}$ , by (3.1) we have  $\lambda' \geq |\lambda_n|$ , then by the case of  $n = 2$  there exist the  $2 \times 2$  nonnegative matrix  $B$ , with spectrum  $\sigma_2 = \{\lambda', \lambda_n\}$  in the following form

$$B = \begin{pmatrix} 0 & -\lambda'\lambda_n \\ 1 & \lambda' + \lambda_n \end{pmatrix}.$$

It is clear that  $\lambda'$  is Perron eigenvalue of nonnegative matrix  $B$  and orthonormal eigenvector corresponding to  $\lambda'$  is

$$s = \begin{pmatrix} \frac{-\lambda_n}{\sqrt{1+\lambda_n^2}} \\ \frac{1}{\sqrt{1+\lambda_n^2}} \end{pmatrix}.$$

The nonnegative matrices  $A$  and  $B$  are satisfied on theorem 2.1. then the nonnegative matrix,

$$C = \begin{pmatrix} A_1 & as^* \\ sb^T & B \end{pmatrix},$$

with spectrum  $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  is a solution of our problem.  $\square$

**Theorem 3.2.** Assume  $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  and  $n \geq 3$ . If  $\lambda_1$  be positive real number and  $\lambda_2$  and  $\lambda_3$  be pair complex numbers and the other elements of  $\sigma$  be negative or zero real numbers and the conditions (3.1) and (3.2) satisfy and furthermore elements of  $\sigma$  satisfy in the following condition,

$$\alpha_1 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + |\lambda_2|^2 \leq 0.$$

Then there exists an  $n \times n$  nonnegative matrix such that  $\sigma$  is its spectrum.

*Proof.* We prove by induction on  $n$ . For  $n = 3$ , the following nonnegative matrix,

$$A = \begin{pmatrix} 0 & \lambda_1\lambda_2\lambda_3 & 0 \\ 0 & 0 & 1 \\ 0 & -\alpha_1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix},$$

is a solution of the problem.

Assume the proposition for  $n - 1$  satisfies, then so as to construct on  $n \times n$  nonnegative matrix with spectrum  $\sigma = \{\lambda_1, \dots, \lambda_n\}$ , we can use the process of Theorem 3.1.  $\square$

**Theorem 3.3.** Assume  $\sigma = \{\lambda_1, \dots, \lambda_n\}$ , such that  $\sigma$  has only one negative number  $\lambda_2$  and other elements of it, are nonnegative real numbers, furthermore assume the conditions (3.1) and (3.2) satisfy, then there exists an  $n \times n$  nonnegative matrix, such that  $\sigma$  is its spectrum.

*Proof.* If  $n = 2$ , the  $2 \times 2$  nonnegative matrix (3.3) is a solution of this problem. If  $n > 2$ , we consider the following matrix,

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where  $A$  is the matrix of (3.3) and  $B$  is nonnegative diagonal matrix of  $n - 2$  order in the following form,

$$B = \text{diag}(\lambda_3, \dots, \lambda_n)$$

consequently  $C$  is solution of the problem.  $\square$

**Example 3.4.**  $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ ,  $\mu = \sqrt{1 + \lambda_3^2}$ ,  $\beta = \sqrt{1 + \lambda_4^2}$

$$A_1 = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\mu} & \frac{\lambda_1\lambda_2\lambda_4}{\mu\beta} & \frac{-\lambda_1\lambda_2}{\mu\beta} \\ \frac{-\lambda_3}{\mu} & 0 & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_4}{\beta} & \frac{-(\lambda_1+\lambda_2)\lambda_3}{\beta} \\ \frac{-\lambda_4}{\mu\beta} & \frac{-\lambda_4}{\beta} & 0 & -(\lambda_1 + \lambda_2 + \lambda_3)\lambda_4 \\ \frac{1}{\mu\beta} & \frac{1}{\beta} & 1 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{pmatrix}$$

$$\sigma_2 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}, \gamma = \sqrt{1 + \lambda_5^2}$$

$$A_2 = \begin{pmatrix} 0 & \frac{\lambda_1 \lambda_2 \lambda_3}{\mu} & \frac{\lambda_1 \lambda_2 \lambda_4}{\mu \beta} & \frac{\lambda_1 \lambda_2 \lambda_5}{\mu \beta \gamma} & \frac{-\lambda_1 \lambda_2}{\mu \beta \gamma} \\ -\lambda_3 & 0 & \frac{(\lambda_1 + \lambda_2) \lambda_3 \lambda_4}{\beta} & \frac{(\lambda_1 + \lambda_2) \lambda_3 \lambda_5}{\beta \gamma} & \frac{-(\lambda_1 + \lambda_2) \lambda_3}{\beta \gamma} \\ \frac{\mu}{\beta} & \frac{-\lambda_4}{\beta} & 0 & \frac{(\lambda_1 + \lambda_2 + \lambda_3) \lambda_4 \lambda_5}{\gamma} & \frac{-(\lambda_1 + \lambda_2 + \lambda_3) \lambda_4}{\gamma} \\ \frac{-\lambda_5}{\beta} & \frac{-\lambda_5}{\beta} & \frac{-\lambda_5}{\beta} & 0 & -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \lambda_5 \\ \frac{\mu \beta \gamma}{1} & \frac{\beta \gamma}{1} & \frac{\gamma}{1} & 1 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \end{pmatrix},$$

$$\sigma_3 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}, \eta = \sqrt{1 + \lambda_5^2}, \lambda = \lambda_1 + \lambda_2 + \lambda_3, \lambda' = \lambda + \lambda_4 + \lambda_5$$

$$A_3 = \begin{pmatrix} 0 & \frac{\lambda_1 \lambda_2 \lambda_3}{\mu} & \frac{\lambda_1 \lambda_2 \lambda_4}{\mu \beta} & \frac{\lambda_1 \lambda_2 \lambda_5}{\mu \beta \gamma} & \frac{\lambda_1 \lambda_2 \lambda_6}{\mu \beta \gamma \eta} & \frac{-\lambda_1 \lambda_2}{\mu \beta \gamma \eta} \\ -\lambda_3 & 0 & \frac{(\lambda_1 + \lambda_2) \lambda_3 \lambda_4}{\beta} & \frac{(\lambda_1 + \lambda_2) \lambda_3 \lambda_5}{\beta \gamma} & \frac{(\lambda_1 + \lambda_2) \lambda_3 \lambda_6}{\beta \gamma \eta} & \frac{-(\lambda_1 + \lambda_2) \lambda_3}{\beta \gamma \eta} \\ \frac{\mu}{\beta} & \frac{-\lambda_4}{\beta} & 0 & \frac{\lambda \lambda_4 \lambda_5}{\gamma} & \frac{\lambda \lambda_4 \lambda_6}{\gamma \eta} & \frac{-\lambda \lambda_4}{\gamma \eta} \\ \frac{-\lambda_5}{\beta} & \frac{-\lambda_5}{\beta} & \frac{-\lambda_5}{\beta} & 0 & \frac{(\lambda + \lambda_4) \lambda_5 \lambda_6}{\eta} & \frac{-(\lambda + \lambda_4) \lambda_5}{\eta} \\ \frac{\mu \beta \gamma}{1} & \frac{\beta \gamma}{1} & \frac{\gamma}{1} & 0 & \frac{\eta}{1} & \frac{\eta}{1} \\ \frac{-\lambda_6}{\beta} & \frac{-\lambda_6}{\beta} & \frac{-\lambda_6}{\beta} & \frac{-\lambda_6}{\beta} & 0 & -\lambda' \lambda_6 \\ \frac{\mu \beta \gamma \eta}{1} & \frac{\beta \gamma \eta}{1} & \frac{\gamma \eta}{1} & \frac{\eta}{1} & 1 & \lambda' + \lambda_6 \end{pmatrix}.$$

We select the next example from [15] and try to find a nonsymmetric nonnegative matrix for the given  $\sigma$ .

**Example 3.5.** Assume given

$$\sigma = \{\lambda_1 = 15, \lambda_2 = -1, \lambda_3 = -2, \lambda_4 = -3, \lambda_5 = -4, \lambda_6 = -5\},$$

since  $\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \leq 0$  and  $\sum_{i=1}^6 \lambda_i \geq 0$  then by Theorem (3.1) we construct a solution

for  $\sigma$ . At first it is easy to see that the symmetric matrix  $C_1 = \begin{pmatrix} 0 & 15 \\ 1 & 14 \end{pmatrix}$  has eigenvalues

$\begin{pmatrix} 15 \\ -1 \end{pmatrix}$  and the matrix  $B = \begin{pmatrix} 0 & 28 \\ 1 & 12 \end{pmatrix}$  has eigenvalues  $\begin{pmatrix} 14 \\ -2 \end{pmatrix}$ , with normalized Perron

eigenvector  $s = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ , then let  $a = \begin{pmatrix} 15 \\ 1 \end{pmatrix}$  and  $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  then the  $3 \times 3$  following matrix

$$C_2 = \begin{pmatrix} A_2 & as^* \\ sb^T & B \end{pmatrix} = \begin{pmatrix} 0 & 6\sqrt{5} & 3\sqrt{5} \\ 2/5\sqrt{5} & 0 & 28 \\ 1/5\sqrt{5} & 1 & 12 \end{pmatrix},$$

has eigenvalues

$$\begin{pmatrix} 15 \\ -2 \\ -1 \end{pmatrix},$$

and it is necessary to mention that the member (3,3) of the matrix  $C_1$  is equal to 12, and

this makes it possible to continue the algorithm. Again the matrix  $B = \begin{pmatrix} 0 & 36 \\ 1 & 9 \end{pmatrix}$  has

eigenvalues  $\begin{pmatrix} 12 \\ -3 \end{pmatrix}$  with normalized Perron eigenvector  $s = \begin{pmatrix} 3/10\sqrt{10} \\ 1/10\sqrt{10} \end{pmatrix}$ , in this case

$a = \begin{pmatrix} 3\sqrt{5} \\ 28 \end{pmatrix}$  and  $b = \begin{pmatrix} 1/5\sqrt{5} \\ 1 \end{pmatrix}$  so that

$$C_3 = \begin{pmatrix} A_2 & as^* \\ sb^T & B \end{pmatrix} = \begin{pmatrix} 0 & 6\sqrt{5} & 9/2\sqrt{2} & 3/2\sqrt{2} \\ 2/5\sqrt{5} & 0 & \frac{42}{5}\sqrt{10} & \frac{14}{5}\sqrt{10} \\ 3/10\sqrt{2} & 3/10\sqrt{10} & 0 & 36 \\ 1/10\sqrt{2} & 1/10\sqrt{10} & 1 & 9 \end{pmatrix},$$

has eigenvalues

$$\begin{pmatrix} 15 \\ -3 \\ -2 \\ -1 \end{pmatrix}.$$

With continue this method we have

$$C_4 = \begin{pmatrix} 0 & 6\sqrt{5} & 9/2\sqrt{2} & \frac{6}{17}\sqrt{2}\sqrt{17} & \frac{3}{34}\sqrt{2}\sqrt{17} \\ 2/5\sqrt{5} & 0 & \frac{42}{5}\sqrt{10} & \frac{56}{85}\sqrt{10}\sqrt{17} & \frac{14}{85}\sqrt{10}\sqrt{17} \\ 3/10\sqrt{2} & 3/10\sqrt{10} & 0 & \frac{144}{17}\sqrt{17} & \frac{36}{17}\sqrt{17} \\ \frac{2}{85}\sqrt{2}\sqrt{17} & \frac{2}{85}\sqrt{10}\sqrt{17} & \frac{4}{17}\sqrt{17} & 0 & 36 \\ \frac{1}{170}\sqrt{2}\sqrt{17} & \frac{1}{170}\sqrt{10}\sqrt{17} & 1/17\sqrt{17} & 1 & 5 \end{pmatrix}$$

with eigenvalues

$$\begin{pmatrix} 15 \\ -4 \\ -3 \\ -2 \\ -1 \end{pmatrix}$$

and finally with round the solution with 4 floating point, we have

$$C_5 = \begin{pmatrix} 0.0 & 13.42 & 6.363 & 2.057 & 0.5047 & 0.1009 \\ 0.8944 & 0.0 & 26.56 & 8.591 & 2.106 & 0.4212 \\ 0.4242 & 0.9486 & 0.0 & 34.93 & 8.559 & 1.712 \\ 0.1372 & 0.3068 & 0.9701 & 0.0 & 35.30 & 7.062 \\ 0.03364 & 0.07519 & 0.2377 & 0.9805 & 0.0 & 25.0 \\ 0.006729 & 0.01504 & 0.04755 & 0.1961 & 1.0 & 0.0 \end{pmatrix}$$

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