

ON SUBSPACE BALANCED CONVEX-CYCLIC OPERATORS

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ABSTRACT. Let X be a separable Banach space and M be a subspace of X . A bounded Linear operator T on X is subspace balanced convex-cyclic for a subspace M , if there exists a vector $x \in X$ such that the intersection of balanced convex hull of $\text{orb}(T, x)$ with M is dense in M . We give an example of subspace balanced convex-cyclic operator that is not balanced convex-cyclic. Also we give an improvement of the Kitai-like criterion for subspace balanced convex-cyclicity and bring on with the Hahn-Banach characterization for subspace balanced convex-cyclicity.

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1. INTRODUCTION

Dynamics of linear operators are mainly with the behaviour of them. The study of subspace hypercyclic operators was started by Mador and Martinez in [6].

Convex-cyclic operators was introduced by Rezaei in [7]. The study of subspace convex-cyclic operators was started in [1]. Baseri, Kashkooly and Rezaei start balanced convex-cyclic operators in [3].

A bounded linear operator T on Banach space X is balanced convex-cyclic (see [3]) if there exists a vector $x \in X$ such that the balanced convex hull of $\text{orb}(T, x)$ is dense in X . The vector x is said to be balanced convex-cyclic vector for T .

In [3] it is mentioned that between a set and its linear span there is a balanced convex hull, from this we get that every hypercyclic operator is balanced convex-cyclic and every balanced convex-cyclic operator is cyclic. In this paper we introduce and study the subspace balanced convex-cyclic operators.

2. BASIC DEFINITIONS AND THEOREMS

Let H be a real or complex separable Hilbert space. Let M be a subspace of H that is closed and $B(H)$ be the algebra of all linear continuous operators on H .

Definition 2.1. The set of all balanced convex polynomials is define as $\mathcal{BCP} = \{P(z) = \sum_{j=0}^n a_j z^j : \sum_{j=0}^n |a_j| \leq 1\}$.

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Definition 2.2. Let $T \in B(H)$ and let M be a non-zero subspace of H . T is said to be subspace balanced convex-cyclic operator, if there exists $x \in H$ such that $BC - orb(T, x) \cap M$ is dense in M , where;

$$\begin{aligned} BC - orb(T, x) &= \{P(T)x : P \text{ is balanced convex polynomial}\} \\ &= \{P(T)x : P(T) = a_0 + a_1T + a_2T^2 \dots + a_nT^n, n \in \mathbb{N}, \sum_{j=0}^n |a_j| \leq 1\} \end{aligned}$$

Such a vector x is said to be a subspace balanced convex-cyclic vector. we will use M -balanced convex-cyclic instead of subspace balanced convex-cyclic. we define $BC(T, M) := \{x \in H : BC - orb(T, x) \cap M \text{ is dense in } M\}$ as the set of all subspace balanced convex-cyclic vectors for M .

Proposition 2.3. Let $T : X \rightarrow X$ and $S : Y \rightarrow Y$ be bounded operators, If $S \oplus T$ is hypercyclic then so are S and T .

Proof. See [5] □

Proposition 2.4. Let X be a Banach space and $T \in B(X)$ be a balanced convex-cyclic operator, then

1. $\|T\| > 1$.
2. $\sup\{\|T^n\| : n \geq 1\} = +\infty$.
3. $\sup\{\|T^{*n}\Lambda\| : n \in \mathbb{N}\} = +\infty$, for every $\Lambda \neq 0$ in X^* .
4. every component of $\sigma(T)$ must intersect the set $\mathbb{C} \setminus \mathbb{D}$.
5. If X is an infinite-dimensional complex Banach space, then T is not compact.

Proof. See [3] □

Example 2.5. Let T be a balanced convex-cyclic operator on \mathcal{H} and I be the identity operator on \mathcal{H} . Then $T \oplus I : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}$ is subspace balanced convex-cyclic operator for subspace $M = \mathcal{H} \oplus \{0\}$ with subspace balanced convex-cyclic vector $x \oplus 0$, moreover $T \oplus I$ is not balanced convex-cyclic operator.

Proof. Since T is balanced convex-cyclic operator on \mathcal{H} , so there exists $x \in \mathcal{H}$ such that $BC - orb(T, x)$ is dense in \mathcal{H} . let $M = \mathcal{H} \oplus \{0\} \subseteq \mathcal{H} \oplus \mathcal{H}$ and $m = x \oplus 0$ thus, $BC - orb(T \oplus I, m) = \{P(T \oplus I)m : P \text{ is a balanced convex polynomial}\} = \{P(T)x \oplus 0 : P \text{ is a balanced convex polynomial}\} \subseteq \mathcal{H} \oplus \{0\} = M$.

Since $BC - orb(T, x)$ is dense in \mathcal{H} , so $BC - orb(T \oplus I, m) \cap M$ is dense in M . Therefore $T \oplus I$ is a subspace balanced convex-cyclic operator.

Assume that $T \oplus I$ is a subspace balanced convex-cyclic on $\mathcal{H} \oplus \mathcal{H}$, then by proposition 2.3, the identity operator must be balanced convex-cyclic on $\{0\}$, which is impossible.

Because $\|I\| = 1$ and by proposition 2.4 we get a contradiction. □

We will define subspace balanced convex-transitive operators and we will show that they will be subspace balanced convex-cyclic operators.

We will use the idea from [4,5,6] changing them for balanced convex polynomial spans and generalizing them.

Definition 2.6. Let $T \in B(\mathcal{H})$ and M be a non-zero subspace of \mathcal{H} . T is said to be M -balanced convex-transitive with respect to M if for all non-empty sets $U, V \subseteq M$ both

are relatively open, there exists a balanced convex–polynomial P such that $P(T)(U) \cap V \neq \emptyset$ or $U \cap P(T)^{-1}(V) \neq \emptyset$ contains a relatively open non–empty subset of M .

Proposition 2.7. Let $T \in B(\mathcal{H})$ and M be a non–zero subspace of \mathcal{H} .

Then $BC(T, M) = \bigcap_{j=1}^{\infty} \bigcup_{P \in \mathcal{BCP}} P(T)^{-1}(V_j)$ where \mathcal{BCP} is the collection of all balanced convex polynomials and $\{V_j\}$ countable open basis for relative topology of M as a subspace of \mathcal{H} .

Proof. If $x \in \bigcap_{j=1}^{\infty} \bigcup_{P \in \mathcal{BCP}} P(T)^{-1}(V_j)$ then for all $j=1,2,\dots$ there exists a balanced convex polynomial P such that $x \in P(T)^{-1}(V_j)$ if and only if $P(T)x \in V_j$. But since $\{V_j\}$ is a basis for the relative topology of M , this occurs if and only if $BC - orb(T, M) \cap M$ is dense in M , that is $x \in BC(T, M)$. \square

Lemma 2.8. Let $T \in B(\mathcal{H})$ and M be a non–zero subspace of \mathcal{H} . Then the following are equivalent:

- (1) T is M –balanced convex–transitive with respect to M .
- (2) For each relatively open subsets U and V of M , there exists $P \in \mathcal{BCP}$ such that $P(T)^{-1}(U) \cap V$ is relatively open subset in M .
- (3) For each relatively open subsets U and V of M , there exists $P \in \mathcal{BCP}$ such that $P(T)^{-1}(U) \cap V \neq \emptyset$ and $P(T)(M) \subseteq M$.

Proof. (3) \implies (2) Since $P(T) : M \rightarrow M$ is continuous and V is relatively open in M , then $P(T)^{-1}(U)$ is also relatively open in M . Now, if V be any open subset of M , then $P(T)^{-1}(U) \cap V$ is open and $P(T)^{-1}(U) \cap V \subseteq M$.

(2) \implies (1) Since for each relatively open subsets U and V , $P(T)^{-1}(U) \cap V$ is relatively open subset in M , so $P(T)^{-1}(U) \cap V \neq \emptyset$ and $P(T)^{-1}(U) \cap V$ is open in M .

(1) \implies (3) By definition of M –balanced convex–transitive, there exists U and V relatively open subsets in M , such that $W = P(T)^{-1}(U) \cap V \neq \emptyset$ and W is relatively open in M , and $W \subseteq P(T)^{-1}(U)$. So $P(T)(W) \subseteq U$ and $U \subseteq M$, therefore we have $P(T)(W) \subseteq M$. We must show that $P(T)(M) \subseteq M$.

Let $x \in M$ and $x_0 \in W$, since W is relatively open in M , so there exists $r > 0$ such that $x_0 + rx \in W$.

Since $P(T)(W) \subseteq M$, so $P(T)(x_0 + rx) = P(T)(x_0) + rP(T)x \in M$.

Thus $P(T)(x_0) \in M$ and since M is subspace so $r^{-1}(-P(T)(x_0) + P(T)(x_0) + rP(T)x) \in M$, that is $P(T)(x) \in M$. This is true for any $x \in M$, hence for $P(T)(x) \in M$, so $P(T)(M) \subseteq M$. \square

Theorem 2.9. Let $T \in B(\mathcal{H})$ and M be a non–zero subspace of \mathcal{H} . If T is M –balanced convex–transitive, then T is M –balanced convex–cyclic.

Proof. By proposition 2.7 and lemma 2.8 proof is clear. \square

3. SUBSPACE BALANCED CONVEX–CYCLIC CRITERION

In this section similar to [3] we give easily applicable Kitai-like criterion for an operator to be subspace balanced convex–cyclic. Also we will relate it with invariant subspaces and we will see that the converse of theorem 2.9 in general is not true.

Proposition 3.1. Let $T \in B(\mathcal{H})$ and let M be a non–zero subspace of \mathcal{H} . If there exist X and Y , dense subsets of M such that for every $x \in X$ and $y \in Y$ there exists a sequence $\{P_k\}_{k=1}^{\infty}$ of balanced convex polynomials such that;

- (1) $P_k(T)y \rightarrow 0$; $\forall y \in Y$.
- (2) For each $x \in X$, there exists a sequence $\{y_k\} \subseteq M$, such that $y_k \rightarrow 0$ and $P_k(T)y_k \rightarrow x$.
- (3) M is an invariant subspace for $P_k(T)$ for all $k \geq 1$.

Then T is M –balanced convex–cyclic operator.

Proof. We will use lemma 2.8 and theorem 2.9.

Let U and V be non–empty relatively open subsets of M . We will show that there exists $k \geq 1$ such that $P_k(T)(U) \cap V \neq \emptyset$. Since X, Y are dense in M , there exists $v \in Y \cap V$ and $u \in X \cap U$. Since U and V are relatively open, there exists $\varepsilon > 0$ such that the M –ball centered at v of radius ε is contained in V and the M –ball centered at u of radius ε is contained in U .

By hypothesis, for $u \in X$ and $v \in Y$, we can choose k large enough such that there exists $y_k \in M$ with $\|P_k(T)v\| < \frac{\varepsilon}{2}$, $\|y_k\| < \varepsilon$ and $\|P_k(T)y_k - u\| < \frac{\varepsilon}{2}$. So we have;

a) Since $v \in M$ and $y_k \in M$, so $v + y_k \in M$. Also since $\|(v + y_k) - v\| = \|y_k\| < \varepsilon$ it follows that $v + y_k$ is in M –ball centered at v of radius ε and hence $v + y_k \in V$.

b) Since v and y_k are in M and M is invariant under $P_k(T)$, it follows that $P_k(T)(v + y_k) \in M$. Also $\|P_k(T)(v + y_k) - u\| \leq \|P_k(T)v\| + \|P_k(T)y_k - u\| < \varepsilon$.

Hence $P_k(T)(v + y_k)$ is in the M –ball centered at u of radius ε and thus $P_k(T)(v + y_k) \in U$. So by (a) and (b), T is M –balanced convex–transitive and by theorem 2.9 we get that $v + y_k \in P_k(T)^{-1}(U) \cap V$, thus $P_k(T)^{-1}(U) \cap V \neq \emptyset$ which means that T is M –balanced convex–cyclic operator. \square

4. HAHN–BANACH CHARACTERIZATION

A necessary and sufficient condition for an operator so as to have a balanced convex–cyclic vector was established in [3].

Now in this section we give a similar necessary and sufficient condition for subspace balanced convex–cyclic operators. Also we prove some properties about subspace balanced convex–cyclic operators.

Theorem 4.1. (Hahn–Banach characterization for subspace balanced convex–cyclicity)

Let X be a separable infinite dimensional Banach space and M be a non–trivial closed subspace of X and $T \in B(X)$. Then for a vector $x \in X$ the following conditions are equivalent:

- (1) The vector x is an M –balanced convex–cyclic vector for T .
- (2) For every non–zero functional $f \in M^*$ we have $\sup\{|f(P(T)x)| : P \text{ is balanced convex polynomial and } P(T)x \in M\} = +\infty$.

Proof. Let $S = BC - orb(T, x) \cap M$. So S is a balanced convex set and the proof is similar to proposition 2.2 in [3] and so we omit it. \square

Theorem 4.2. Let X be a separable infinite dimensional Banach space. If T is M -balanced convex-cyclic operator, then;

- (1) $\|T\| > 1$.
- (2) $\sup\{\|T^{*n}f\| : n \in \mathbb{N}\} = +\infty$ for each non-zero $f \in M^*$.

Proof. (1) If $\|T\| \leq 1$, then for each $x \in X$, $BC - orb(T, x) \cap M$ is norm bounded and can not dense in M . So T is not M -balanced convex-cyclic which is a contradiction.

(2) By way of contradiction, suppose there exists a non-zero linear functional $f \in M^*$ such that $\|T^{*n}f\| \leq b$ for some positive number b and all $n \geq 1$. Let $x \in X$ so $|f(T^n x)| = |(T^{*n}f)x| \leq \|T^{*n}f\| \|x\| \leq b \|x\|$. Hence for every balanced convex polynomial P , we have $|f(P(T)x)| \leq b \|x\|$. So $BC - orb(T, x) \cap M$ can not dense in M and T is not M -balanced convex-cyclic which is a contradiction. □

Proposition 4.3. If $T \in B(X)$ is an M -balanced convex-cyclic operator, then $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$.

Proof. Let x be a subspace balanced convex-cyclic vector for T and λ be an eigenvalue for T^* . So there exists a non-zero bounded linear functional f on X such that $T^* \circ f = \lambda f$.

Hence $f(T^n x) = T^{*n}(f(x)) = \lambda^n f(x)$. So for every balanced convex polynomial P we have $f(P(T)x) = P(\lambda)f(x)$ and consequently $\sup_{P \in \mathcal{BCP}} |f(P(T)x)| = \sup_{P \in \mathcal{BCP}} |P(\lambda)| \|f\| = +\infty$ if and only if $|\lambda| > 1$. This completes the proof. □

As a result of theorem 4.1 we have the next corollary.

Corollary 4.4. If $T \in B(X)$ is an M -balanced convex-cyclic operator, then T has dense range.

The next theorem similar to theorem 2.6 in [2] shows that if an operator $T \in B(X)$ be balanced convex-cyclic, then there exists a special subspace M of X such that T is an M -balanced convex-cyclic operator.

Theorem 4.5. Let $x \in X$ be a balanced convex-cyclic vector for an operator $T \in B(X)$, then x is an M -balanced convex-cyclic vector for a non-trivial closed subspace M of X .

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