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Research Paper

ON CONFORMAL TRANSFORMATION OF SOME NON-RIEMANNIAN CURVATURES IN FINSLER GEOMETRY

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Abstract. In this paper, we study the conformal transformation of some important and effective non-Riemannian curvatures in Finsler Geometry. We find the necessary and sufficient condition under which the conformal transformation preserves the Berwald curvature **B**, mean Berwald curvature **E**, Landsberg curvature **L**, mean Landsberg curvature **J**, and the non-Riemannian curvature **H**.

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Keywords: Berwald curvature, mean Berwald curvature, Landsberg curvature, mean Landsberg curvature, the quantity **H**.

1. Introduction

The notion of conformal transformations is an important concept in Riemannian-Finsler geometry. The famous Liouville's theorem made clear already in the 19th century that in dimensions $n \geq 3$ conformal mappings are more rigid than in dimension 2. In the field of general relativity, conformal transformations are vital and important because they preserve the causal structure up to time orientation and light-like geodesics up to parametrization.

The theory of conformal transformation (or change) of Finsler metrics has been studied by many Finsler geometers [2][3][4][5][6][10][11][12][14][15]. But, Knebelman is the first person that studied the conformal theory of general Finsler metrics in [4]. He gave a geometrical criterion according to two Finsler metrics $\mathbf{g}(x,y) = g_{ij}(x,y)dx^i dx^j$ and $\tilde{\mathbf{g}}(x,y) = \tilde{g}_{ij}(x,y)dx^i dx^j$ to be conformal; this reduces to the usual requirement that $g_{ij} = e^{\kappa}\tilde{g}_{ij}$. In [5], he proved that the mentioned condition implies that $\kappa = \kappa(x)$ is a function of position, merely. Indeed, two Finsler metric functions $F = F(x,y)$ and $\bar{F} = \bar{F}(x,y)$ as conformal if the length of an arbitrary vector in the one is proportional to the length in the other. The classical Weyl theorem states that the projective and conformal properties of a Finsler metric determine the metrics properties uniquely [9]. Thus, the conformal properties of the class of Finsler metric deserve extra attention.

In Finsler geometry, there are several important non-Riemannian quantities: the Berwald curvature **B**, the mean Berwald curvature **E** and the Landsberg curvature **L**, the mean Landsberg curvature **J**, the non-Riemannian curvature **H**, etc. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian [13]. In order to understand the conformal

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Finsler geometry, one can consider the conformal transformation of these non-Riemannian quantities.

For a Finsler metric F on a manifold M, the second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ are inner product $\mathbf{g}_y$ and symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM. F is said to be Landsbergian if $\mathbf{L}_y = 0$. Taking a trace of Landsberg curvature give us mean Landsberg curvature $\mathbf{J}_y$. A Finsler metric F is said to be weakly Landsbergian if $\mathbf{J}_y = 0$.

The geodesics of F are characterized locally by the equation

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where G^i are coefficients of a spray defined on M denoted by

$$\mathbf{G}(x, y) = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

Taking three vertical derivation of geodesic coefficients of F give us the Berwald curvature $\mathbf{B}$. A Finsler metric F is called a Berwald metric if $\mathbf{B} = 0$. In this case, $G^i = \Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y \in T_xM$ for any $x \in M$. Every Berwald metric is a Landsberg metric. Taking a trace of Berwald curvature yields mean Berwald curvature $\mathbf{E}$. Taking a horizontal derivation of the mean of Berwald curvature $\mathbf{E}$ give us the H-curvature $\mathbf{H}$. In the class of Weyl metrics, vanishing this quantity results that the Finsler metric is of constant flag curvature and this fact clarifies its geometric meaning [1][7]. By the definition, if $\mathbf{E} = 0$ then $\mathbf{H} = 0$.

In this paper, we study the conformal transformation of some important and effective non-Riemannian curvatures in Finsler Geometry. We find the necessary and sufficient condition under which the conformal transformation preserves the Berwald curvature $\mathbf{B}$ (Theorem 3.2), mean Berwald curvature $\mathbf{E}$ (Theorem 3.3), Landsberg curvature $\mathbf{L}$ (Theorem 4.1), mean Landsberg curvature $\mathbf{J}$ (Theorem 4.2), and the non-Riemannian curvature $\mathbf{H}$ (Theorem 5.1).

There are many connections in Finsler geometry. Throughout this paper, we set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x \in M} T_xM$ the tangent space and $TM_0 := TM - \{0\}$ the slit tangent space of M. A Finsler structure on M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$; (iii) The quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positive-definite on T_xM

$$\mathbf{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \Big[F^2(y + su + tv) \Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then, the pair (M, F) is called a Finsler manifold.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[\mathbf{g}_{y+tw}(u, v) \right]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion.

Let (M, F) be a Finsler manifold. For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke's theorem, every positive-definite Finsler metric F is Riemannian if and only if $\mathbf{I}_y = 0$.

Let (M, F) be a Finsler manifolds and $\mathbf{G} = y^i\delta/\delta x^i$ be its induced spray on TM which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

For a vector $y \in T_xM_0$, the Berwald curvature $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ is defined by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\partial/\partial x^i|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

Taking a trace of Berwald curvature $\mathbf{B}$ give us the mean of Berwald curvature $\mathbf{E}$ which is defined by $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$, where

$$\mathbf{E}_y(u, v) := \frac{1}{2}\sum_{i=1}^{n} g^{ij}(y)g_y\Big(\mathbf{B}_y(u, v, e_i), e_j\Big). \tag{2.1}$$

In local coordinates, $\mathbf{E}_y(u, v) := E_{ij}(y)u^iv^j$, where

$$E_{ij} := \frac{1}{2}B^m{}_{mij}.$$

Taking a horizontal derivation of the mean of Berwald curvature $\mathbf{E}$ give us the H-curvature $\mathbf{H}$ which is defined by $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$, where

$$H_{ij} := E_{ij|m}y^m.$$

Here, "|" denotes the horizontal covariant differentiation with respect to the Berwald connection of F.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u, v, w), y\big).$$

In local coordinates, $\mathbf{L}_y(u, v, w) := L_{\ ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := -\frac{1}{2}y_lB^l{}_{ijk}.$$

$\mathbf{L}_y(u,v,w)$ is symmetric in u, v and w and $\mathbf{L}_y(y,v,w)=0$. $\mathbf{L}$ is called the Landsberg curvature. A Finsler metric F is called a Landsberg metric if $\mathbf{L}=0$.

For $y\in T_xM$, define $\mathbf{J}_y: T_xM\to\mathbb{R}$ by $\mathbf{J}_y(u):=J_i(y)u^i$, where

$$J_i:=g^{jk}L_{ijk}.$$

By definition, $\mathbf{J}_y(y)=0$. $\mathbf{J}$ is called the mean Landsberg curvature or J-curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J}_y=0$. By definition, every Landsberg metric is a weakly Landsberg metric. Mean Landsberg curvature can be defined as following

$$J_i:=y^m\frac{\partial I_i}{\partial x^m}-I_m\frac{\partial G^m}{\partial y^i}-2G^m\frac{\partial I_i}{\partial y^m}.$$

By definition, we get

$$\mathbf{J}_y(u):=\frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y\in T_xM$, $\sigma=\sigma(t)$ is the geodesic with $\sigma(0)=x$, $\dot\sigma(0)=y$ and $U(t),V(t),W(t)$ are linearly parallel vector fields along σ with $U(0)=u,V(0)=v,W(0)=w$. Then the mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y\in T_xM_0$.

3. Conformal Transformation of Berwald and Mean Berwald Curvatures

In this section, we find the necessary and sufficient condition under which the conformal transformation preserves the Berwald curvature $\mathbf{B}$ and mean Berwald curvature $\mathbf{E}$. For this aim, we need the following.

Theorem 3.1. (Rapcsák [8]) *Let F and $\bar F$ be two Finsler metrics on a manifold M. Then*

$$\bar G^i=G^i+\frac{g^{ij}}{4}\Big\{(\bar F^2)_{|k,j}y^k-(\bar F^2)_{|j}\Big\},\tag{3.1}$$

where G^i and $\bar G^i$ are the geodesic spray coefficients of F and $\bar F$, respectively, and "|" and "," denote the horizontal and vertical derivation with respect to the Berwald connection of F.

Now, we can study the conformal transformation of Berwald curvature. We prove the following.

Theorem 3.2. *Let F and $\bar F$ be two Finsler metrics on a manifold M. If $\bar F(x,y)=e^\sigma F(x,y)$, then the conformal transformation preserves the Berwald curvature if and only if the conformal factor $\sigma=\sigma(x)$ satisfies following equation:*

$$\begin{aligned}&2C_{jkl}\sigma^i-2g_{jk}\sigma^mC^i{}_{ml}+4\sigma^pC^m_{pl}(y_jC^i{}_{mk}+y_kC^i{}_{mj})-2\sigma^m(g_{jl}C^i{}_{mk}+g_{kl}C^i{}_{mj}+y_jC^i{}_{mk,l}\\&+y_kC^i{}_{mj,l})+4y_l\sigma^pC^m_{pk}C^i{}_{mj}-4F^2\sigma^sC^p_{sl}C^m_{pk}C^i{}_{mj}+2F^2\sigma^p(C^m_{pk,l}C^i{}_{mj}+C^m_{pk}C^i{}_{mj,l})\\&-2y_l\sigma^mC^i{}_{mj,k}+2F^2\sigma^sC^m_{sl}C^i{}_{mj,k}-F^2\sigma^mC^i{}_{mj,k,l}=0.\end{aligned}\tag{3.2}$$

In particular, if $\sigma(x)=constant$, then $\bar{\mathbf{B}}=\mathbf{B}$.

Proof. Let F and $\bar F$ be two Finsler metrics on a manifold M. By using the Rapcsák s identity, the following relationship between G^i and $\bar G^i$ holds

$$\bar G^i=G^i+\frac{\bar F_{;m}y^m}{2\bar F}y^i+\frac{\bar F}{2}\bar g^{il}\Big\{\bar F_{;k,l}y^k-\bar F_{;l}\Big\},\tag{3.3}$$

where ";" and "," denote the horizontal and vertical derivations with respect to the Berwald connection of F. Suppose that F is conformally related to a $\bar{F}$, namely, $\bar{F} = e^{\sigma}F$, where $\sigma = \sigma(x)$ is a scalar function on M. Since $F_{;m} = 0$, then the following hold

$$\bar{F}_{;m} = \sigma_m e^{\sigma}F, \quad \bar{F}_{,i} = e^{\sigma}F_{,i}, \quad \bar{F}_{;m,l} = \sigma_m e^{\sigma}F_{,l}, \quad \bar{g}_{ij} = e^{2\sigma}g_{ij}, \quad \bar{g}^{ij} = e^{-2\sigma}g^{ij}. \tag{3.4}$$

where we put

$$\sigma_m := \frac{\partial \sigma}{\partial x^m}.$$

By putting (3.4) in (3.3), we get

$$\bar{G}^i = G^i + \sigma_0 y^i - \frac{1}{2}F^2\sigma^i, \tag{3.5}$$

where we define

$$\sigma_0 := \sigma_i y^i \quad \sigma^i := g^{im}\sigma_m.$$

Then, (3.5) can be written as follows

$$\bar{G}^i = G^i + Py^i - Q^i, \tag{3.6}$$

where

$$P := \sigma_k y^k, \quad Q^i := \frac{1}{2}F^2\sigma^i. \tag{3.7}$$

Let us define

$$G^i_j := \frac{\partial G^i}{\partial y^j}, \quad G^i_{jk} := \frac{\partial G^i_j}{\partial y^k}, \quad \bar{G}^i_j := \frac{\partial \bar{G}^i}{\partial y^j}, \quad \bar{G}^i_{jk} := \frac{\partial \bar{G}^i_j}{\partial y^k},$$
$$P_j := \frac{\partial P}{\partial y^j}, \quad P_{jk} := \frac{\partial P_j}{\partial y^k}, \quad Q^i_j := \frac{\partial Q^i}{\partial y^j}, \quad Q^i_{jk} := \frac{\partial Q^i_j}{\partial y^k}, \quad Q^i_{jkl} := \frac{\partial Q^i_{jk}}{\partial y^l}.$$

Taking vertical derivations of (3.6) imply that

$$\bar{G}^i_j = G^i_j + P_j y^i + P\delta^i_j - Q^i_j, \tag{3.8}$$

$$\bar{G}^i_{jk} = G^i_{jk} + P_{jk}y^i + P_j\delta^i_k + P_k\delta^i_j - Q^i_{jk}, \tag{3.9}$$

$$\bar{B}^i_{jkl} = B^i_{jkl} + P_{jkl}y^i + P_{jk}\delta^i_l + P_{jl}\delta^i_k + P_{kl}\delta^i_j - Q^i_{jkl}, \tag{3.10}$$

The following hold

$$P_i = \sigma_i, \quad P_{ij} = P_{ijk} = 0. \tag{3.11}$$

By (3.10) and (3.11), we get

$$\bar{B}^i_{jkl} = B^i_{jkl} - Q^i_{jkl}. \tag{3.12}$$

The following holds

$$(g^{im})_{,j} = -2C^{im}_{\ \ j}, \quad (\sigma^i)_{,j} = -2\sigma_m C^{mi}_{\ \ j} = -2\sigma^m C^i_{\ mj}. \tag{3.13}$$

By (3.7) and (3.13), we get

$$Q^i_j = y_j\sigma^i - F^2\sigma^m C^i_{\ mj}, \tag{3.14}$$

$$Q^i_{jk} = g_{jk}\sigma^i - 2\sigma^m(y_j C^i_{\ mk} + y_k C^i_{\ mj}) + 2F^2\sigma^p C^m_{\ pk}C^i_{\ mj} - F^2\sigma^m C^i_{\ mj,k} \tag{3.15}$$

Using (3.13) we have

$$\sigma_m C^{im}_{j,k} = \left(\sigma_m C^{im}_{j}\right)_{,k} = \left(\sigma^m C^{i}{}_{mj}\right)_{,k} = -2\sigma^p C^{m}_{pk} C^{i}{}_{mj} + \sigma^m C^{i}{}_{mj,k}$$
$$= \sigma^m (C^{i}{}_{mj,k} - 2C^{p}{}_{mk} C^{i}{}_{pj}). \tag{3.16}$$

Thus

$$Q^i_{jkl} = 2C_{jkl}\sigma^i - 2g_{jk}\sigma^m C^i{}_{ml} + 4\sigma^p C^m_{pl}(y_j C^i{}_{mk} + y_k C^i{}_{mj}) - 2\sigma^m (g_{jl} C^i{}_{mk} + y_j C^i{}_{mk,l}$$
$$+ g_{kl} C^i{}_{mj} + y_k C^i{}_{mj,l}) + 4y_l \sigma^p C^m_{pk} C^i{}_{mj} - 4F^2 \sigma^s C^p_{sl} C^m_{pk} C^i{}_{mj} + 2F^2 \sigma^p (C^m_{pk,l} C^i{}_{mj}$$
$$+ C^m_{pk} C^i{}_{mj,l}) - 2y_l \sigma^m C^i{}_{mj,k} + 2F^2 \sigma^s C^m_{sl} C^i{}_{mj,k} - F^2 \sigma^m C^i{}_{mj,k,l}. \tag{3.17}$$

By (3.12) and (3.17), we can get the proof. □

Theorem 3.3. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x, y) = e^{\sigma} F(x, y)$, then the conformal transformation preserves the mean Berwald curvature if and only if the conformal transformation is homothetic or the conformal factor $\sigma = \sigma(x)$ satisfies following equation:*

$$4I_p(y_i C^p{}_{kj} + y_j C^p{}_{ki}) + 2F^2(C^p{}_{kj} I_{p,i} + C^p{}_{ki} I_{p,j}) - 2(y_i I_{k,j} + y_j I_{k,i} + g_{ij} I_k)$$
$$+ F^2(2I_s C^s{}_{ki,j} - I_{k,i,j} - 4I_s C^s{}_{pi} C^p{}_{kj}) = 0. \tag{3.18}$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{E}} = \mathbf{E}$.

Proof. Taking a trace of (3.12) implies that

$$\bar{E}_{ij} = E_{ij} - \frac{1}{2} Q^m_{mij}, \tag{3.19}$$

where $Q^m_{mij} := trac(Q^k_{lij})$. (3.14) yields

(3.20) $Q^m_m = y_m \sigma^m - F^2 \sigma^p I_p$.

(3.21) $Q^m_{mi} = \sigma^k [2F^2 C^s{}_{ki} I_s + g_{ki} - 2y_i I_k - F^2 I_{k,i}]$,

$$Q^m_{mij} = \sigma^k \Big[4I_p(y_i C^p{}_{kj} + y_j C^p{}_{ki}) + 2F^2(C^p{}_{kj} I_{p,i} + C^p{}_{ki} I_{p,j}) - 2(y_i I_{k,j} + y_j I_{k,i} + g_{ij} I_k)$$
$$+ 2F^2 I_s C^s{}_{ki,j} - F^2 I_{k,i,j} - 4F^2 I_s C^s{}_{pi} C^p{}_{kj}\Big]. \tag{3.22}$$

Then, by (3.19) and (3.22) we get the proof. □

4. Conformal Transformation of Landsberg and Mean Landsberg Curvatures

In this section, we find the necessary and sufficient condition under which the conformal transformation preserves the Landsberg curvature $\mathbf{L}$ and mean Landsberg curvature $\mathbf{J}$.

Theorem 4.1. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x, y) = e^{\sigma} F(x, y)$, then the conformal transformation preserves the Landsberg curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation:*

$$\sigma_0 C_{jkl} + \sigma^s \Big[y_j C_{skl} + y_k C_{sjl} + y_l C_{sjk} + F^2 C_{jkl,s} - F^2(C_{mjl} C^m{}_{sk} + C_{mjk} C^m{}_{sl} + C_{mkl} C^m{}_{sj})\Big] = 0.$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{L}} = \mathbf{L}$.

Proof. Since

$$\bar{y}_i = \bar{F}\bar{F}_{y^i} = e^{2\sigma} y_i,$$

contracting (3.12) with $\bar{y}_i$ implies that

$$\bar{L}_{jkl} = e^{2\sigma}\Big(L_{jkl} + \frac{1}{2} y_i Q^i_{jkl}\Big), \tag{4.1}$$

where

$$\begin{aligned} y_i Q^i_{jkl} = {} & 2C_{jkl} y_i \sigma^i - 2\sigma^m y_i (y_j C^i{}_{mk,l} + y_k C^i{}_{mj,l}) + 2F^2 \sigma^p C^m_{pk} y_i C^i{}_{mj,l} - 2y_l \sigma^m y_i C^i{}_{mj,k} \\ & + 2F^2 \sigma^s C^m_{sl} y_i C^i{}_{mj,k} - F^2 \sigma^m y_i C^i{}_{mj,k,l}. \end{aligned} \tag{4.2}$$

The following holds

$$y_i C^i{}_{mk,l} = -C_{mkl}. \tag{4.3}$$

Also, we have

$$0 = (y_i C^i{}_{mj})_{,k,l} = 2C_{ikl} C^i{}_{mj} + g_{ik} C^i{}_{mj,l} + g_{il} C^i{}_{mj,k} + y_i C^i{}_{mj,k,l} \tag{4.4}$$

and

$$g_{ik} C^i{}_{mj,l} = C_{mjk,l} - 2C_{ikl} C^i{}_{mj}. \tag{4.5}$$

By (4.4) and (4.5), we get

$$y_i C^i{}_{mj,k,l} = 2C_{ikl} C^i{}_{mj} - C_{mjk,l} - C_{mjl,k}. \tag{4.6}$$

Thus

$$\begin{aligned} y_i Q^i_{jkl} = {} & 2\sigma_0 C_{jkl} + 2\sigma^s (y_j C_{skl} + y_k C_{sjl} + y_l C_{sjk}) + F^2 \sigma^s (C_{sjk,l} + C_{sjl,k}) \\ & - 2F^2 \sigma^s (C_{mjl} C^m_{sk} + C_{mjk} C^m_{sl} + C_{mkl} C^m_{sj}). \end{aligned} \tag{4.7}$$

Since $C_{ijk,l} = C_{jkl,i}$, then (4.7) reduces to

$$\begin{aligned} y_i Q^i_{jkl} = {} & 2\sigma_0 C_{jkl} + 2\sigma^s (y_j C_{skl} + y_k C_{sjl} + y_l C_{sjk}) + 2F^2 \sigma^s C_{jkl,s} \\ & - 2F^2 \sigma^s (C_{mjl} C^m_{sk} + C_{mjk} C^m_{sl} + C_{mkl} C^m_{sj}). \end{aligned} \tag{4.8}$$

By (4.8), we get the proof. □

Theorem 4.2. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma} F(x,y)$, then the conformal transformation preserves the mean Landsberg curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation:*

$$\sigma_0 I_l + \sigma^s \Big[I_s y_l + F^2 (I_{l,s} + 2C^{jk}_s C_{jkl} - C^k{}_{ml} C^m_{sk} - C^j{}_{ml} C^m_{sj} - I_m C^m_{sl}) \Big] = 0. \tag{4.9}$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{J}} = \mathbf{J}$.

Proof. The following holds

$$(g^{jk})_{,s} = -2C^{jk}_s.$$

Thus

$$g^{jk} C_{jkl,s} = I_{l,s} + 2C^{jk}_s C_{jkl}.$$

Multiplying (4.1) with $\bar{g}^{jk}$ implies that

$$\bar{J}_l = J_l + \frac{1}{2} y_i g^{jk} Q^i_{jkl}, \tag{4.10}$$

where

$y_i g^{jk} Q^i_{jkl} = 2\sigma_0 I_l + 2\sigma^s I_s y_l + 2F^2\sigma^s(I_{l,s} + 2C^{jk}_{s} C_{jkl}) - 2F^2\sigma^s(C^k_{\ ml} C^m_{\ sk} + C^j_{\ ml} C^m_{\ sj} + I_m C^m_{\ sl})$.

By Theorem 4.1, we get the proof. □

5. Conformal Transformation of H-Curvature

In this section, we find the necessary and sufficient condition under which the conformal transformation preserves the H-Curvature $\mathbf{H}$.

Theorem 5.1. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma}F(x,y)$, then the conformal transformation preserves the H-curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation:*

$$\begin{aligned} Q^m_{mij|s} y^s = \sigma^p \Big[& 2F^2 E_{ij,p} + 2(E_{ip}y_j + E_{jp}y_i) - 2F^2(E_{im}C^m_{\ pj} + E_{jm}C^m_{\ pi}) \\ & - F^2 Q^m_{mij,p} - (Q^m_{mip}y_j + Q^m_{mjp}y_i) + F^2(Q^m_{mis}C^s_{\ pj} + Q^m_{mjs}C^s_{\ pi})\Big]. \end{aligned} \tag{5.1}$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{H}} = \mathbf{H}$.

Proof. We have

$$\bar{H}_{ij} = \bar{E}_{ij||s} y^s = (E_{ij} - \frac{1}{2}Q^m_{mij})_{||s} y^s = E_{ij||s}y^s - \frac{1}{2}Q^m_{mij||s}y^s, \tag{5.2}$$

where "$||$" denotes the horizontal derivation with respect to the Berwald connection of $\bar{F}$. We get

$$\begin{aligned} E_{ij||s} &= \frac{\partial E_{ij}}{\partial x^s} - \bar{G}^m_s \frac{\partial E_{ij}}{\partial y^m} - E_{im}\bar{G}^m_{js} - E_{jm}\bar{G}^m_{is} \\ &= E_{ij|s} - \Big(P_s y^m + P\delta^m_s - Q^m_s\Big)\frac{\partial E_{ij}}{\partial y^m} - E_{im}(P_{js}y^m + P_j\delta^m_s + P_s\delta^m_j - Q^m_{js}) \\ & \qquad - E_{jm}(P_{is}y^m + P_i\delta^m_s + P_s\delta^m_i - Q^m_{is}). \end{aligned} \tag{5.3}$$

Then

$$\begin{aligned} \bar{H}_{ij} &= H_{ij} - 2(Py^m - Q^m)\frac{\partial E_{ij}}{\partial y^m} - E_{im}(P\delta^m_j - Q^m_j) - E_{jm}(P\delta^m_i - Q^m_i) - \frac{1}{2}Q^m_{mij||s}y^s \\ &= H_{ij} + \sigma^p\Big[F^2\frac{\partial E_{ij}}{\partial y^p} + (E_{ip}y_j + E_{jp}y_i) - F^2(E_{im}C^m_{\ pj} + E_{jm}C^m_{\ pi})\Big] - \frac{1}{2}Q^m_{mij||s}y^s. \end{aligned} \tag{5.4}$$

Now, we are going to compute $Q^m_{mij||s}y^s$. For simplicity, let us put

$$Q^m_m := X.$$

Then we have

$$\begin{aligned} X_{ij||s} &= \frac{\partial X_{ij}}{\partial x^s} - \bar{G}^m_s \frac{\partial X_{ij}}{\partial y^m} - X_{im}\bar{G}^m_{js} - X_{jm}\bar{G}^m_{is} \\ &= X_{ij|s} - \Big(P_s y^m + P\delta^m_s - Q^m_s\Big)\frac{\partial X_{ij}}{\partial y^m} - X_{im}(P_{js}y^m + P_j\delta^m_s + P_s\delta^m_j - Q^m_{js}) \\ & \qquad - X_{jm}(P_{is}y^m + P_i\delta^m_s + P_s\delta^m_i - Q^m_{is}). \end{aligned} \tag{5.5}$$

Thus

$$X_{ij||s}y^s \;=\; X_{ij|s}y^s + 2Q^m\frac{\partial X_{ij}}{\partial y^m} + X_{im}Q^m_j + X_{jm}Q^m_i.$$

It follows that

$$Q^m_{mij||s}y^s = Q^m_{mij|s}y^s + \sigma^m\Big[F^2\frac{\partial Q^p_{pij}}{\partial y^m} + (y_iQ^p_{pjm} + y_jQ^p_{pim}) - F^2(Q^s_{sip}C^p{}_{mj} + Q^s_{sjp}C^p{}_{mi})\Big]. \tag{5.6}$$

By (5.4) and (5.6), we get

$$\begin{aligned}\bar{H}_{ij} = H_{ij} + \sigma^p\Bigg[&F^2\frac{\partial E_{ij}}{\partial y^p} + (E_{ip}y_j + E_{jp}y_i) - F^2(E_{im}C^m_{\ pj} + E_{jm}C^m_{\ pi})\\ &-\frac{1}{2}\Big\{F^2\frac{\partial Q^m_{mij}}{\partial y^p} + (Q^m_{mip}y_j + Q^m_{mjp}y_i) - F^2(Q^m_{mis}C^s{}_{pj} + Q^m_{mjs}C^s{}_{pi})\Big\}\Bigg] - \frac{1}{2}Q^m_{mij|s}y^s.\end{aligned} \tag{5.7}$$

Let us put

$$Q^m_{mijp} := \frac{\partial Q^m_{mij}}{\partial y^p}.$$

Then, we get the proof. □

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