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Research Paper

A COLLECTION OF LOCAL SPECTRA PRESERVING MAPS

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Abstract. We collected some results about maps on the algebra of all bounded operators that preserve the local spectrum and local spectral radius at nonzero vectors. Also, we describe maps that preserve operators of local spectral radius zero at points and discuss several problems in this direction. Finally, we collect maps that preserve the local spectral subspace of operators associated with any singleton.

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1. Introduction and Background

Linear preserver problems, in the most general setting, demands the characterization of maps between algebras that leave a certain property, a particular relation, or even a subset invariant. In all cases that have been studied by now, the maps are either supposed to be linear, or proved to be so. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [25], who characterized linear maps that preserve the determinant of matrices. The aforesaid Frobenius' work was generalized by J. Dieudonné [21], who characterized linear maps preserving singular matrices. The goal is to describe the general form of linear maps between two Banach algebras that preserve a certain property, or a certain class of elements, or a certain relation. One of the most famous problems in this direction is Kaplansky's problem [32] asking whether every surjective unital invertibility preserving linear map between two semisimple Banach algebras is a Jordan homomorphism. His question was motivated by two classical results, the result of Marcus and Moyls [34] on linear maps preserving eigenvalues of matrices and the Gleason-Kahane-Zelazko theorem [26, 31] stating that that Every unital invertibility preserving linear functional on a unital complex Banach algebra is necessarily multiplicative. This result was obtained independently by Gleason in [26] and Kahane-Zelazko in [31], and was refined by Zelazko in [37]. In the non-commutative case, the best known results so far are due to Aupetit [2] and Sourour [36]. They showed that the answer to the Kaplansky question is in the affirmative for von Neumann algebras [2] and for bijective unital linear invertibility preserving maps acting on the algebra of all bounded operators on a Banach space [36]. Linear and nonlinear preserver problems cleared the way for several authors to describe maps on matrices or operators that preserve invertibility, spectrum, spectral radius and spectrally bounded; see for instance

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[28, 29, 35] and the references therein.

2. preservers of local spectra

Throughout this paper, X and Y denote infinite-dimensional complex Banach spaces, and $B(X,Y)$ denotes the space of all bounded linear maps from X into Y. As usual, when $X=Y$, we simply write $B(X)$ for $B(X,X)$. Also, let $M_n(\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. The local resolvent set, $\rho_T(x)$, of an operator $T\in B(X)$ at some point $x\in X$ is the set of all $\lambda\in\mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and a X-valued analytic function $f:U\longrightarrow X$ such that $(\mu I-T)f(\mu)=x$ for all $\mu\in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$. The local resolvent set, $\rho_T(x)$, of an operator $T\in B(X)$ at some point $x\in X$ is the set of all $\lambda\in\mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and a X-valued analytic function $f:U\longrightarrow X$ such that $(\mu I-T)f(\mu)=x$ for all $\mu\in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$. The local spectral radius of T at x is given by $r_T(x):=\limsup_{n\longrightarrow\infty}\|T^n(x)\|^{\frac{1}{n}}$, and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. Recall that an operator $T\in B(X)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) if, for every open subset U of $\mathbb{C}$, there exists no nonzero analytic solution, $f:U\longrightarrow X$, of the equation

$$(\mu I-T)f(\mu)=0, \quad \forall\ \mu\in U.$$

Every operator $T\in B(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property. For every subset $F\subseteq\mathbb{C}$ the local spectral subspace $X_T(F)$ is defined by

$$X_T(F)=\{x\in X:\sigma_T(x)\subseteq F\}.$$

Clearly, if $F_1\subseteq F_2$ then $X_T(F_1)\subseteq X_T(F_2)$. The remarkable books by Aiena [1] and by Laursen and Neumann [33] provide an excellent exposition as well as a rich bibliography of the local spectral theory.

In this paper, we survey some results about preservers of local spectra on the algebra of all bounded operators.

The first lemma summarizes some basic properties of the local spectrum.

Lemma 2.1. [1], [33] *Let X be a Banach space and $T\in B(X)$. For every $x,y\in X$ and a scalar $\alpha\in\mathbb{C}$ the following statements hold.*

(i) *If T has SVEP, then $\sigma_T(x)\neq\emptyset$ provided that $x\neq 0$.*
(ii) *$\sigma_T(\alpha x)=\sigma_T(x)$ if $\alpha\neq 0$, and $\sigma_{\alpha T}(x)=\alpha\sigma_T(x)$.*
(iii) *If $Tx=\lambda x$ for some $\lambda\in\mathbb{C}$, then $\sigma_T(x)\subseteq\{\lambda\}$. Further, if $x\neq 0$ and T has SVEP, then $\sigma_T(x)=\{\lambda\}$.*
(v) *If $S\in B(X)$ commutes with T, then $\sigma_T(Sx)\subseteq\sigma_T(x)$.*
(iv) *$\sigma_{T^n}(x)=\{\sigma_T(x)\}^n$ for all $x\in X$ and $n\in\mathbb{N}$.*

Let $\mathcal{A}=B(X)$ (resp. $\mathcal{A}=M_n(\mathbb{C})$) and $x\in X$ (resp. $x\in\mathbb{C}^n$) be a fixed nonzero vector. A map $\varphi:\mathcal{A}\to\mathcal{A}$ is said to preserve

(1) the local spectrum at x if

$$\sigma_{\varphi(T)}(x) = \sigma_T(x) \qquad (T \in B(X).$$

(2) the local spectral radius at x if

$$r_{\varphi(T)}(x) = r_T(x) \qquad (T \in B(X).$$

The study of linear and nonlinear local spectra preserver problems attracted the attention of a number of authors. The problem of characterizing additive maps on $B(X)$ preserving local spectrum was initiated by Bourhim and Ransford in [15]. They showed that the only additive map φ from $B(X)$ into itself for which

$$\sigma_{\varphi(T)}(x) = \sigma_T(x), \qquad (T \in B(X),\ x \in X),$$

is the identity.

Theorem 2.2. [15] *Let $\varphi : B(X) \to B(X)$ be an additive map such that*

$$\sigma_{\varphi(T)}(x) = \sigma_T(x), \qquad (T \in B(X),\ x \in X).$$

Then $\varphi(T) = T$ for all $T \in B(X)$.

Note that in the above theorem, unlike many results in this general area, the map φ do not require to be surjective, or even linear, just additivity suffices.

Gonzalez and Mbekhta [27] characterized linear maps on $M_n(\mathbb{C})$ that preserving the local spectrum at only a fixed nonzero vector $x_0 \in \mathbb{C}^n$. They proved that a linear map φ preserves the local spectrum at x_0 if and only if there exists an invertible matrix A in $M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\varphi(T) = ATA^{-1}$ for all $T \in M_n(C)$.

Theorem 2.3. [27] *Let $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map and $x_0 \in \mathbb{C}^n$ be a fixed nonzero vector. Then φ preserves the local spectrum at x_0 if and only if there exists an invertible $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\varphi(T) = ATA^{-1}$ for every $T \in M_n(\mathbb{C})$.*

In the case when the vector x_0 is not fixed, Costara is given the following theorem.

Theorem 2.4. [18] *Let $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Then the following statement are equivalent;*
(i) For each $T \in M_n(\mathbb{C})$ there exists a nonzero vector $x_T \in \mathbb{C}^n$ such that

$$\sigma_{\varphi(T)}(x_T) = \sigma_T(x_T).$$

(ii) For each $T \in M_n(\mathbb{C})$ there exists a nonzero vector $x_T \in \mathbb{C}^n$ such that

$$\sigma_{\varphi(T)}(x_T) \cap \sigma_T(x_T) \neq \varnothing.$$

(iii) There exists an invertible matrix $A \in M_n(\mathbb{C})$ such that either

$$\varphi(T) = ATA^{-1}, \qquad (T \in M_n(\mathbb{C})),$$

or

$$\varphi(T) = AT^tA^{-1}, \qquad (T \in M_n(\mathbb{C})),$$

where T^t is the transpose of T.

Bourhim and Miller [14] described linear maps on $M_n(C)$ preserving the local spectral radius at a fixed nonzero vector in $\mathbb{C}^n$.

Theorem 2.5. [14] *Let x_0 be a nonzero fixed vector of $\mathbb{C}^n$. A linear map φ from $M_n(\mathbb{C})$ into itself preserves the local spectral radius at $x_0 \in \mathbb{C}^n$, i.e.,*

$$r_T(x_0) = r_{\varphi(T)}(x_0), \quad (T \in M_n(\mathbb{C})),$$

if and only if there exist a scalar α of modulus 1 and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\varphi(T) = \alpha ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

The corresponding result for the case when x_0 is not fixed is given by Costara in the next theorem.

Theorem 2.6. [18] *Let φ be a linear map from $M_n(\mathbb{C})$ into itself. Then for each $T \in M_n(\mathbb{C})$ there exists a nonzero vector $x_T \in \mathbb{C}^n$ such that $r_{\varphi(T)}(x_T) = r_T(x_T)$ if and only if there exist an invertible matrix $A \in M_n(\mathbb{C})$ and an unimodular α such that either*

$$\varphi(T) = \alpha ATA^{-1}, \qquad (T \in M_n(\mathbb{C})),$$

or

$$\varphi(T) = \alpha AT^tA^{-1}, \qquad (T \in M_n(\mathbb{C})),$$

where T^t is the transpose of T.

Bracic and Muller [16] extended the both Theorem 2.3 and Theorem 2.5 to infinite dimensional Banach space by characterizing surjective continuous linear maps φ on $B(X)$ that preserve the local spectrum and the local spectral radius at a fixed nonzero vector in X.

Theorem 2.7. [16] *Let $x_0 \in X$ be a fixed nonzero vector and let $\varphi : B(X) \to B(X)$ be a surjective continuous linear mapping. Then $\sigma_{\varphi(T)}(x_0) = \sigma_T(x_0)$ for all $T \in B(X)$ if and only if there exists an invertible operator $A \in B(X)$ such that $Ax_0 = x_0$ and $\varphi(T) = ATA^{-1}$ for all $T \in B(X)$.*

Theorem 2.8. [16] *Let $x_0 \in X$ be a fixed nonzero vector. Let $\varphi : B(X) \to B(X)$ be a surjective continuous linear mapping. Then $r_{\varphi(T)}(x_0) = r_T(x_0)$ for all $T \in B(X)$ if and only if there exist an invertible operator $A \in B(X)$ and $c \in \mathbb{C}$ of modulus 1 such that $Ax_0 = x_0$ and $\varphi(T) = cATA^{-1}$ for all $T \in B(X)$.*

Bracic and Muller [16] also asked whether their results remain true without continuity assumption on φ.

problem.[16] Is it possible to omit the assumption of continuity of φ in Theorem 2.7 and Theorem 2.8?

Costara [20] gave an affirmative answer to this problem, in the case when φ preserves the local spectral radius at a fixed nonzero vector. He showed that a linear mapping on $B(X)$ that decreases the local spectral radius at a fixed nonzero vector in X is automatically continuous.

Bendaoud [6] characterized surjective maps φ on $M_n(\mathbb{C})$ which satisfy $\sigma_{\varphi(T)+\varphi(S)}(x_0) \subseteq \sigma_{T+S}(x_0)$ for a fixed nonzero vector x_0 in $\mathbb{C}^n$ and all matrices T and S. He arrived at the same conclusion by supposing that $\sigma_{T+S}(x_0) \subseteq \sigma_{\varphi(T)+\varphi(S)}(x_0)$ for a fixed nonzero vector x_0 in $\mathbb{C}^n$ and all matrices T and S, without the surjectivity assumption on φ.

Theorem 2.9. [6] *Let x_0 be a fixed nonzero vector in $\mathbb{C}^n$. A map φ from $M_n(\mathbb{C})$ into itself satisfies*

$$\sigma_{T+S}(x_0) \subseteq \sigma_{\varphi(T)+\varphi(S)}(x_0), \qquad (T, S \in M_n(\mathbb{C})),$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\varphi(T) = ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

The following result shows that the same conclusion holds when the reverse set inclusion in

$$\sigma_{T+S}(x_0) \subseteq \sigma_{\varphi(T)+\varphi(S)}(x_0), \qquad (T, S \in M_n(\mathbb{C})),$$

occurs but at the price of the additional assumption that φ is surjective.

Theorem 2.10. [6] *Let x_0 be a fixed nonzero vector in $\mathbb{C}^n$. A surjective map φ from $M_n(\mathbb{C})$ into itself satisfies*

$$\sigma_{\varphi(T)+\varphi(S)}(x_0) \subseteq \sigma_{T+S}(x_0), \qquad (T, S \in M_n(\mathbb{C}))$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\varphi(T) = ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

However, Bendaoud characterized continuous maps from $M_n(\mathbb{C})$ onto itself that preserve the local spectral radius of the sum of matrices.

Theorem 2.11. [6] *Let x_0 be a fixed nonzero vector in $\mathbb{C}^n$. A continuous map φ from $M_n(\mathbb{C})$ onto itself satisfies*

$$r_{\varphi(T)+\varphi(S)}(x_0) = r_{T+S}(x_0), \qquad (T, S \in M_n(\mathbb{C}))$$

if and only if there exist a scalar c of modulus one and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\varphi(T) = cATA^{-1}$ for all $T \in M_n(\mathbb{C})$, or $A\overline{x_0} = x_0$ and $\varphi(T) = cA\overline{T}A^{-1}$ for all $T \in M_n(\mathbb{C})$. Here $\overline{T}$ (resp. $\overline{x}$) denotes the matrix (resp. the vector) obtained from T (resp. x) by entrywise complex conjugation.

Bendaoud et al. [7] in the following theorem characterized nonlinear maps on $M_n(\mathbb{C})$ that preserve the local spectrum of the product of matrices at a fixed nonzero vector.

Theorem 2.12. [7] *Let x_0 be a fixed nonzero vector in $\mathbb{C}^n$. A map φ from $M_n(\mathbb{C})$ into itself satisfies*

$$\sigma_{\varphi(T)\varphi(S)}(x_0) = \sigma_{TS}(x_0), \qquad (T, S \in M_n(\mathbb{C})) \qquad (2.1)$$

if and only if there exist a scalar $\varepsilon = \pm 1$ and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$, and $\varphi(T) = \varepsilon ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

The following theorem shows that unlike the finite dimensional case when φ satisfies (2.1) at a fixed nonzero vector x_0, the only map φ on $B(X)$ which satisfies the equality (2.1) at all vectors $x \in X$ is the identity or the negative of the identity.

Theorem 2.13. [7] *A map φ from $B(X)$ into itself satisfies*

$$\sigma_{\varphi(T)\varphi(S)}(x) = \sigma_{TS}(x), \qquad (T, S \in B(X), x \in X)$$

if and only if there exists a scalar $\varepsilon = \pm 1$ such that $\varphi(T) = \varepsilon T$ for all $T \in B(X)$.

In the case of two Banach spaces X, Y are different, Bourhim and Mashreghi [12] characterized surjective maps on $B(X)$ that preserve the local spectrum of product operators at fixed nonzero vector.

Theorem 2.14. [12] *Let $x_0 \in X$ and $y_0 \in Y$ be two nonzero vectors. A map φ from $B(X)$ onto $B(Y)$ satisfies*

$$\sigma_{\varphi(T)\varphi(S)}(y_0) = \sigma_{TS}(x_0), \qquad (T, S \in B(X)),$$

if and only if there exists a bijective bounded linear operator $A \in B(X, Y)$ such that $Ax_0 = y_0$ and either $\varphi(T) = ATA^{-1}$ for all $T \in B(X)$ or $\varphi(T) = -ATA^{-1}$ for all $T \in B(X)$.

Bendaoud [5] in the following theorem characterized nonlinear maps on $M_n(\mathbb{C})$ that preserve the local spectrum of triple product matrices at a fixed nonzero vector.

Theorem 2.15. [5] *Let x_0 be a fixed nonzero vector in $\mathbb{C}^n$. A map φ from $M_n(\mathbb{C})$ into itself satisfies*

$$\sigma_{\varphi(T)\varphi(S)\varphi(T)}(x_0) = \sigma_{TST}(x_0), \qquad (T, S \in M_n(\mathbb{C}))$$

if and only if there are a third root of unity ε and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$, and $\varphi(T) = \varepsilon ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

Bourhim and Mashreghi [13] characterized surjective maps on the algebra of all bounded linear operators on a complex Banach space that preserve the local spectrum of triple product operators at fixed nonzero vector.

Theorem 2.16. [13] *Let $x_0 \in X$ and $y_0 \in Y$ be two nonzero vectors. A map φ from $B(X)$ onto $B(Y)$ satisfies*

$$\sigma_{\varphi(T)\varphi(S)\varphi(T)}(y_0) = \sigma_{TST}(x_0), \qquad (T, S \in B(X)),$$

if and only if there exists a bijective bounded linear mapping A from X into Y such that $Ax_0 = y_0$ and either $\varphi(T) = ATA^{-1}$ for all $T \in B(X)$ or $\varphi(T) = -ATA^{-1}$ for all $T \in B(X)$.

Bourhim and Mabrouk [9] described the form of all maps preserving the local spectrum of Jordan product of operators on a complex Banach space.

Theorem 2.17. [9] *Let $x_0 \in X \setminus \{0\}$ and $y_0 \in Y \setminus \{0\}$. A map φ from $B(X)$ onto $B(Y)$ satisfies*

$$\sigma_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(y_0) = \sigma_{TS+ST}(x_0) \qquad (T, S \in B(X))$$

if and only if there exists a bijective mapping $A \in B(X, Y)$ such that $Ax_0 = y_0$ and either $\varphi(T) = ATA^{-1}$ for all $T \in B(X)$ or $\varphi(T) = -ATA^{-1}$ for all $T \in B(X)$.

Note that if X and Y are isomorphic Banach spaces, then the statements of the above result can be reduced to the case when $X = Y$ and $x_0 = y_0$. Bourhim and Mabrouk [10] showed that Theorem 2.17 remains valid without the surjectivity condition on φ but when $X = Y = \mathbb{C}^n$.

Theorem 2.18. [10] *Let x_0 be a nonzero fixed vector of $\mathbb{C}^n$. A linear map φ from $M_n(\mathbb{C})$ into itself satisfies*

$$\sigma_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(x_0) = \sigma_{TS+ST}(x_0) \qquad (T, S \in M_n(\mathbb{C}))$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\varphi(T) = \pm ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

3. maps preserving operators of local spectral radius zero

In [17], C. Costara described surjective linear maps on $B(X)$ which preserve operators of local spectral radius zero at points of X. He showed that if $\varphi : B(X) \to B(X)$ is a linear and surjective map such that for every $x \in X$ and $T \in B(X)$

$$r_T(x) = 0 \Leftrightarrow r_{\varphi(T)}(x) = 0,$$

then there exists a nonzero scalar $c \in \mathbb{C}$ such that $\varphi(T) = cT$ for all $T \in B(X)$.

Theorem 3.1. [17] *Let $\varphi : B(X) \to B(X)$ be linear and surjective map such that for every $x \in X$ and $T \in B(X)$,*

$$r_{\varphi(T)}(x) = 0 \Leftrightarrow r_T(x) = 0,$$

then there exists a nonzero complex number $c \in \mathbb{C}$ such that $\varphi(T) = cT$ for all $T \in B(X)$.

In the case of two different Banach spaces X and Y, it was proved by Bourhim and Ransford at [15, Theorem 1.5] that if B is a bounded linear operator from Y into X and $\varphi : B(X) \to B(Y)$ is linear and surjective such that $\sigma_{\varphi(T)}(y) = \sigma_T(By)$ for all $T \in B(X)$ and for all $y \in Y$, then B is invertible and $\varphi(T) = B^{-1}TB$ for all $T \in B(X)$. The following theorem is the corresponding result in the case of maps preserving operators of local spectral radius zero.

Theorem 3.2. [17] *Let X and Y be complex Banach spaces and $\varphi : B(X) \to B(Y)$ linear and bijective for which there exists $B \in B(Y, X)$ such that for every $y \in Y$*

$$r_T(By) = 0 \Leftrightarrow r_{\varphi(T)}(y) = 0.$$

Then B is invertible and there exists a nonzero complex number c such that $\varphi(T) = cB^{-1}TB$ for all $T \in B(X)$.

This result has been extended by Bourhim and Mashreghi in [11] where it is shown that if φ is a surjective (not necessarily linear) map on $B(X)$ that satisfies

$$r_{T-S}(x) = 0 \Leftrightarrow r_{\varphi(T)-\varphi(S)}(x) = 0$$

for every $x \in X$ and $T \in B(X)$, then there are a nonzero scalar $c \in \mathbb{C}$ and an operator $A \in B(X)$ such that $\varphi(T) = cT + A$ for all $T \in B(X)$.

Theorem 3.3. [11] *Let φ be a surjective map on $B(X)$ which satisfies*

$$r_{T-S}(x) = 0 \Leftrightarrow r_{\varphi(T)-\varphi(S)}(x) = 0, \qquad (x \in X,\ T \in B(X)),$$

then there are a nonzero scalar $c \in \mathbb{C}$ and an operator $A \in B(X)$ such that $\varphi(T) = cT + A$ for all $T \in B(X)$.

Elhodaibi and Jaatit [24] established a similar result to the one given by [11, Theorem 4.1]. The only difference is that the map φ is not assumed surjective.

Theorem 3.4. [24] *Let $\varphi : B(X) \to B(X)$ be a map. Then the following assertions are equivalent.*
(i) $r_{\varphi(T)-\varphi(S)}(x) = 0$ if and only if $r_{T-S}(x) = 0$ for all $x \in X$ and $T \in B(X)$.
(ii) There exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\varphi(T) = \mu T + \varphi(0)$ for all $T \in B(X)$.

Bourhim and Costara [8] considered the more general problem of describing linear maps φ on $M_n(\mathbb{C})$ preserving operators of local spectral radius zero at a nonzero fixed vector $x_0 \in \mathbb{C}^n$. Their aim was to characterize linear maps φ on $M_n(\mathbb{C})$ such that

$$r_T(x_0) = 0 \Leftrightarrow r_{\varphi(T)}(x_0) = 0,\ (T \in M_n(\mathbb{C})).$$

Since this problem is trivial for the case when $n = 1$, they supposed that $n \geq 2$. For the special case when $n = 2$, they obtained the following result.

Theorem 3.5. [8] *Let x_0 be a nonzero fixed vector of $\mathbb{C}^2$. A linear map φ on $M_2(\mathbb{C})$ into itself satisfies*

$$r_T(x_0) = 0 \Leftrightarrow r_{\varphi(T)}(x_0) = 0, \qquad (T \in M_2(\mathbb{C})),$$

if and only if there exists a nonzero scalar α, an invertible matrix $U \in M_2(\mathbb{C})$ for which $Ux_0 = x_0$, and a matrix $Q \in M_2(\mathbb{C})$ satisfying $Qx_0 = x_0$ and $tr(Q) \neq -1$ such that $\varphi(T) = \alpha(UTU^{-1} + tr(T)Q)$ for all $T \in M_2(\mathbb{C})$.

In [14], Bourhim and Miller showed that a linear map φ on $M_n(\mathbb{C})$ preserves the local spectral radius at a nonzero vector $x_0 \in \mathbb{C}^n$ if and only if φ is an automorphism (up to a multiple factor of modulus one) and x_0 is an eigenvector of the intertwining matrix; see also [19] for nonlinear local spectral radius preservers. For the special case when $n = 2$, the above theorem shows that there are nontrivial linear maps on $M_2(\mathbb{C})$ that do not preserve the local spectral radius at x_0, even after a re-scaling that preserves matrices of local spectral radius zero at x_0. However, Bourhim and Costara [8] in the next theorem showed that if n is an integer greater than 2 and φ is a linear map on $M_n(\mathbb{C})$ satisfying

$$r_T(x_0) = 0 \Leftrightarrow r_{\varphi(T)}(x_0) = 0,\ (T \in M_n(\mathbb{C})),$$

then φ is, up to a nonzero multiple factor, a local spectral radius preserver at x_0.

Theorem 3.6. [8] *Let $n \geq 3$ be a natural number and fix a nonzero vector $x_0 \in \mathbb{C}^n$. A linear map $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies*

$$r_T(x_0) = 0 \Leftrightarrow r_{\varphi(T)}(x_0) = 0, \qquad (T \in M_n(\mathbb{C})),$$

if and only if there exists a nonzero scalar α and an invertible matrix $U \in M_n(\mathbb{C})$ such that $Ux_0 = x_0$ and $\varphi(T) = \alpha UTU^{-1}$ for all $T \in M_n(\mathbb{C})$.

Bourhim and Costara in [8] maked some remarks and comments on linear and nonlinear preservers of local spectral radius and discussed some further challenging problems, which are suggested by the main results of paper [8].

In the sequel, let $x_0 \in X$ and $y_0 \in Y$ be two nonzero vectors.

problem 1. [8] Which maps φ from $B(X)$ onto $B(Y)$ satisfy

$$r_T(x_0) = 0 \Leftrightarrow r_{\varphi(T)}(y_0) = 0, \qquad (T \in B(X))?$$

When X and Y are infinite-dimensional Banach spaces, Bourhim and Costara conjectured that a linear map φ from $B(X)$ onto $B(Y)$ satisfies

$$r_T(x_0) = 0 \Leftrightarrow r_{\varphi(T)}(y_0) = 0, \qquad (T \in B(X))$$

if and only if there are a nonzero scalar $\alpha \in \mathbb{C}$ and a bijective bounded linear mapping A from X into Y such that $Ax_0 = y_0$ and $\varphi(T) = \alpha ATA^{-1}$ for all $T \in B(X)$. Note that the injectivity of any linear map φ satisfying

$$r_T(x_0) = 0 \Leftrightarrow r_{\varphi(T)}(y_0) = 0, \qquad (T \in B(X))$$

follows from [11, Theorem 3.1]. But, unlike for the infinite-dimensional case, the surjectivity assumption of such a map φ is necessary.

As far as for the nonlinear local spectral radius preservers, we first state the following problem.

problem 2. [8] Which maps φ from $B(X)$ onto $B(Y)$ satisfy

$$r_{T\pm S}(x_0) = 0 \Leftrightarrow r_{\varphi(T)\pm\varphi(S)}(y_0) = 0, \qquad (S, T \in B(X)) \qquad (3.1)?$$

Obviously, (3.1) holds for any map φ from $B(X)$ onto $B(Y)$ satisfying

$$r_{T\pm S}(x_0) = r_{\varphi(T)\pm\varphi(S)}(x_0), \qquad (S, T \in B(X)) \qquad (3.2).$$

In the finite-dimensional case, the description of such maps is known as shown by Costara in [19]. He proved that a surjective map φ on $M_n(\mathbb{C})$ satisfies (3.2) with $\varphi(0) = 0$ if and only if φ is an automorphism multiplied by a scalar of modulus one and the intertwining matrix sends x_0 to y_0. However, when X and Y are infinite-dimensional spaces, the characterization of maps satisfying (3.2) is unknown and remains an open problem as well.

In [12], Bourhim and Mashreghi showed that a map φ from $B(X)$ onto $B(Y)$ satisfies

$$\sigma_{TS}(x_0) = \sigma_{\varphi(T)\varphi(S)}(y_0) \qquad (S, T \in B(X)),$$

if and only if there exists a bijective bounded linear mapping A from X into Y such that $Ax_0 = y_0$ and either $\varphi(T) = ATA^{-1}$ for all $T \in B(X)$ or $\varphi(T) = -ATA^{-1}$ for all $T \in B(X)$. Naturally, this result suggests the problem of describing all maps φ from $B(X)$ onto $B(Y)$ for which

$$r_{TS}(x_0) = r_{\varphi(T)\varphi(S)}(y_0), \qquad (S, T \in B(X)).$$

Even more, one may ask the following more general question of describing all maps on $B(X)$ preserving the product of operators of local spectral radius zero at some fixed nonzero vector of X.

problem 3. [8] Which maps φ from $B(X)$ onto $B(Y)$ satisfy

$$r_{TS}(x_0) = 0 \Leftrightarrow r_{\varphi(T)\varphi(S)}(y_0) = 0, \qquad (S, T \in B(X))?$$

Similar questions can be asked when replacing the usual product by triple or Jordan product.

problem 4. [8] Describe all maps φ from $B(X)$ onto $B(Y)$ satisfying either

$$r_{STS}(x_0) = 0 \Leftrightarrow r_{\varphi(S)\varphi(T)\varphi(S)}(y_0) = 0, \qquad (S, T \in B(X))$$

or

$$r_{TS+TS}(x_0) = 0 \Leftrightarrow r_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(y_0) = 0, \qquad (S, T \in B(X))$$

4. local spectral subspace preserver

Motivated by the result from the theory of linear preservers proved by Jafarian and Sourour [30], Dolinar et al. [22], characterised the form of maps preserving the lattice of sum of operators, they showed that maps (not necessarily linear) $\varphi : B(X) \to B(X)$ satisfy $Lat(\varphi(A) + \varphi(B)) = Lat(A + B)$ for all $A, B \in B(X)$, if and only if there are a non zero scalar α and a map $\phi : B(X) \to \mathbb{F}$ such that $\varphi(A) = \alpha A + \phi(A)I$ for all $A \in B(X)$, where $\mathbb{F}$ is the complex field $\mathbb{C}$ or the real field $\mathbb{R}$ and $Lat(A)$ is denoted the lattice of A, that is, the set of all invariant subspaces of A. They proved also, that a not necessarily linear maps $\varphi : B(X) \to B(X)$ satisfies $Lat(\varphi(A)\varphi(B)) = Lat(AB)$ (resp. $Lat(\varphi(A)\varphi(B)\varphi(A)) = Lat(ABA)$, resp. $Lat(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)) = Lat(AB + BA)$) for all $A, B \in B(X)$, if and only if there is a map $\phi : B(X) \to \mathbb{F}$ such that $\varphi(A) \neq 0$ if $A \neq 0$ and $\varphi(A) = \phi(A)A$ for all $A \in B(X)$. Recall that $X_T(\Omega)$, the local spectral subspace of T associated with a subset Ω of $\mathbb{C}$, is an element of $Lat(T)$, so one can replace the lattice preserving property by the local spectral subspace preserving property.

To the nonlinear maps preserving the local spectral subspace, Elhodaibi and Jaatit [23] showed that the only additive map which preserving the local spectral subspace associated with any singletons is the identity.

Theorem 4.1. [23] *Let $\varphi : B(X) \longrightarrow B(X)$ be an additive map such that*

$$X_{\varphi(T)}(\{\lambda\}) = X_T(\{\lambda\}), \quad (T \in B(X),\ \lambda \in \mathbb{C}).$$

Then $\varphi(T) = T$ for all $T \in B(X)$.

In [4], Benbouziane et al. characterized the form of surjective weakly continuous maps φ from $B(X)$ into $B(X)$ which satisfy

$$X_{\varphi(T)-\varphi(S)}(\{\lambda\}) = X_{T-S}(\{\lambda\}), \quad \forall\, T, S \in B(X), \quad (T, S \in B(X),\ \lambda \in \mathbb{C}).$$

Theorem 4.2. [3] *Let X be a complex Banach space. Suppose $\varphi : B(X) \longrightarrow B(X)$ is a surjective weakly continuous map which satisfies the following condition,*

$$X_{\varphi(T)-\varphi(S)}(\{\lambda\}) = X_{T-S}(\{\lambda\}), \quad (T, S \in B(X),\ \lambda \in \mathbb{C}).$$

Then there is a nonzero scalar $c \in \mathbb{C}$ such that $\varphi(T) = cT + \varphi(0)$ for all $T \in B(X)$.

As a continuation in this direction, Benbouziane et al. In [3], determined the forms of all maps preserving the local spectral subspace of sum, product and triple product of operators associated with non-fixed singletons.

Theorem 4.3. [3] *A surjective map $\varphi : B(X) \longrightarrow B(X)$ satisfies*

$$X_{\varphi(T)+\varphi(S)}(\{\lambda\}) = X_{T+S}(\{\lambda\}), \quad (T, S \in B(X),\ \lambda \in \mathbb{C})$$

if and only if $\varphi(T) = T$ for all $T \in B(X)$.

Furthermore, they investigated the product case as well as the triple product case.

Theorem 4.4. [3] *A surjective map $\varphi : B(X) \longrightarrow B(X)$ satisfies*

$$X_{\varphi(T)\varphi(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}), \quad (T, S \in B(X),\ \lambda \in \mathbb{C}),$$

if and only if $\varphi(T) = \epsilon T$ for all $T \in B(X)$, where $\epsilon = \pm 1$.

We end this paper with the following theorem in which Benbouziane et al. characterized maps on $B(X)$ preserving the local spectral subspace of the triple product of operator associated with a singleton.

Theorem 4.5. [3] *A surjective map φ from $B(X)$ into itself satisfies*

$$X_{\varphi(T)\varphi(S)\varphi(T)}(\{\lambda\}) = X_{TST}(\{\lambda\}), \quad (T, S \in B(X),\ \lambda \in \mathbb{C})$$

if and only if there exists a scalar $\alpha \in \mathbb{C}$ such that $\alpha^3 = 1$ and $\varphi(T) = \alpha T$ for all $T \in B(X)$.

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