**Mathematical Analysis & Convex Optimization**

Vol. 3 (2022), No. 1, 43–48
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.3.1.5

**Research Paper**

# AUTOMATIC CONTINUITY OF SURJECTIVE ALMOST DERIVATIONS ON FRECHET $Q$-ALGEBRAS

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Abstract. In 1971 R. L. Carpenter proved that every derivation $T$ on a semisimple commutative Frechet algebra $\Lambda$ with identity is continuous. By relaxing the commutativity assumption on $\Lambda$ and adding the surjectivity assumption on $T$, we derive a corresponding continuity result, for a new concept of almost derivations on Frechet algebras in this article. Also, it is further proved that every surjective almost derivation $T$ on non commutative semisimple Frechet $Q$-algebras $\Lambda$ with an additional condition on $\Lambda$, is continuous. Moreover, an example is provided to illustrate our main result.

**MSC(2020):** Primary 46H40; Secondary 46H05.
**keywords:** Almost derivation, Automatic continuity, Frechet $Q$-algebras.

## 1. Introduction

We provide a brief outline of definitions and known outcomes in this section. For more details, one may refer to [2, 7]. All vector spaces are considered over the complex field, and we assume that all algebras are unital. An Banach algebra $\Lambda$ is a complete normed algebra, where a normed algebra $\Lambda$ is an algebra with a norm $||.||$, which also satisfies $||p.q|| \leq ||p||.||q||$, for all $p, q \in \Lambda$. An algebra with a Hausdorff topology is called a topological algebra, if all algebraic operations are jointly continuous. The Jacobson radical $rad(\Lambda)$ of an algebra $\Lambda$ is the intersection of all maximal right(or left) ideals. An algebra is said to be semisimple, if $rad(\Lambda) = \{0\}$.

**Definition 1.1.** [2] The spectrum $\sigma_\Lambda(p)$ of an element $p$ of an algebra $\Lambda$ is the set of all complex numbers $\gamma$ such that $\gamma.1-p$ is not invertible in $\Lambda$. The spectral radius $r_\Lambda(p)$ of an element $p \in \Lambda$ is defined by $r_\Lambda(p) = \sup\{|\gamma| : \gamma \in \sigma_\Lambda(p)\}$.

If $(\Lambda, ||.||)$ is a Banach algebra, then $r_\Lambda(p) = \lim_{n\to\infty} ||p^n||^{\frac{1}{n}}$. Also, for any Banach algebra $\Lambda$, we have $rad(\Lambda) = \{p \in \Lambda : r_\Lambda(pq) = 0$, for every $q \in \Lambda\}$. See ([14], Lemma 1).

*Date*: Received: December 10, 2021, Accepted: June 19, 2022.
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**Definition 1.2.** Let $\Lambda$ and $\Gamma$ be two complete metrizable topological vector spaces. Let $T : \Lambda \to \Gamma$ be a linear map. Then the separating space of $T$ is defined as the set
$S(T) = \{q \in \Gamma : \text{ there exists } (p_n)_{n=1}^{\infty} \text{ in } \Lambda \text{ such that } p_n \to 0 \text{ and } Tp_n \to q\}$.

Also, $S(T)$ is a closed linear subspace of $\Gamma$. Moreover, $S(T) = \{0\}$ if and only if $T$ is continuous, because of closed graph theorem. For a proof, see ([2], 5.1.2).

**Lemma 1.3.** *([14], Lemma 2) Let $\Lambda$ be a Banach algebra, $P(z)$ a polynomial with coefficients in $\Lambda$ and $R > 0$. Then*
$$r_{\Lambda}^2(P(1)) \leq \sup_{|z|=R} r_{\Lambda}(P(z)). \sup_{|z|=\frac{1}{R}} r_{\Lambda}(P(z)).$$

A complete metrizable topological algebra is called an $F$-algebra. A topological algebra $\Lambda$ is said to be a LMC algebra, if its topology be induced by a separating family of submultiplicative seminorms. A Frechet algebra is a LMC algebra which is also an $F$-algebra. A $Q$-algebra is a topological algebra in which the set of all invertible elements is open. A metrizable LMC algebra is written in the form $(\Lambda, (p_n)_{n=1}^{\infty})$, where $(p_n)_{n=1}^{\infty}$ is a separating sequence and each $p_n$ is a submultiplicative seminorm (i.e. $p_n(u.v) \leq p_n(u).p_n(v)$, for all $u, v \in \Lambda$) satisfying $p_n(u) \leq p_{n+1}(u)$, for all $n \in \mathbb{N}$, and $u \in \Lambda$, in which the topology on $\Lambda$ is induced by the seminorms $p_n, n \in \mathbb{N}$. Also, a sequence $(y_k)$ in the Frechet algebra $(\Lambda, (p_n))$ converges to $y \in \Lambda$ if and only if $p_n(y_k - y) \to 0$, for every $n \in \mathbb{N}$, as $k \to \infty$. In a Frechet $Q$-algebra, spectral radius of every element is a finite number, see [7]. Every Banach algebra is a Frechet $Q$-algebra.

**Remark 1.4.** Let $(\Lambda, (p_n))$ be a Frechet algebra, and $\Lambda_n$ be the completion of the quotient algebra $\Lambda/ker\ p_n$, with respect to the norm ${p_n}'(y + ker\ p_k) = p_n(y), y \in \Lambda$, then $\Lambda_n$ is a Banach algebra.

**Definition 1.5.** [6] Let $\Lambda$ be an algebra. A linear map $T : \Lambda \to \Lambda$ is called derivation, if $T(\mu.\eta) = \mu.T(\eta) + T(\mu).\eta$, for all $\mu, \eta \in \Lambda$.

Recently T.G. Honary et al introduced the concept of almost multiplicative maps between Frechet algebras in [4]. Next, we introduce almost derivations on Frechet algebras.

**Definition 1.6.** Let $(\Lambda, (p_n))$ be a Frechet algebra. A linear map $T : \Lambda \to \Lambda$ is called almost derivation, if there are $\epsilon_n \geq 0$ such that $p_n(T(\mu.\eta) - \mu.T(\eta) - T(\mu).\eta) \leq \epsilon_n p_n(\mu)\ p_n(\eta)$; for all $n \in \mathbb{N}$, and $\mu, \eta \in \Lambda$.

**Remark 1.7.** If $\epsilon_n = 0$, for every $n$, then almost derivations on $\Lambda$ turn out to be derivations on $\Lambda$, because $(p_n)$ is a separating sequence of seminorms on $\Lambda$. Also, every derivation is an almost derivation, for every $\epsilon_n \geq 0$.

A conjecture of Kaplansky[6] can be stated in the following question form. Is every derivation on semisimple Banach algebra continuous?. Kaplansky conjecture was proved by Johnson and Sinclair[5] in 1968. In 1971, R.

L. Carpenter[1] proved that every derivation on a semisimple commutative Frechet algebra with identity is continuous. There are some recent articles [9, 10, 11, 12, 13] for automatic continuity of derivations in the theory of topological algebras.

In this article, we prove that every surjective almost derivation $T$ on a semisimple Frechet $Q$-algebra $(\Lambda, (p_n))$, with an additional condition on $(\Lambda, (p_n))$, is continuous.

## 2. main results

**Theorem 2.1.** *Let $(\Lambda, (p_n))$ be a semisimple Frechet algebra(not necessarily commutative), and $\Lambda_k$ be the completion of $\Lambda/ker\ p_k$, with respect to the norm ${p_k}'(y + ker\ p_k) = p_k(y), y \in \Lambda$. If $T : \Lambda \to \Lambda$ is a surjective almost derivation such that $r_{\Lambda_k}(Tx + ker\ p_k) \leq p_k(x)$, for all $k \in \mathbb{N}$ and $x \in \Lambda$, then $T$ is continuous.*

*Proof.* Since closed graph theorem, for proving continuity of $T$, we have to prove that $b = 0$ for every arbitrary sequence $(\tau_n)_{n=1}^{\infty}$ in $\Lambda$ such that $\tau_n \to 0$, and such that $T(\tau_n) \to b$. Let us begin with such a sequence $(\tau_n)_{n=1}^{\infty}$ and $b$.

Since $T$ is onto, there exists $a \in \Lambda$ such that $Ta = b$. We define $P_n(z) = zT\tau_n + T(a - \tau_n)$. Since for each $y \in \Lambda$, $r_{\Lambda_k}(y + ker\ p_k) \leq {p_k}'(y + ker\ p_k) = p_k(y)$, we have

$$\begin{aligned} r_{\Lambda_k}(P_n(z) + ker\ p_k) &\leq p_k(P_n(z)) \\ &\leq |z|p_k(T\tau_n) + p_k(b - T\tau_n). \end{aligned}$$

By hypothesis, we also have

$$\begin{aligned} r_{\Lambda_k}(P_n(z) + ker\ p_k) &= r_{\Lambda_k}(T(z\tau_n + a - \tau_n) + ker\ p_k) \\ &\leq p_k(z\tau_n + a - \tau_n) \\ &\leq |z|p_k(\tau_n) + p_k(a - \tau_n). \end{aligned}$$

By Lemma 1.3, we have

$$\begin{aligned} r^2_{\Lambda_k}(b + ker\ p_k) &= r^2_{\Lambda_k}(P_n(1) + ker\ p_k) \\ &\leq \sup_{|z|=R} r_{\Lambda_k}(P_n(z) + ker\ p_k). \sup_{|z|=\frac{1}{R}} r_{\Lambda_k}(P_n(z) + ker\ p_k) \\ &\leq (Rp_k(\tau_n) + p_k(a - \tau_n))(\frac{1}{R}p_k(T\tau_n) + p_k(b - T\tau_n)), \end{aligned}$$

for every fixed $R > 0$. We fix $k$ and take $n \to \infty$ to obtain

$$r^2_{\Lambda_k}(b + ker\ p_k) \leq p_k(a).\frac{1}{R}p_k(b).$$

Now, let $R \to \infty$ to get $r_{\Lambda_k}(b + ker\ p_k) = 0$, for each $k$ and therefore $r_\Lambda(b) = 0$, because $r_\Lambda(b) = \sup_{k\in\mathbb{N}} r_{\Lambda_k}(b + ker\ p_k)$, see, for example, ([8], Corollary 5.13).

Let $c \in \Lambda$. Since $(\tau_n)_{n=1}^{\infty}$ is a sequence in $\Lambda$ such that $\tau_n \to 0$, we have $p_k(c.\tau_n) \leq p_k(c).p_k(\tau_n) \to 0$, for all $k \in \mathbb{N}$, as $n \to \infty$. Let $w = T(c)$. Since $T$ is an almost derivation, we have

$$\begin{aligned} p_k(T(c.\tau_n) - c.b) &\leq p_k(T(c.\tau_n) - c.T(\tau_n) - T(c).\tau_n) \\ &\quad + p_k(c.T(\tau_n) + w.\tau_n - c.b) \\ &\leq p_k(T(c.\tau_n) - c.T(\tau_n) - T(c).\tau_n) \\ &\quad + p_k(c.T(\tau_n) - c.b) + p_k(w.\tau_n) \\ &\leq \epsilon_k p_k(c)\, p_k(\tau_n) + p_k(c)\, p_k(T(\tau_n) - b) + p_k(w.\tau_n). \end{aligned}$$

Since $p_k(T(\tau_n) - b) \to 0$, $p_k(\tau_n) \to 0$ and $p_k(w.\tau_n) \leq p_k(w).p_k(\tau_n) \to 0$, for all $k \in \mathbb{N}$, as $n \to \infty$, we have $p_k(T(c.\tau_n) - c.b) \to 0$, for every $k$, and hence $T(c.\tau_n) \to c.b$, when $c.\tau_n \to 0$. By the same argument mentioned at the beginning of the proof, we have $r_\Lambda(c.b) = 0$. Since $c \in \Lambda$ is arbitrary, we conclude this $b \in rad(\Lambda) = \{0\}$, and this proves the theorem. □

**Corollary 2.2.** *Let $(\Lambda, (p_n))$ be a semisimple Frechet $Q$-algebra. If $T : \Lambda \to \Lambda$ is a surjective almost derivation with $r_\Lambda(Ta) \leq r_\Lambda(a)$, for all $a \in \Lambda$. Then $T$ is continuous.*

*Proof.* We known that $r_\Lambda(x) \leq p_m(x)$, for some $m \in \mathbb{N}$, and for all $x \in \Lambda$, because $\Lambda$ is a $Q$-algebra. See, for example, ([3], Theorem 6.18). Since $p_k(x) \leq p_{k+1}(x)$, for all $x$, and $k \in \mathbb{N}$, we have

$$r_{\Lambda_k}(Tx + ker\ p_k) \leq r_\Lambda(Tx) \leq r_\Lambda(x) \leq p_m(x) \leq p_k(x),$$

for all $k \geq m$ and $x \in \Lambda$. So, by Theorem 2.1, $T$ is continuous. □

**Corollary 2.3.** *Let $\Lambda$ be a semisimple Banach algebra. If $T : \Lambda \to \Lambda$ is a surjective almost derivation with $r_\Lambda(Ta) \leq r_\Lambda(a)$, for all $a \in \Lambda$, then $T$ is continuous.*

**Example 2.4.** Let $(\Lambda, (p_n))$ be a semisimple Frechet $Q$-algebra. A linear map $T : \Lambda \to \Lambda$ is defined by $T(a) = \beta a$, for all $a \in \Lambda$ where $\beta \in (0, 1]$. Since

$$\begin{aligned} p_n(T(\mu.\eta) - \mu.T(\eta) - T(\mu).\eta) &= p_n(\beta\mu.\eta - \mu.\beta\eta - \beta\mu.\eta) \\ &= p_n(-\beta\mu.\eta) \leq |-\beta| p_n(\mu).p_n(\eta), \end{aligned}$$

for all $\mu, \eta \in \Lambda$, hence $T$ is an almost derivation but not a derivation on $(\Lambda, (p_n))$. Since $\Lambda$ is a $Q$-algebra, there exists $k \in \mathbb{N}$ such that $r_\Lambda(a) = \lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}}$, for all $a \in \Lambda$. See, for example ([3], Theorem 6.18). So

$$r_\Lambda(Ta) = r_\Lambda(\beta a) = \lim_{n\to\infty}(p_k((\beta a)^n))^{\frac{1}{n}} = |\beta| \lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}} \leq r_\Lambda(a).$$

So, all hypothesis of Corollary 2.2 are satisfied and $T$ is continuous.

## 3. conclusion

R. L. Carpenter result motivates us to ask an open question: Is every surjective almost derivation on semisimple Frechet algebras continuous? Moreover, a partial answer to this open question is derived in the sense that every surjective almost derivation $T$ on semisimple Frechet $Q$-algebras $\Lambda$, with an additional condition on $\Lambda$, is continuous.

## Acknowledgement

The authors first like to thank the reviewers and the editor for their valuable comments and corrections for improvement of this article. The Council of Scientific and Industrial Research(CSIR) of India funded the work of the second author (G. Siva).

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