

**APPLICATION OF DIFFERENTIAL TRANSFORMATION METHOD TO
THE DULLIN-GOTTWALD-HOLM EQUATION**

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ABSTRACT. Nonlinear problems in partial differential equations are open problems in many fields of mathematics and engineering. So associated with the structure of the problems, many analytical and numerical methods are obtained. We show that the differential transformation method is an appropriate method for the Dullin-Gottwald-Holm equation (DGH), which is a nonlinear partial differential equation arise in many physical phenomena. Hence in this paper, the differential transform method (DTM) is applied to the Dullin-Gottwald-Holm equation. We obtain the exact solutions of Dullin-Gottwald-Holm equation by using the DTM. In addition, we give some examples to illustrate the sufficiency of the method for solving such nonlinear partial differential equations. These results show that this technique is easy to apply and provide a suitable method for solving differential equations. To our best knowledge, the theorem presented in Section 2 has been not introduced previously. We presented and proved this new theorem which can be very effective for formulating the nonlinear forms of partial differential equations.

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1. Introduction and Background

There are various techniques for solving nonlinear equations such as ordinary differential equations, partial differential equations and integral equations. Some of these techniques are based on numerical methods that solve the problem after linearization with Newton's method [3, 4, 5, 6]. Some other methods solve the problem analytically, such as the DTM method. In this work, we will solve the DGH problem using the DTM method. The Dullin-Gottwald-Holm (DGH) equation has many applications in physics and mathematics fields. In this paper, we consider the following Dullin-Gottwald-Holm equation

$$(1.1) \quad u_t + \alpha u_x - \beta u_{xxt} + \gamma uu_x + \delta u_x u_{xx} + \delta u u_{xxx} = 0,$$

where $u(x, t)$ is the fluid velocity in the x direction, α is related to the critical shallow water wave speed and β , γ and δ are arbitrary constants.

Authors of [12, 8, 9, 13, 10] utilized different methods such as exact traveling-wave solutions, linearly solitary wave solutions, soliton-like solutions and Exp-function methods for solving equation (1.1).

In 1986, Zhou proposed a new method for solving linear and nonlinear problems in electrical

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circuit problems [23]. Authors of [11] and [2] used this method for solving partial differential equations and system of differential equations. A. Borhanifar and R. Abazari applied this method for Burgers and Schrödinger equations [1, 7]. In [20, 19, 21, 15, 18], this method has been utilized for solving some important equations with initial and boundary conditions. Similar discussions can be found in [14, 22].

2. THE TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM

The basic definitions and operations of one-dimensional DT were introduced in [23, 11, 2]. In order to speed up the convergence rate and improve the accuracy of calculation, the entire domain of t needs to be split into sub-domains [16, 17].

Now we introduce the basic definition of the two-dimensional differential transform. To this end, consider a function of two variables $w(x, t)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $w(x, t) = f(x)g(t)$. Based on the properties of two-dimensional differential transform, the function $w(x, t)$ can be represented as

$$(2.1) \quad w(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} W(i, j) x^i t^j.$$

where $W(i, j)$ is called the spectrum of $w(x, y)$. Now we introduce the basic definitions and operations of two-dimensional DT as follows [14].

Definition 2.1. If $w(x, t)$ is analytic and continuously differentiable with respect to time t in the domain of interest, then

$$(2.2) \quad W(h, k) = \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x, t) \right]_{x=x_0, t=t_0},$$

where the spectrum function $W(k, h)$ is the transformed function, which is also called the T-function. Let $w(x, y)$ be the original function while the uppercase $W(k, h)$ stands for the transformed function. Now we define the differential inverse transform of $W(k, h)$ as follows

$$(2.3) \quad w(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h) (x - x_0)^k (t - t_0)^h.$$

Using Eq. (2.3) in (2.2), we have

$$(2.4) \quad w(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x, t) \right]_{x_0=0, t_0=0} x^k t^h = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h) x^k t^h.$$

From the above definitions and equations (2.1) and (2.2), we can obtain some of the fundamental mathematical operations performed by two-dimensional differential transform in Table 1.

Remark 2.2. We denote the arithmetical numbers $\mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$ by \mathbb{A} . For a $k \in \mathbb{A}$, we define $\mathbb{A}_k := \{0, \dots, k\}$. For $k \in \mathbb{N}$, suppose $\mathbb{A}_{k0} := \{2n | n = 0, \dots, [\frac{k}{2}]\}$, and $\mathbb{A}_{k1} := \{2n - 1 | n = 1, \dots, [\frac{k+1}{2}]\}$, then $\mathbb{A}_k = \mathbb{A}_{k0} \cup \mathbb{A}_{k1}$, and $\mathbb{A}_{k0} \cap \mathbb{A}_{k1} = \{\}$.

Original function: $w(x, t)$	Transformed function: $W(k, h)$
$u(x, t) \pm v(x, t)$	$U(k, h) \pm V(k, h)$
$cu(x, t)$	$cU(k, h)$
$\frac{\partial}{\partial x}u(x, t)$	$(k+1)U(k+1, h)$
$\frac{\partial^{r+s}}{\partial x^r \partial t^s}u(x, t)$	$\frac{(k+r)!(h+s)!}{k!h!}U(k+r, h+s)$
$u(x, t)v(x, t)$	$\sum_{r=0}^k \sum_{s=0}^h U(r, h-s)V(k-r, s)$
$\frac{\partial}{\partial x}u(x, t)\frac{\partial}{\partial t}v(x, t)$	$\sum_{r=0}^k \sum_{s=0}^h (k-r+1)(h-s+1)U(k-r+1, s)V(r, h-s+1)$

TABLE 1. Operations of the two-dimensional differential transform.

Theorem 2.3. If $\omega(x, t) = \frac{\partial^n}{\partial x^n}u(x, t) \cdot \frac{\partial^m}{\partial x^m}v(x, t)$ then

$$(2.5) \quad W(k, h) = \sum_{s=0}^h \sum_{r=0}^k \prod_{i=1}^m (r+i) \prod_{j=1}^n (k-r+j) U(r+m, h-s) V(k-r+n, s),$$

for all $n, m \in \mathbb{A}$, if $m = 0$ then $\prod_{i=1}^m (r+i) = 1$ and if $n = 0$ then $\prod_{j=1}^n (k-r+j) = 1$.

Proof. suppose $n, m \in \mathbb{N}$ then

$$\begin{aligned} W(k, h) &= \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x, t) \right]_{x_0=0, t_0=0} = \frac{1}{k!h!} \frac{\partial^k}{\partial x^k} \left\{ \frac{\partial^h}{\partial t^h} \omega(x, t) \right\} \\ &= \frac{1}{k!h!} \frac{\partial^k}{\partial x^k} \left\{ \sum_{s=0}^h \binom{h}{s} \frac{\partial^s}{\partial t^s} \left(\frac{\partial^n u}{\partial x^n} \right) \frac{\partial^{h-s}}{\partial t^{h-s}} \left(\frac{\partial^m v}{\partial x^m} \right) \right\} \\ &= \sum_{s=0}^h \sum_{r=0}^k \frac{1}{k!h!} \binom{h}{s} \binom{k}{r} \frac{\partial^r}{\partial x^r} \left(\frac{\partial^{m+h-s} u}{\partial t^{h-s} \partial x^m} \right) \frac{\partial^{k-r}}{\partial x^{k-r}} \left(\frac{\partial^{s+n} v}{\partial t^s \partial x^n} \right) \\ &= \sum_{s=0}^h \sum_{r=0}^k \frac{1}{r!(h-s)!s!(k-r)!} \frac{\partial^{r+m+h-s}}{\partial x^{r+m} \partial t^{h-s}} u(x, t) \frac{\partial^{k-r+n+s}}{\partial x^{k-r+n} \partial t^s} v(x, t) \\ &= \sum_{s=0}^h \sum_{r=0}^k \frac{(r+1)\dots(r+m)}{(r+m)!(h-s)!} \frac{\partial^{r+m+h-s}}{\partial x^{r+m} \partial t^{h-s}} u(x, t) \times \frac{(k-r+1)\dots(k-r+n)}{(k-r+n)!s!} \frac{\partial^{k-r+n+s}}{\partial x^{k-r+n} \partial t^s} v(x, t) \\ &= \sum_{s=0}^h \sum_{r=0}^k (r+1)\dots(r+m) U(r+m, h-s) \times (k-r+1)\dots(k-r+n) V(k-r+n, s). \end{aligned}$$

In row 5 we are multiplied numerator and denominator by $(r+1)\dots(r+m)$. This multiplication is need not to use, if $m = 0$. Also in row 6 numerator and denominator are multiplied by $(k-r+1)\dots(k-r+n)$, which is need not to use, if $n = 0$. \square

3. APPLICATION OF THE DTM

In this section, we apply the DTM for solving the presented Dullin-Gottwald-Holm equation. For this aim, consider the equation (1.1) with the following initial and boundary conditions

$$(3.1) \quad u(x, 0) = f(x), \quad x \in \mathbb{R},$$

$$(3.2) \quad u(0, t) = g(t), \quad 0 \leq t,$$

$$(3.3) \quad u_x(0, t) = q(t), \quad 0 \leq t.$$

Let $U(k, h)$ be the differential transform of $u(x, t)$. By using Table 1., Theorem 2.3, Eq. (2.1) and Definition 2.1 when $x_0 = t_0 = 0$, we get the differential transform version of Eq. (1.1) as follows

$$(3.4) \quad \beta U(k+2, h+1)(k+1)(k+2)(h+1) = (h+1)U(k, h+1) + \alpha(k+1)U(k+1, h) \\ + \sum_{r=0}^k \sum_{s=0}^h \left\{ \gamma(r+1)U(r+1, h-s)U(k-r, s) + \delta(r+1)(r+2)(k-r+1)U(r+2, h-s)U(k-r+1, s) \right. \\ \left. + \delta(r+1)(r+2)(r+3)U(r+3, h-s)U(k-r, s) \right\}.$$

By initial and boundary conditions we get

$$(3.5) \quad \sum_{k=0}^{\infty} U(k, 0)x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k, \quad x \in \mathbb{R},$$

$$(3.6) \quad \sum_{h=0}^{\infty} U(0, h)t^h = \sum_{h=0}^{\infty} \frac{g^{(h)}(0)}{h!} t^h, \quad 0 \leq t,$$

$$(3.7) \quad \sum_{h=0}^{\infty} U(1, h)t^h = \sum_{h=0}^{\infty} \frac{q^{(h)}(0)}{h!} t^h, \quad 0 \leq t.$$

For $k, h = 0, 1, 2, \dots$ the values of $U(k, 0), U(0, h), U(1, h)$ can be obtained from Eqs. (3.5)-(3.7). By using the Eq. (3.4), we find the remainder values of U , as illustrated in the following test problems.

Example 3.1. Consider the equation (1.1) with $\alpha = \beta = \frac{1}{2}, \delta = -1, \gamma = 2$, initial and boundary conditions, $f(x) = e^x, g(t) = q(t) = e^{-t}$. From Eqs. (3.5)-(3.7), we have

$$(3.8) \quad U(k, 0) = \frac{1}{k!}, \quad U(0, h) = U(1, h) = \frac{(-1)^h}{h!} \quad k, h \in \mathbb{A}.$$

Eq. (3.4) forces

$$(3.9) \quad \beta \frac{1}{2} U(k+2, h+1)(k+1)(k+2)(h+1) = (h+1)U(k, h+1) + \frac{1}{2}(k+1)U(k+1, h) \\ + \sum_{r=0}^k \sum_{s=0}^h \left\{ 2(r+1)U(r+1, h-s)U(k-r, s) - (r+1)(r+2)(k-r+1)U(r+2, h-s)U(k-r+1, s) \right. \\ \left. - (r+1)(r+2)(r+3)U(r+3, h-s)U(k-r, s) \right\}.$$

For $h = 0$ and $k = 0, 1, 2, \dots$, we get

$$\frac{1}{2}(k+1)(k+2)U(k+2, 1) = U(k, 1) + \frac{1}{2k!},$$

or

$$(3.10) \quad U(k+2, 1) = \frac{2U(k, 1)}{(k+1)(k+2)} + \frac{1}{(k+2)!}.$$

By induction and using the Eqs. (3.8),(3.10), we obtain

$$(3.11) \quad U(k, 1) = -\frac{1}{k!}, \quad k \in \mathbb{A}.$$

From (3.9), for $h = 1$ and $k = 0, 1, 2, \dots$, we get

$$(k+1)(k+2)U(k+2, 2) = 2U(k, 2) - \frac{1}{2k!},$$

or

$$(3.12) \quad U(k+2, 2) = \frac{2U(k, 2)}{(k+1)(k+2)} - \frac{1}{2(k+2)!}.$$

By induction and using the Eqs. (3.8),(3.12), we obtain

$$(3.13) \quad U(k, 2) = \frac{1}{2k!}, \quad k \in \mathbb{A}.$$

From (3.9), for $h = 2$ and $k = 0, 1, 2, \dots$, we get

$$\frac{3}{2}(k+1)(k+2)U(k+2, 3) = 3U(k, 3) + \frac{1}{4k!},$$

or

$$(3.14) \quad U(k+2, 3) = \frac{2U(k, 3)}{(k+1)(k+2)} + \frac{1}{2 \times 3(k+2)!}.$$

By induction and using the Eqs. (3.8),(3.14), we obtain

$$(3.15) \quad U(k, 3) = -\frac{1}{3!k!}, \quad k \in \mathbb{A}.$$

Hence we obtain the $u(x, t)$ as follows

$$(3.16) \quad u(x, t) \simeq \sum_{k=0}^{\infty} \frac{x^k}{k!} - \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) t + \frac{1}{2} \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) t^2 - \frac{1}{3!} \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) t^3 + \dots,$$

which it is the Taylor expansion of the

$$(3.17) \quad u(x, t) = e^{x-t},$$

where it is the exact solution of this example.

Example 3.2. Consider the equation (1.1) with $\alpha = 0, \beta = 4, \delta = -2, \gamma = 1$, initial and boundary conditions, $f(x) = 0, g(t) = 0, q(t) = \frac{\ln(1+t)}{\sqrt{2}}$. From Eqs. (3.5)-(3.7), we have

$$(3.18) \quad U(k, 0) = 0, \quad U(0, h) = 0, \quad k, h \in \mathbb{A},$$

$$(3.19) \quad U(1, 0) = 0, \quad U(1, h) = \frac{(-1)^{h-1}}{\sqrt{2}h}, \quad h \in \mathbb{N}.$$

Eq. (3.4) forces

$$(3.20) \quad 4(k+1)(k+2)(h+1)U(k+2, h+1) = (h+1)U(k, h+1) \\ + \sum_{r=0}^k \sum_{s=0}^h \left\{ (r+1)U(r+1, h-s)U(k-r, s) - 2(r+1)(r+2)(k-r+1)U(r+2, h-s)U(k-r+1, s) \right. \\ \left. - 2(r+1)(r+2)(r+3)U(r+3, h-s)U(k-r, s) \right\}.$$

For $h = 0$ and $k \in \mathbb{A}$, from (3.18),(3.20) we get

$$(3.21) \quad U(k+2, 1) = \frac{U(k, 1)}{4(k+1)(k+2)}.$$

By induction and using the Eqs. (3.18)-(3.19),(3.21), we obtain

$$(3.22) \quad U(k, 1) = \begin{cases} 0 & k = 2n, n \in \mathbb{A}, \\ \frac{1}{\sqrt{24^n \times (2n+1)!}} & k = 2n+1, n \in \mathbb{A}. \end{cases}$$

From (3.18),(3.20), for $h = 1$ and $k = 0, 1, 2, \dots$, we get

$$(3.23) \quad U(k+2, 2) = \frac{U(k, 2)}{4(k+1)(k+2)}.$$

By induction and using the Eqs. (3.18)-(3.19),(3.23), we obtain

$$(3.24) \quad U(k, 2) = \begin{cases} 0 & k = 2n, n \in \mathbb{A}, \\ -\frac{1}{2\sqrt{24^n \times (2n+1)!}} & k = 2n+1, n \in \mathbb{A}. \end{cases}$$

From (3.20), for $h = 2$ and $k = 0, 1, 2, \dots$, we get

$$(3.25) \quad 3 \times 4(k+1)(k+2)U(k+2, 3) = 3U(k, 3) + \sum_{r=0}^k \left\{ (r+1)U(r+1, 1)U(k-r, 1) \right. \\ \left. - 2(r+1)(r+2)(k-r+1)U(r+2, 1)U(k-r+1, 1) - 2(r+1)(r+2)(r+3)U(r+3, 1)U(k-r, 1) \right\}.$$

For $k = 2n, n \in \mathbb{A}$, from (3.22),(3.25), we obtain $U(2n, 3) = 0$. For $k = 2n+1, n \in \mathbb{A}$, from (3.22),(3.25), we get

$$(3.26) \quad 3 \times 4(2n+2)(2n+3)U(2n+3, 3) = 3U(2n+1, 3) + \frac{1}{2 \times 4^n} \sum_{r=0}^n \frac{2r-n}{(2r+1)!(2(n-r)+1)!}.$$

The last term in (3.26) vanishes. For this purpose put $u_r = (2r+1)!(2(n-r)+1)!$ therefor

$$\frac{u_r}{u_{r-1}} = \frac{2r(2r+1)}{(2(n-r)+2)(2(n-r)+3)} \equiv \frac{\varphi(r)}{h(r)},$$

then

$$(3.27) \quad \frac{\varphi(r)}{u_r} = \frac{h(r)}{u_{r-1}} \equiv v_r, \\ v_{r+1} - v_r = \frac{h(r+1)}{u_r} - \frac{\varphi(r)}{u_r} = 2(2n+1) \times \frac{n-2r}{u_r}.$$

Thus

$$(3.28) \quad 0 = v_{n+1} - v_0 = \sum_{r=0}^n (v_{r+1} - v_r) = 2(2n+1) \sum_{r=0}^n \frac{n-2r}{u_r}.$$

Equations (3.26),(3.28) forces

$$(3.29) \quad 4(2n+2)(2n+3)U(2n+3,3) = U(2n+1,3).$$

By induction and using the Eqs. (3.19),(3.29), we obtain

$$(3.30) \quad U(2n+1,3) = \frac{1}{3\sqrt{2}4^n \times (2n+1)!}.$$

Similar evaluations forces

$$(3.31) \quad U(k,4) = \begin{cases} 0 & k = 2n, n \in \mathbb{A}, \\ \frac{-1}{4\sqrt{2}4^n \times (2n+1)!} & k = 2n+1, n \in \mathbb{A}. \end{cases}$$

Hence we obtain the $u(x,t)$ as follows

$$(3.32) \quad \begin{aligned} u(x,t) &\simeq \frac{t}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{4^n(2n+1)!} - \frac{t^2}{2\sqrt{2}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{4^n(2n+1)!} + \frac{t^3}{3\sqrt{2}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{4^n(2n+1)!} \\ &- \frac{t^4}{4\sqrt{2}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{4^n(2n+1)!} + \dots, \end{aligned}$$

which it is the Taylor expansion of the

$$(3.33) \quad u(x,t) = \sqrt{2} \sinh\left(\frac{x}{2}\right) \ln(1+t),$$

where it is the exact solution of this example.

Example 3.3. In accordance with Can et al. (2009), for $0 < \gamma = 2\delta$, $\alpha = 1 + \beta$, the function $u(x,t) = \cos(x-t)$ is a solution of the equation (1.1) [10]. With the initial and boundary conditions $f(x) = \cos x$, $g(t) = \cos t$, $q(t) = \sin t$, this is the only solution. For simplicity of discussion, suppose $\beta = \delta = 1$, and hence $\alpha = \gamma = 2$. From Eqs. (3.5)-(3.7), we have

$$(3.34) \quad U(0,h) = \begin{cases} \frac{(-1)^n}{(2n)!} & h = 2n, n \in \mathbb{A}, \\ 0 & h = 2n+1, n \in \mathbb{A}. \end{cases},$$

$$(3.35) \quad U(k,0) = \begin{cases} \frac{(-1)^n}{(2n)!} & k = 2n, n \in \mathbb{A}, \\ 0 & k = 2n+1, n \in \mathbb{A}. \end{cases},$$

$$(3.36) \quad U(1,h) = \begin{cases} \frac{(-1)^n}{(2n+1)!} & h = 2n+1, n \in \mathbb{A}, \\ 0 & h = 2n, n \in \mathbb{A}. \end{cases}$$

Eq. (3.4) forces

$$(3.37) \quad (k+1)(k+2)(h+1)U(k+2, h+1) = (h+1)U(k, h+1) + 2(k+1)U(k+1, h) \\ + \sum_{r=0}^k \sum_{s=0}^h \left\{ 2(r+1)U(r+1, h-s)U(k-r, s) + (r+1)(r+2)(k-r+1)U(r+2, h-s)U(k-r+1, s) \right. \\ \left. + (r+1)(r+2)(r+3)U(r+3, h-s)U(k-r, s) \right\}.$$

By a mathematical induction we show that $U(2m, 1) = 0$, $m \in \mathbb{A}$. Eqs. (3.34)-(3.36) and Eq. (3.37) with $h = 0$, $k = 0$, forces $U(2, 1) = 0$. This equation and Eq. (3.34) show that the property are true for $m = 0, 1$. Now suppose $m \in \mathbb{N}$ and the property is true for $0, 1, \dots, m$. Eqs. (3.34)-(3.36) and Eq. (3.37) with $h = 0$, $k = 2m$, imply that

$$(3.38) \quad (2m+1)(2m+2)U(2m+2, 1) = U(2m, 1) + 2(2m+1)U(2m+1, 0) \\ + \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor = m} \left\{ 2(2n+1)U(2n+1, 0)U(2m-2n, 0) \right. \\ \left. + (2n+1)(2n+2)(2m-2n+1)U(2n+2, 0)U(2m-2n+1, 0) \right. \\ \left. + (2n+1)(2n+2)(2n+3)U(2n+3, 0)U(2m-2n, 0) \right\} \\ + \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor = m} \left\{ 2(2n)U(2n, 0)U(2m-2n+1, 0) + (2n)(2n+1)(2m-2n+2)U(2n+1, 0)U(2m-2n+2, 0) \right. \\ \left. + (2n)(2n+1)(2n+2)U(2n+2, 0)U(2m-2n+1, 0) \right\} = 0.$$

Hence the property $U(2m, 1) = 0$ is true for all $m \in \mathbb{A}$. Note that in accordance with Remark 2.2, $\sum_{r=0}^{k=2m} a_r = \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor = m} a_{2n} + \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor = m} a_{2n-1}$. Now suppose $m \in \mathbb{N}$ and put $k = 2(m-1) + 1 \in \mathbb{N}$, $h = 0$, in (3.37), then

$$(3.39) \quad 2m(2m+1)U(2m+1, 1) = U(2(m-1) + 1, 1) + 2(2m)U(2m, 0) \\ + \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor = m-1} \left\{ 2(2n+1)U(2n+1, 0)U(2(m-n-1) + 1, 0) \right. \\ \left. + (2n+1)(2n+2)(2m-2n)U(2n+2, 0)U(2m-2n, 0) \right. \\ \left. + (2n+1)(2n+2)(2n+3)U(2n+3, 0)U(2(m-n-1) + 1, 0) \right\} \\ + \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor = m} \left\{ 2(2n)U(2n, 0)U(2m-2n, 0) + (2n)(2n+1)(2m-2n+1)U(2n+1, 0)U(2m-2n+1, 0) \right. \\ \left. + (2n)(2n+1)(2n+2)U(2n+2, 0)U(2m-2n, 0) \right\}.$$

In accordance with (3.34)-(3.36), we obtain

$$\begin{aligned}
2m(2m+1)U(2m+1,1) &= U(2(m-1)+1,1) + \frac{(-1)^m}{(2m-1)!} \left\{ 2 - \sum_{n=1}^m \binom{2m-1}{2n-2} + \sum_{n=1}^m \binom{2m-1}{2n-1} \right\} \\
(3.40) \qquad &= U(2(m-1)+1,1) + \frac{2(-1)^m}{(2m-1)!}, \quad m \in \mathbb{N}.
\end{aligned}$$

Hence we can write

$$(3.41) \qquad U(2m+1,1) = \frac{U(2(m-1)+1,1)}{2m(2m+1)} + \frac{2(-1)^m}{(2m+1)!}, \quad m \in \mathbb{N}.$$

By a mathematical induction we show that $U(2m+1,1) = \frac{(-1)^m}{(2m+1)!}$, $m \in \mathbb{A}$. According to (3.36), $U(1,1) = 1$ and hence the property is true for $m = 0$. Put $m = 1$ in Eq. (3.41), then we obtain $U(3,1) = \frac{-1}{3!}$ and hence the property is true for $m = 1$. Now suppose $m \in \mathbb{N}$ and the property is true for $0, 1, \dots, m$, then these hypotheses and Eq. (3.41) imply that

$$\begin{aligned}
U(2m+1,1) &= \frac{U(2(m-1)+1,1)}{2m(2m+1)} + \frac{2(-1)^m}{(2m+1)!} \\
(3.42) \qquad &= \frac{(-1)^{m-1}}{(2(m-1)+1)!2m(2m+1)} + \frac{2(-1)^m}{(2m+1)!} = \frac{(-1)^m}{(2m+1)!}.
\end{aligned}$$

So the property is true for all $m \in \mathbb{A}$. In accordance with the above discussion, we obtain

$$(3.43) \qquad U(k,1) = \begin{cases} \frac{(-1)^m}{(2m+1)!} & k = 2m+1, m \in \mathbb{A}, \\ 0 & k = 2m, m \in \mathbb{A}. \end{cases}$$

For $h = 1$, $k \in \mathbb{A}$, Eq. (3.37) forces

$$\begin{aligned}
2(k+1)(k+2)U(k+2,2) &= 2U(k,2) + 2(k+1)U(k+1,1) \\
&+ \sum_{r=0}^k \left\{ 2(r+1)(U(r+1,1)U(k-r,0) + U(r+1,0)U(k-r,1)) \right. \\
&+ (r+1)(r+2)(k-r+1)(U(r+2,1)U(k-r+1,0) + U(r+2,0)U(k-r+1,1)) \\
(3.44) &+ (r+1)(r+2)(r+3)(U(r+3,1)U(k-r,0) + U(r+3,0)U(k-r,1)) \left. \right\}.
\end{aligned}$$

Eqs. (3.35),(3.43) and Eq. (3.44) with $k = 2m+1$, $m \in \mathbb{A}$, imply that

$$(3.45) \qquad 2(2m+2)(2m+3)U(2m+3,2) = 2U(2m+1,2).$$

In accordance with Eq. (3.36), $U(1,2) = 0$, and hence using a mathematical induction on Eq. (3.45) implies that $U(2m+1,2) = 0$, $m \in \mathbb{A}$. By a mathematical induction we shall show that $U(2m,2) = \frac{(-1)^{m-1}}{2(2m)!}$, $m \in \mathbb{A}$. In accordance with Eq. (3.34), $U(0,2) = -\frac{1}{2}$, and hence the property is true for $m = 0$. Eqs. (3.34)-(3.36) and Eq. (3.44) with $k = 0$ imply that $U(2,2) = \frac{1}{4}$. Hence the property is true for $m = 1$. Now suppose $m \in \mathbb{N}$ and the property is

true for $0, 1, \dots, m$. Eq. (3.43) and Eq. (3.44) with $k = 2m$, imply that

$$\begin{aligned}
(3.46) \quad & 2(2m+1)(2m+2)U(2m+2, 2) = 2U(2m, 2) + 2(2m+1)\frac{(-1)^m}{(2m+1)!} \\
& + \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor = m} \left\{ 2(2n+1)\frac{(-1)^n}{(2n+1)!} \times \frac{(-1)^{m-n}}{(2m-2n)!} + (2n+1)(2n+2)(2m-2n+1)\frac{(-1)^{n+1}}{(2n+2)!} \times \frac{(-1)^{m-n}}{(2m-2n+1)!} \right. \\
& \left. + (2n+1)(2n+2)(2n+3)\frac{(-1)^{n+1}}{(2n+3)!} \times \frac{(-1)^{m-n}}{(2m-2n)!} \right\} \\
& + \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor = m} \left\{ 2(2n)\frac{(-1)^n}{(2n)!} \times \frac{(-1)^{m-n}}{(2m-2n+1)!} + (2n)(2n+1)(2m-2n+2)\frac{(-1)^n}{(2n+1)!} \times \frac{(-1)^{m-n+1}}{(2m-2n+2)!} \right. \\
& \left. + (2n)(2n+1)(2n+2)\frac{(-1)^{n+1}}{(2n+2)!} \times \frac{(-1)^{m-n}}{(2m-2n+1)!} \right\} = 2U(2m, 2) + 2\frac{(-1)^m}{(2m)!}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(3.47) \quad & U(2m+2, 2) = \frac{U(2m, 2)}{(2m+1)(2m+2)} + \frac{(-1)^m}{(2m+2)!} \\
& = \frac{(-1)^{m-1}}{2(2m)!(2m+1)(2m+2)} + \frac{(-1)^m}{(2m+2)!} = \frac{(-1)^m}{2(2m+2)!}.
\end{aligned}$$

This means the property is true for $m+1$, and hence the property is true for all $m \in \mathbb{A}$. In accordance with the above discussion, we obtain

$$(3.48) \quad U(k, 2) = \begin{cases} \frac{(-1)^{m-1}}{2(2m)!} & k = 2m, m \in \mathbb{A}, \\ 0 & k = 2m+1, m \in \mathbb{A}. \end{cases}$$

In accordance with Eq. (3.34), $U(0, 3) = 0$. Eqs. (3.34)-(3.36) and Eq. (3.37) with $h = 2$, $k = 0$, imply that $U(2, 3) = 0$. Eqs. (3.35), (3.43), (3.48) and Eq. (3.37) with $h = 2$, $k = 2m$, $m \in \mathbb{N}$, imply that

$$(3.49) \quad U(2m+2, 3) = \frac{U(2m, 3)}{(2m+1)(2m+2)},$$

Therefore by using a mathematical induction we obtain $U(2m, 3) = 0$, $m \in \mathbb{A}$. Eqs. (3.35), (3.43) and Eq. (3.37) with $h = 2$, $k = 2m+1$, $m \in \mathbb{A}$, imply that

$$(3.50) \quad U(2m+3, 3) = \frac{U(2m+1, 3)}{(2m+2)(2m+3)} + \frac{(-1)^m}{3(2m+3)!}.$$

Therefore by using a mathematical induction we obtain

$$(3.51) \quad U(2m+1, 3) = \begin{cases} \frac{-2}{3(2m+1)!} & m = 2n-1, n \in \mathbb{N}, \\ \frac{-1}{(2m+1)!} & m = 2n, n \in \mathbb{N}. \end{cases}$$

In accordance with the above discussion, we obtain

$$(3.52) \quad U(k, 3) = \begin{cases} 0 & k = 2n, n \in \mathbb{A}, \\ \frac{-2}{3k!} & k = 4n-1, n \in \mathbb{N}, \\ \frac{-1}{3!} & k = 1, \\ \frac{-1}{k!} & k = 4n+1, n \in \mathbb{N}. \end{cases}$$

Eqs. (3.35),(3.43),(3.48) and (3.52) imply that

$$(3.53) \quad \sum_{k=0}^{\infty} U(k, h)x^k = \begin{cases} \cos x & h = 0, \\ \sin x & h = 1, \\ \frac{-1}{2} \cos x & h = 2, \\ \frac{5x - \sin x - 5 \sinh x}{6} & h = 3. \end{cases}$$

In accordance with Eq. (2.1), we can write $u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t)$, where

$$(3.54) \quad u_n(x, t) = \sum_{h=0}^n \left(\sum_{k=0}^{\infty} U(k, h)x^k \right) t^h.$$

Figure 1 shows variations of $u(x, t)$ and $u_3(x, t)$ in rectangle $\{(x, t) : |x| \leq 1, |t| \leq 0.5\}$.

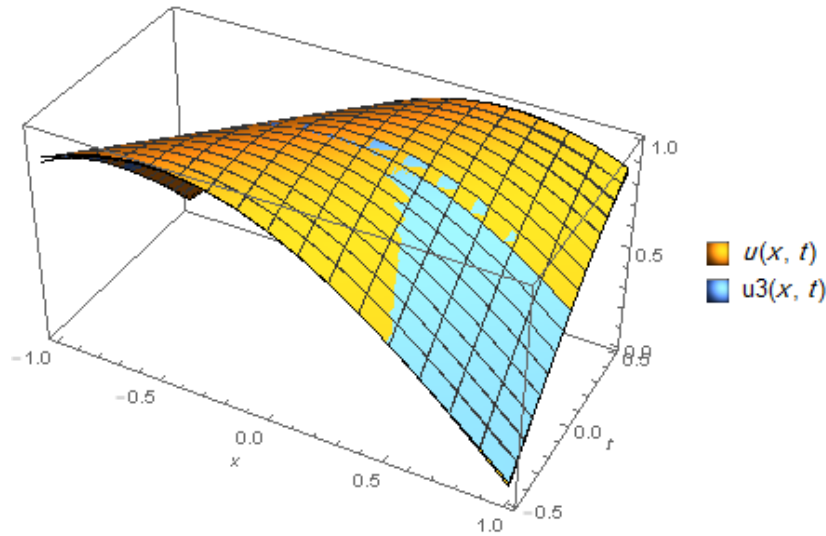


FIGURE 1. Variations of $u(x, t)$ and $u_3(x, t)$ as functions of x, t for the Example 3.3.

Conclusion

In this research, the exact solutions of the Dullin-Gottwald-Holm equation has been found by using differential transform method. This equation has lots of applications in physics. Obtaining the exact solution for this equation was the strong point of our method. As we saw in Examples 3.1-3.3, and according to Theorem 2.3 and Table 1, any DGH problem can be solved analytically by the DTM method. Of course, the scope of Theorem 2.3 goes beyond DGH problems and covers any similar problem in which $\frac{\partial^n}{\partial x^n} u(x, t) \cdot \frac{\partial^m}{\partial x^m} v(x, t)$ appears. In addition, similar to what we saw in the two-variable Taylor expansion at the origin, the analytical answer can be obtained at any point where the answer has a Taylor expansion.

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