

**EQUIVALENCE RELATIONS ON BEST COAPPROXIMATION AND  
WORST COAPPROXIMATION**

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**ABSTRACT.** A kind of approximation, called best coapproximation was introduced and discussed in normed linear spaces by C. Franchetti and M. Furi in 1972. Subsequently, this study was taken up by several researchers in different abstract spaces.

In this paper, we define relations on best coapproximation and worst coapproximation. We show that these relations are equivalence relation. We obtain cosets sets of best coapproximation and worst approximation. We obtain some results on these sets, compactness and weakly compactness and define coqproximal and coqremotal.

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**1. Introduction and Background**

As a counter part to best approximation, a kind of approximation called best coapproximation was introduced in normed linear spaces by C. Franchetti and M. Furi [3] to study some characteristic properties of real Hilbert spaces.

Subsequently, this theory has been developed to a large extent in normed linear spaces and in Hilbert spaces by C. Franchetti and M. Furi, H. Mazaheri, P.L.Papini and I. Singer, Geetha S. Rao and by many others (see e.g. [3, 5, 6, 12, 13] and references cited therein).

In a series of papers, G. Albinus, G.G. Lorentz, T.D. Narang, G. Pantelidis, K. Schnatz, A.I. Vasilev and others (see e.g. [1, 4, 7, 11, 14, 16, 19] and references cited therein) have tried to extend various results on best approximation available in normed linear spaces to metric linear spaces. The situation in case of best coapproximation is somewhat different. Whereas some attempts have been made to discuss best coapproximation in metric linear spaces (see e.g. [9, 10]) but still in these spaces this theory is less developed as compared to the theory of best approximation. The present paper is also a step in this direction.

The paper mainly deals with some results on the existence and uniqueness of best coapproximation in quotient spaces when the underlying spaces are metric linear spaces. We also show how coqproximality is transmitted to and from quotient spaces. The results proved in the paper extend and generalize various known results on the subject.

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Let  $(X, \|\cdot\|)$  be a normed linear space,  $W$  a non-empty subset of  $X$ . A point  $y_0 \in W$  is said to be a best coapproximation point for  $x \in X$ , if

$$\|y - y_0\| \leq \|x - y\|,$$

Suppose  $g \in W$ , we set coset best coapproximation

$$R_g = \{x \in X : \|y - g\| \leq \|x - y\| \text{ for } y \in W\},$$

Let  $(X, \|\cdot\|)$  be a normed linear space,  $W$  a non-empty subset of  $X$ ,  $x \in X$  and  $0 \in W$ . We set

$$R_W(x) = \{g_0 \in W : \|y - g_0\| \leq \|x - y\| \text{ for all } y \in W\}.$$

The set  $W$  is coproximal if for all  $x \in X$ ,  $R_W(x)$  is non-empty. The set  $W$  is cochebyshev if for all  $x \in X$ ,  $R_W(x)$  is singleton.

Let  $X$  be a normed linear space and  $W$  a non-empty subset of  $X$ . A point  $q(x) \in W$  is said to be a cofarthest point for  $x \in X$  if

$$\|y - q(x)\| \geq \|x - y\|,$$

we set

$$G_g = \{x \in X : \|y - g\| \geq \|x - y\| \text{ for all } y \in W\},$$

Let  $(X, \|\cdot\|)$  be a normed linear space,  $W$  a non-empty subset of  $X$ ,  $x \in X$  and  $0 \in W$ . We have

$$G_W(x) = \{g_0 \in W : \|y - g_0\| \geq \|x - y\| \text{ for all } y \in W\}.$$

The set  $W$  is coremotal if for all  $x \in X$ ,  $G_W(x)$  is non-empty. The set  $W$  is uniquely coremotal if for all  $x \in X$ ,  $G_W(x)$  is singleton.

**Definition 1.1.** Let  $X$  be a normed linear space,  $W$  a subset of  $X$ .

i)  $W$  is called qcproximal if for every  $x \in X$ , the set  $(x - W) \cap R_0$  is a non-empty compact subset of  $X$ .

ii)  $W$  is called qcoremotal if for every  $x \in X$ , the set  $(x - W) \cap G_0$  is a non-empty compact subset of  $X$ .

## 2. EQUIVALANCE RELATIONS ON COAPPROXIMATE SETS

In this section we define two equivalence relations on coapproximate sets. We obtain some results on these relations.

**Definition 2.1.** Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a coproximal subset in  $X$  and  $x, y \in X$ . We define two relations on  $X$ , with

(i)

$$x \triangleright_1 y \Leftrightarrow \forall w \in W : \|x - w\| = \|y - w\|.$$

(ii)

$$x \triangleright_2 y \Leftrightarrow \text{for some } g \in W : g \in R_W(x) \cap R_W(y).$$

We denote the equivalence class of  $x \in X$  under relation  $\triangleright_1(\triangleright_2)$  by  $[x]_1([x]_2)$

**Theorem 2.2.** Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a proximal subset in  $X$ . The relations  $\triangleright_1$  is equivalence relation.

*Proof.* These relation is reflexive and symmetric. We show that transitivity relation  $\triangleright_1$ . For all elements  $a, b, c \in X$ , if  $a \triangleright_1 b$  and  $b \triangleright_1 c$ , then  $\forall w \in W : \|a - w\| = \|b - w\| = \|c - w\|$ . It follows that  $a \triangleright_1 c$ .  $\square$

**Theorem 2.3.** *Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a proximal subset in  $X$ . The relations  $\triangleright_2$  is equivalence relation.*

*Proof.* These relation is reflexive and symmetric. We show that trnsitivity relation  $\triangleright_1$ . For all elements  $a, b, c \in X$ , if  $a \triangleright_2 b$  and  $b \triangleright_2 c$ , then  $R_W(a) = R_W(b) = R_W(c)$ . It follows that  $a \triangleright_2 c$ .  $\square$

**Theorem 2.4.** *Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a coproximal subset in  $X$ . Then for every  $x \in X$ , there exists a  $g \in W$  such that  $[x]_1 \subseteq R_g$ .*

*Proof.* Suppose  $x \in X$ , since  $W$  is coproximal, there exists a  $g \in R_W(x)$ . Now

$$\begin{aligned} y \in [x]_1 &\iff x \triangleright_1 y \\ &\iff \forall w \in W : \|x - w\| = \|y - w\|. \end{aligned}$$

Also  $\forall w \in W : \|w - g\| \leq \|w - x\|$ . Therefore

$$[x]_1 \subseteq R_g.$$

$\square$

**Theorem 2.5.** *Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a coproximal subset in  $X$ . Then for every  $x \in X$ , there exists a  $g \in W$  such that  $[x]_2 \subseteq R_g$ .*

*Proof.* Suppose  $x \in X$ , Since  $W$  is coproximal, there exists  $g \in W$  such that  $g \in R_W(x)$ . Now

$$\begin{aligned} y \in [x]_2 &\Rightarrow x \triangleright_2 y \\ &\Rightarrow R_W(x) = R_W(y) \\ &\Rightarrow g \in R_W(y) \\ &\Rightarrow y \in R_g. \end{aligned}$$

$\square$

**Example 2.6.** Suppose  $X = \mathbb{R}^2$  with the norm  $\|(x, y)\| = \sqrt{x^2 + y^2}$  and  $W = \{(x, 0) : x \in \mathbb{R}\}$ .  $W$  is coproximal, becasue if  $(x, y) \in \mathbb{R}^2$ , we set  $g_0 = (0, y)$ . It is clear that  $g_0 \in R_W((x, y))$ . Also

$$\{(a, 1) : a \in \mathbb{R}\} \subseteq [(0, 1)]_1$$

**Theorem 2.7.** *Let  $(X, \|\cdot\|)$  be a normed linear space and  $W$  a cochebyshev subspace of  $X$ ,  $x, y \in X$  and  $g_0 \in W$ . If  $g_0 = R_W(x)$  and  $y \in [g_0, x]$ , then  $g_0 = R_W(y)$ . (where  $[g_0, x] = \{\lambda g_0 + (1 - \lambda)x : \lambda \geq 0\}$ .)*

*Proof.* Since  $g_0 = R_W(x)$  and  $y \in [g_0, x]$ , for some  $\lambda > 0$ ,  $y = \lambda g_0 + (1 - \lambda)x$  and for all  $w \in W$ , we have  $\|g_0 - w\| \leq \|x - w\|$ . For all  $w \in W$

$$\begin{aligned} \|y - w\| &= \|\lambda g_0 + (1 - \lambda)x - \lambda w - (1 - \lambda)w\| \\ &= \|(1 - \lambda)(x - w)\| + \lambda\|g_0 - w\| \\ &\geq (1 - \lambda)\|w - g_0\| + \lambda\|g_0 - w\| \\ &\geq \|w - g_0\| \end{aligned}$$

Therefore  $g_0 = R_W(y)$ . .

$\square$

**Definition 2.8.** ([14],[17]) For any two elements  $x$  and  $y$  in normed linear space  $X$ ,  $x$  is said to be orthogonal to  $y$  in the sense of Birkhorff-James, written as  $x \perp y$ , if  $\|x + \lambda y\| \geq \|x\|$  for every real scalar  $\lambda$ .

Let  $W$  is closed subspace of a normed space  $X$ . From [2],

$$g_0 \in R_W(x) \iff W \perp x - g_0.$$

Let  $W$  is closed subspace of a normed space  $X$ . We set

$${}^\perp W = \{x \in X : W \perp x\}.$$

**Theorem 2.9.** Let  $(X, \|\cdot\|)$  be a normed linear space and  $W$  a coproximinal subspace of  $X$ ,  ${}^\perp W$  a convex set and  $x, y \in X$ . If  $x \triangleright_2 y$ , then  $x - y \in {}^\perp W$ .

*Proof.* Suppose  $x \triangleright_2 y$ , then there exists a  $g_0 \in R_W(x) \cap R_W(y)$ . Then  $x - g_0, y - g_0 \in {}^\perp W$ , therefore  $x - y = 2 \frac{x - g_0 + y - g_0}{2} \in {}^\perp W$ .  $\square$

**Theorem 2.10.** Let  $(X, \|\cdot\|)$  be a normed linear space. If  $W$  is a subset of  $X$ . Then

- i) for every  $g \in W$ ,  $R_g$  and  $G_g$  are closed set;
- ii) the set  $W$  is cochebyshev if and only if for every  $g_1, g_2 \in W$  and  $g_1 \neq g_2$ , we have  $R_{g_1} \cap R_{g_2} = \emptyset$ ;
- iii) the set  $W$  is uniquely coremotal if and only if for every  $g_1, g_2 \in W$  and  $g_1 \neq g_2$ , we have  $G_{g_1} \cap G_{g_2} = \emptyset$ ;
- iv) if  $W$  is cochebyshev,  $R_g \cap R_{-g} \neq \emptyset$  then  $g = 0$ ;
- v) if  $W$  is uniquely coremotal  $G_g \cap G_{-g} \neq \emptyset$  then  $g = 0$ .

*Proof.* (i) Suppose  $g \in W$  and  $\{x_n\} \subset R_g$  and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . Then  $\forall w \in W$ ,  $\|w - g\| \geq \|x_n - w\|$ . Therefore  $\forall w \in W$ ,  $\|w - g\| \geq \|x - w\|$ , it follows that  $x \in R_g$ . Similarly  $G_g$  is closed.

(ii) suppose  $W$  is cochebyshev,  $g_1, g_2 \in W$ ,  $g_1 \neq g_2$  and  $x \in R_{g_1} \cap R_{g_2}$ . Then  $g_1, g_2 \in R_W(x)$  and  $g_1 = g_2$ , that is a contraction.

On converse, if  $g_1, g_2 \in W$  and  $g_1 \neq g_2$  and  $R_{g_1} \cap R_{g_2} = \emptyset$ . Suppose for some  $x \in X$ , there exists  $h_1, h_2 \in R_W(x)$  and  $h_1 \neq h_2$ . Then  $x \in R_{h_1} \cap R_{h_2}$ . That is a contraction, it follows that  $W$  is cochebyshev.

iii) The prove is similar to (ii).

iv) There exists a  $x \in X$  such that  $x \in R_g \cap R_{-g}$ . Therefore  $g, -g \in R_W(x)$  and  $g = -g$ . It follows that for all  $g = 0$ .

v) There exists a  $x \in X$  such that  $x \in G_g \cap G_{-g}$ . Therefore  $g, -g \in G_W(x)$  and  $g = -g$ . It follows that for all  $g = 0$ .  $\square$

### 3. EQUIVALANCE RELATIONS ON WORST COAPPROXIMATE

In this section we define two equivalence relations on worst coapproximate. We obtains some results on these relations.

**Definition 3.1.** Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a coremotal subset in  $X$  and  $x, y \in X$ . We define two relations on  $X$ , with

(i)

$$x \triangleleft_1 y \iff \forall w \in W : \|x - w\| = \|y - w\|.$$

(ii)

$$x \triangleleft_2 y \iff \text{for some } g \in W : g \in G_W(x) \cap G_W(y).$$

We denoted the equivalence class of  $x \in X$  under relation  $\triangleleft_1(\triangleleft_2)$  by  ${}_1[x]({}_2[x])$

**Theorem 3.2.** *Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a proximal subset in  $X$ . The relations  $\triangleleft_1$  is equivalence relation.*

*Proof.* These relation is reflexive and symmetric. We show that trnsitivity relation  $\triangleleft_1$ . For all elements  $a, b, c \in X$ , if  $a \triangleleft_1 b$  and  $b \triangleleft_1 c$ , then  $\forall w \in W : \|a - w\| = \|b - w\| = \|c - w\|$ . It follows that  $a \triangleleft_1 c$ .  $\square$

**Theorem 3.3.** *Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a proximal subset in  $X$ . The relations  $\triangleleft_2$  is equivalence relation.*

*Proof.* These relation is reflexive and symmetric. We show that trnsitivity relation  $\triangleleft_1$ . For all elements  $a, b, c \in X$ , if  $a \triangleleft_2 b$  and  $b \triangleleft_2 c$ , then  $R_W(a) = R_W(b) = R_W(c)$ . It follows that  $a \triangleleft_2 c$ .  $\square$

**Theorem 3.4.** *Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a coproximal subset in  $X$ . Then for every  $x \in X$ , there exists a  $g \in W$  such that  $[x]_1 \subseteq P_g$ .*

*Proof.* Suppose  $x \in X$ , since  $W$  is coproximal, there exists a  $g \in R_W(x)$ . Now

$$\begin{aligned} y \in [x]_1 &\iff x \triangleright_1 y \\ &\iff \forall w \in W : \|x - w\| = \|y - w\|. \end{aligned}$$

Also  $\forall w \in W : \|w - g\| \leq \|w - x\|$ . Therefore

$$[x]_1 \subseteq R_g.$$

$\square$

**Theorem 3.5.** *Let  $(X, \|\cdot\|)$  be a normed space,  $W$  a coproximal subset in  $X$ . Then for every  $x \in X$ , there exists a  $g \in W$  such that  ${}_2[x] \subseteq R_g$ .*

*Proof.* Suppose  $x \in X$ , Since  $W$  is coproximal, there exists  $g \in W$  such that  $g \in R_W(x)$ . Now

$$\begin{aligned} y \in {}_2[x] &\Rightarrow x \triangleleft_2 y \\ &\Rightarrow R_W(x) = R_W(y) \\ &\Rightarrow g \in R_W(y) \\ &\Rightarrow y \in R_g. \end{aligned}$$

$\square$

**Example 3.6.** Suppose  $X = \mathbb{R}^2$  with the norm  $\|(x, y)\| = \sqrt{x^2 + y^2}$  and  $W = \{(x, 0) : x \in \mathbb{R}\}$ .  $W$  is coproximal, becasue if  $(x, y) \in \mathbb{R}^2$ , we set  $g_0 = (0, y)$ . It is clear that  $g_0 \in R_W((x, y))$ . Also

$$\{(a, 1) : a \in \mathbb{R}\} \subseteq [(0, 1)]_1$$

**Theorem 3.7.** *Let  $(X, \|\cdot\|)$  be a normed linear space and  $W$  a cochebyshev subspace of  $X$ ,  $x, y \in X$  and  $g_0 \in W$ . If  $g_0 = R_W(x)$  and  $y \in [g_0, x]$ , then  $g_0 = R_W(y)$ . (where  $[g_0, x] = \{\lambda g_0 + (1 - \lambda)x : \lambda \geq 0\}$ .)*

*Proof.* Since  $g_0 = R_W(x)$  and  $y \in [g_0, x)$ , for some  $\lambda > 0$ ,  $y = \lambda g_0 + (1 - \lambda)x$  and for all  $w \in W$ , we have  $\|g_0 - w\| \leq \|x - w\|$ . For all  $w \in W$

$$\begin{aligned} \|y - w\| &= \|\lambda g_0 + (1 - \lambda)x - \lambda w - (1 - \lambda)w\| \\ &= \|(1 - \lambda)(x - w)\| + \lambda\|g_0 - w\| \\ &\geq (1 - \lambda)\|w - g_0\| + \lambda\|g_0 - w\| \\ &\geq \|w - g_0\| \end{aligned}$$

Therefore  $g_0 = R_W(y)$ . . □

#### 4. QCOPROXIMAL AND QCOREMOTAL

In this section we are bring some propeties of qcoproximal and qcoemotal.

**Theorem 4.1.** *Let  $X$  be a Banach space,  $W$  a coproximal subset of  $X$  and for every  $g \in W$ ,  $W - g = W$ . If for all  $g \in W$ ,  $R_0$  is compact. Then  $R_W(x)$  is compact and  $W$  is qcoproximal.*

*Proof.* Suppose  $g \in W$  and  $R_g$  is compact. Since  $W - g = W$  we have  $R_g = -g + R_0$ , because

$$\begin{aligned} x \in R_g &\iff \|g - w\| \leq \|w - x\| \quad \forall w \in W \\ &\iff \|w - g\| \leq \|w - g + x + g\| \\ &\text{iff } \|w\| \leq \|x + g - w\| \\ &\iff x + g \in R_0 \\ &\iff x \in -g + R_0 \end{aligned}$$

Since  $R_g$  is compact, then  $R_0$  is compact and for all  $x \in X$ ,  $(x - W) \cap R_0$  is compact. Therefore  $W$  is quasi proximal. If  $\{g_n\}_{n \geq 1}$  is a sequence in  $P_W(x)$ , then for all  $n \geq 1$ ,  $x \in P_{g_n}$ . Then  $\{x - g_n\}_{n \geq 1}$  is a sequence in  $P_0$ . Since  $P_0$  is compact, there exists a convergence subsequence  $\{x - g_{n_k}\}_{k \geq 1}$  in  $P_0^c$ , therefore there exists a convergence subsequence  $\{g_{n_k}\}_{k \geq 1}$  in  $R_W(x)$ . Therefore  $R_W(x)$  is compact. □

**Theorem 4.2.** *Let  $X$  be a Banach space,  $W$  a coremotal subset of  $X$  and for every  $g \in W$ ,  $W - g = W$ . If for every  $g \in W$ ,  $F_g$  is compact. Then  $W$  is qcoremotal.*

*Proof.* Suppose  $g \in W$  and  $F_g$  is compact. Since  $g - W = W$  we have  $G_g = -g + G_0$ , because

$$\begin{aligned} x \in G_g &\iff \|g - w\| \geq \|w - x\| \quad \forall w \in W \\ &\iff \|w - g\| \geq \|w - g + x + g\| \\ &\text{iff } \|w\| \geq \|x + g - w\| \\ &\iff x + g \in G_0 \\ &\iff x \in -g + G_0 \end{aligned}$$

Since  $G_g$  is compact, then  $G_0$  is compact and for all  $x \in X$ ,  $(x - W) \cap G_0$  is compact. Therefore  $W$  is qcoremotal. if  $\{g_n\}_{n \geq 1}$  is a sequence in  $G_W(x)$ , then for all  $n \geq 1$ ,  $x \in G_{g_n}$ . Then  $\{x - g_n\}_{n \geq 1}$  is a sequence in  $G_0$ . Since  $G_0$  is compact, there exists a convergence subsequence  $\{x - g_{n_k}\}_{k \geq 1}$  in  $G_0$ , therefore there exists a convergence subsequence  $\{g_{n_k}\}_{k \geq 1}$  in  $G_W(x)$ . Therefore  $G_W(x)$  is compact. □

**Theorem 4.3.** *Let  $X$  be a Banach space,  $W$  a coproximinal subset of  $X$  and for every  $x \in X$  the set  $R_W(x)$  is compact, Then  $W$  is qcproximinal.*

*Proof.* for every  $x \in X$  the set  $R_W(x)$  is compact,  $x \in X$  and  $\{x_n\} \subset (x - W) \cap R_0$ . Then  $\{x_n - x\} \subset W$  and for all  $w \in W$ , we have  $\|w\| \leq \|x_n - w\|$ . We set  $g_n = x_n - x$ , we have

$$\|g_n\| \leq \|x_n - g_n\|.$$

Since  $R_W(x)$  is compact. There exists a subsequence  $\{g_{n_k}\}$  such that for  $l_0 \in W$ ,

$$g_{n_k} \rightarrow l_0 \quad k \rightarrow \infty.$$

It follows that

$$x_{n_k} \rightarrow x + l_0 \quad k \rightarrow \infty.$$

□

**Theorem 4.4.** *Let  $X$  be a Banach space,  $W$  a coremotall subset of  $X$  and for every  $x \in X$  the set  $G_W(x)$  is compact, Then  $W$  is qcoremotal.*

*Proof.* or every  $x \in X$  the set  $G_W(x)$  is compact,  $x \in X$  and  $\{x_n\} \subset (x - W) \cap G_0$ . Then  $\{x_n - x\} \subset W$  and for all  $w \in W$ , we have  $\|w\| \leq \|x_n - w\|$ . We set  $g_n = x_n - x$ , we have

$$\|g_n\| \leq \|x_n - g_n\|.$$

Since  $G_W(x)$  is compact. There exists a subsequence  $\{g_{n_k}\}$  such that for  $l_0 \in W$ ,

$$g_{n_k} \rightarrow l_0 \quad k \rightarrow \infty.$$

It follows that

$$x_{n_k} \rightarrow x + l_0 \quad k \rightarrow \infty.$$

□

**Theorem 4.5.** *Let  $X$  be a Banach space and  $W$  a coproximinal hyperplane subspace of  $X$ . Then the following statements are equivalent:*

- i)  $W$  is qcproximinal,
- ii) for every  $g \in W$  and for every sequence  $\{x_n\}_{n \geq 1}$  with  $x_n \in R_g$  has a convergent subsequence.

*Proof.* i)  $\rightarrow$  ii). Since  $\text{codimen}(W) = 1$ , there exists a  $y_0 \in X$  such that  $X = y_0 + W$ . Also  $(y_0 - W) \cap R_0$  is nonempty and compact.

For  $g \in W$ , if the sequence  $\{x_n\}_{n \geq 1} \subseteq R_g$ . Then here exists a sequence  $\{g_n\}$  in  $W$  such that  $x_n - g = y_0 + g_n$ . It is clear  $\{y_0 + g_n\}$  is a sequence in  $(y_0 - W) \cap R_0$ . Therefore  $\{x_n - g\}$  and  $\{x_n\}$  has a convergent subsequence.

ii)  $\rightarrow$  i). If for every  $g \in W$  and sequence  $\{x_n\}_{n \geq 1}$  and  $x_n \in R_g$  has a convergent subsequence. Therefore  $R_g$  is compact, from Theorem 2.4.,  $W$  is qcproximinal. □

**Theorem 4.6.** *Let  $X$  be a Banach space and  $W$  a coremotal hyperplane subset of  $X$ . Then the following statements are equivalent:*

- i)  $W$  is qcoremotal,
- ii) for every  $g \in W$  and for every sequence  $\{x_n\}_{n \geq 1}$  with  $x_n \in G_g$  has a convergent subsequence.

*Proof.*  $i) \rightarrow ii)$ . Since  $\text{codim}(W) = 1$ , there exists a  $y_0 \in X$  such that  $X = y_0 + W$ . Also  $(y_0 - W) \cap G_0$  is nonempty and compact.

For  $g \in W$ , if the sequence  $\{x_n\}_{n \geq 1} \subseteq G_g$ . Then here exists a sequence  $\{g_n\}$  in  $W$  such that  $x_n - g = y_0 + g_n$ . It is clear  $\{y_0 + g_n\}$  is a sequence in  $(y_0 - W) \cap G_0$ . Therefore  $\{x_n - g\}$  and  $\{x_n\}$  has a convergent subsequence.

$ii) \rightarrow i)$ . If for every  $g \in W$  and sequence  $\{x_n\}_{n \geq 1}$  and  $x_n \in G_g$  has a convergent subsequence.  $\square$

**Theorem 4.7.** *Let  $X$  be a Banach space and  $W$  a coproximal subspace of  $X$ . Then the following statements are equivalent:*

*i)  $W$  is qcproximal.*

*ii) for every  $g \in W$ , for every subspace of  $X$  of form  $W_x = W + \text{span}\{x\}$  and for every sequence  $\{x_n\}_{n \geq 1} \subset W_x$  with  $x_n \in P_g^{W_x}$  has a convergent subsequence.*

*Proof.*  $i) \rightarrow ii)$  If  $W$  is qcproximal in  $X$ . Then  $W$ , is qcproximal in every  $W_x$  ( $x \in X \setminus W$ ), Since  $\text{codim}(W) = 1$  in every  $W_x$ . From Theorem 2.6, for every sequence  $\{x_n\}_{n \geq 1} \subset W_x$  with  $x_n \in P_g^{W_x}$  has a convergent subsequence.

$ii) \rightarrow i)$  Assume that we have (ii),  $\text{codim}(W) = 1$  in every  $W_x$ . Also  $W$  is coproximal in  $W_x$  and  $X = \cup_{x \in X \setminus W} W_x$ . It follows that  $W$  is qcproximal in  $X$ .  $\square$

**Theorem 4.8.** *Let  $X$  be a Banach space,  $W$  a coproximal subspace of  $X$ ,  $x \in X$ ,  $g \in W$ . Then the following statements are equivalent:*

*i)  $x \in R_g$ ,*

*ii) for  $w \in W$ ,  $\|w - g_0\| \leq \|x - w\|_{W^\perp}$ ,*

where

$$\|x - g_0\|_{W^\perp} = \sup\{|f(x - g_0)| : \|f\| \leq 1, f \in W^\perp\}.$$

*Proof.*  $i) \rightarrow ii)$  Suppose  $x \in R_g$ , from Lemma 1.1, for all  $w \in W$ , there exists a  $f^w \in X^*$  such that  $\|f^w\| = 1$ ,  $f^w(x) = f^w(g)$  and  $f^w(g) \geq \|g\|$ . Therefore  $|f(x - g_0)| \geq f(x - g_0) \geq \|x - g_0\| - \epsilon$  and  $\|x - g_0\|_{W^\perp} \geq \|x - g_0\| - \epsilon$ .

$ii) \rightarrow i)$  Suppose  $f \in W^\perp$ ,  $\|f\| \leq 1$  and  $h \in W$ , then  $|f(x - g_0)| = |f(x - h)| \leq f\|x - h\| \leq \|x - h\|$ . Therefore

$$\|x - g_0\|_{W^\perp} \leq \|x - h\|.$$

We have  $\|x - g_0\| \leq \|x - g_0\|_{W^\perp} \leq \|x - h\| + \epsilon$ ,  $x \in P_g$ .  $\square$

**Theorem 4.9.** *Let  $X$  be a Banach space,  $W$  a coproximal subspace of  $X$ ,  $x \in X$ ,  $g \in W$ ,  $E \subseteq X$ . Then the following statements are equivalent:*

*i) for  $g \in W$ ,  $E \subseteq R_g$ ,*

*ii) for  $x \in E$ ,  $\|x - g_0\| \leq \|x - g_0\|_{W^\perp}$ .*

*Proof.*  $i) \rightarrow ii)$  Suppose  $g \in W$ , if  $x \in E$ , then  $x \in P_g$ . From Theorem 2.2,  $\|x - g_0\| \leq \|x - g_0\|_{W^\perp}$ .

$ii) \rightarrow i)$  Suppose  $x \in E$  then  $\|x - g_0\| \leq \|x - g_0\|_{W^\perp}$ , from Theorem 2.2,  $x \in P_{g_0}$ . Then from Lemma 1.1, there exists a  $f \in X^*$  such that  $\|f\| = 1$ ,  $f|_W = 0$  and  $f(x - g_0) \geq \|x - g_0\|$ . It follows that for  $g \in W$

$$f(x - g) = f(x) = f(x - g_0) \geq \|x - g_0\|_{W^\perp}.$$

From Lemma 1.1,  $x \in R_g$ .  $\square$

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