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Research Paper

GENERALIZED MAPPINGS RELATED TO HERMITE-HADAMARD INEQUALITY

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ABSTRACT. In this paper we introduce two new mappings in connection to Hermite-Hadamard type inequality. Then we presents some results concerning these mappings associated to the well-known Hermite-Hadamard integral inequality for preinvex functions.

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1. Introduction and preliminary

Let $I = [c, d]$ be an interval on the real line $\mathbb{R}$ $a, b \in I, a < b$, and $f : I \to \mathbb{R}$ be a convex function. We consider the well-known Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

In which both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

In order to provide various refinements of this result, S.S. Dragomir in [1] introduced the mapping $H : [0, 1] \to \mathbb{R}$, as follows

$$H(t) := \frac{1}{b-a}\int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)dx,$$

and established several results about Hermite-Hadamard inequality which some of the main results of H are given below.

Theorem 1.1. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then, we have*
(i) H is convex on $[0, 1]$.
(ii) One has the following bounds;

$$\inf_{t\in[0,1]} H(t) = H(0) = f\big(\frac{a+b}{2}\big) \tag{1.2}$$

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and

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.3}$$

(iii) *H increases monotonically on* $[0,1]$.
(iv) *The following inequalities hold*

$$f\big(\frac{a+b}{2}\big) \leq \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx$$
$$\leq \int_0^1 H(t)dt$$
$$\leq \frac{1}{2}\Big(f\big(\frac{a+b}{2}\big) + \frac{1}{b-a}\int_a^b f(x)dx\Big).$$

The corresponding double integral mapping $F:[0,1]\to\mathbb{R}$, about Hermite-Hadamard inequality was considered first in [2] and it was defined as

$$F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f\Big(tx+(1-t)y\Big)dxdy.$$

Some of the main results concerning this mapping are as follows.

Theorem 1.2. *Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function. Then we have*
(i) $F(\frac{1}{2}+\tau) = F(\frac{1}{2}-\tau)$ *for every* $\tau\in[0,\frac{1}{2}]$ *and*
$F(t) = F(1-t)$, *for all* $t\in[0,1]$
(ii) *F is a convex function on* $[0,1]$.
(iii) *We have the bounds*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)dx.$$

and

$$\inf_{t\in[0,1]} F(t) = F(\frac{1}{2})$$
$$= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\big(\frac{x+y}{2}\big)dxdy.$$

(iv) The following inequality holds

$$f\big(\frac{x+y}{2}\big) \leq F(\frac{1}{2})$$

(v) F decreases monotonically on $[0,\frac{1}{2}]$ and increases monotonically on $[0,\frac{1}{2}]$.
(v) *For every $t\in[0,1]$, the following inequality holds*

$$H(t) \leq F(t).$$

Numerous articles have appeared in the literature reflecting further applications and properties of mappings H, F, (see [3-8] and references therein). In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is the invex functions introduced by Hanson in [9]. Weir and Mond in [10] introduced the concept of preinvex functions and applied it to prove the sufficient optimality conditions and duality in nonlinear programming. There have been some works in the literature which are devoted to investigating preinvex functions (see [10-17]) and references therein).

Now, we recall some notions which will be used throughout the paper. A set $S \subseteq \mathbb{R}$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}$, if for every $x, y \in S$ and $t \in [0,1]$, $y + t\eta(x,y) \in S$. The mapping η is said to be satisfies the condition C if for every $x, y \in S$ and $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y),$$
$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y).$$

For every $x, y \in S$ and every $t_1, t_2 \in [0,1]$ from condition C we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y). \tag{1.4}$$

Recall that for every $x, y \in S$, the $\eta-$path P_{xy} is a subset of S defined by

$$P_{xy} := \{x + t\eta(x,y) |\ 0 \le t \le 1\}.$$

It is obvious that every convex set is invex with respect to the map $\eta(x,y) = x - y$, but there exist invex sets which are not convex. Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Then the function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0,1]$,

$$f(y + t\eta(x,y)) \le tf(x) + (1-t)f(y). \tag{1.5}$$

Every convex function is preinvex with respect to the map $\eta(x,y) = x - y$ but the converse does not holds. The Hermite-Hadamard's inequality for preinvex functions is introduced in [18] by M.A. Noor as,

$$f(a + \frac{1}{2}\eta(b,a)) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \le \frac{f(a) + f(b)}{2}, \tag{1.6}$$

where $a, b \in S$. Since then numerus articles have been published in this category see, for example (see [20-23] and references therein).

The analogue of the arithmetic mean in the context of finite measure spaces (X, Σ, μ) is the integral arithmetic mean, with, for a $\mu-$integrable function $g : X \to \mathbb{R}$ is the number

$$M_1(g) := \frac{1}{\mu(X)} \int_X g d\mu.$$

We recall the following Jensen's type inequality for preinvex functions from [19].

Theorem 1.3. *Let (X, Σ, μ) be a finite measure space and $g : X \to \mathbb{R}$ be a $\mu-$integrable function. Suppose that $S \subseteq \mathbb{R}$ is an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and S includes the image of g. If $f : S \to \mathbb{R}$ is a preinvex function then,*
(i) $M_1(g) \in S$.
(ii) If $\psi(x) := \eta(g(x), M_1(g))$ and $\psi(x) \neq 0$ for every $x \in X$, such that

$g(x) \neq M_1(g)$ then, there exists $K \in \mathbb{R}$ such that the following inequality holds

$$f\left(\frac{1}{\mu(X)}\int_X g d\mu\right) \leq \frac{1}{\mu(X)}\int_X (fog) d\mu - K\frac{1}{\mu(X)}\int_X \psi d\mu, \tag{1.7}$$

provided that ψ and fog are $\mu-$integrable.

In this paper corresponding to mappings H and F, we introduce two new mappings and establish several results in connection to Hermite-Hadamard's type inequality (1.6) for preinvex functions.

2. Main results

In this section we introduce two new mappings which are generalizations of mappings H and F. Motivated by [1], we define the mapping $\mathsf{H} : [0,1] \to \mathbb{R}$, as follow,

$$\mathsf{H}(t) := \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f\Big(a + \frac{1}{2}\eta(b,a) + t\eta(y, a + \frac{1}{2}\eta(b,a))\Big)dy.$$

where S is an invex set S with respect to $\eta : S \times S \to \mathbb{R}$, and f is a real valued function defined on S.
Note that if S is an interval in $\mathbb{R}$ and $\eta(y,x) = y - x$, for every $x, y \in S$, then $\mathsf{H} = H$.

Theorem 2.1. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to [0, +\infty)$. Suppose that $a, b \in S$ with $\eta(b,a) \neq 0$ and $c := a + \eta(b,a)$. If $f : S \to \mathbb{R}$ is a preinvex function then*
(i) H is a convex function on $[0,1]$.
(ii) If η satisfies condition C then, the following bounds hold

$$\sup_{t\in[0,1]} \mathsf{H}(t) = \mathsf{H}(1) = \frac{1}{\eta(b,a)}\int_a^c f(x)dx, \tag{2.1}$$

and

$$\inf_{t\in[0,1]} \mathsf{H}(t) = \mathsf{H}(0) = f\big(a + \frac{1}{2}\eta(b,a)\big). \tag{2.2}$$

(iii) If η satisfies condition C then, H increases monotonically on $[0,1]$.
(iv) If η satisfies condition C then, the following inequalities hold

$$\begin{aligned}
&f(a + \frac{1}{2}\eta(b,a)) \\
&\leq \frac{1}{\eta(b,a)}\int_a^c f\big(\frac{c+y}{2}\big)dy \\
&= \frac{2}{\eta(b,a)}\int_{a+\frac{1}{4}\eta(b,a)}^{a+\frac{3}{4}\eta(b,a)} f(u)du \\
&\leq \frac{1}{2}\Big(f\big(a + \frac{1}{2}\eta(b,a)\big) + \frac{1}{\eta(b,a)}\int_a^c f(x)dx\Big).
\end{aligned}$$

Proof. (i) It is an immediate consequence of definition of H and preinvexity of f.
(ii) By preinvexity of f we have

$$\begin{aligned}\mathsf{H}(t) &\leq \frac{1}{\eta(b,a)}\int_a^c \big((1-t)f(a+\frac{1}{2}\eta(b,a))+tf(x)\big)dx\\ &=(1-t)f(a+\frac{1}{2}\eta(b,a))+t\frac{1}{\eta(b,a)}\int_a^c f(x)dx.\end{aligned}$$

Suppose that the function $h:[0,1]\to\mathbb{R}$ is defined by

$$h(t):=(1-t)f(a+\frac{1}{2}\eta(b,a))+t\frac{1}{\eta(b,a)}\int_a^c f(x)dx.$$

Then, by Hermite-Hadamard's inequality (1.6) we get

$$h'(t)=\frac{1}{\eta(b,a)}\int_a^c f(x)dx-f(a+\frac{1}{2}\eta(b,a))\geq 0,$$

which shows that h is monotonically increasing on $[0,1]$. Hence, for every $t\in[0,1]$ we get

$$\mathsf{H}(t)\leq h(1)=\frac{1}{\eta(b,a)}\int_a^c f(x)dx,$$

thus

$$\sup_{t\in[0,1]} H(t)\leq\frac{1}{\eta(b,a)}\int_a^c f(x)dx. \tag{2.3}$$

On the other hand, if we put the change of variable $y:=a+s\eta(b,a)$, $s\in[0,1]$, and using (1.4) then, we have

$$\begin{aligned}\mathsf{H}(1)=&\frac{1}{\eta(b,a)}\int_a^c f\Big(a+\frac{1}{2}\eta(b,a)+\eta(y,a+\frac{1}{2}\eta(b,a))\Big)dy\\ &=\int_0^1 f\big(a+s\eta(b,a)\big)ds=\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx.\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) implies that

$$\sup_{t\in[0,1]} H(t)=H(1),$$

and the equality (2.1) is proved. Again, by putting change of variable $x:=a+s\eta(b,a)$, $s\in[0,1]$, and using (1.4) we obtain

$$\begin{aligned}&\int_a^c \eta(x,a+\frac{1}{2}\eta(b,a))dx\\ &=\eta(b,a)^2\int_0^1(s-\frac{1}{2})ds=0.\end{aligned} \tag{2.5}$$

Now, we define the function $g:P_{ab}\to P_{ab}$ as

$$g(x):=a+\frac{1}{2}\eta(b,a)+t\eta(x,a+\frac{1}{2}\eta(b,a)).$$

Integrating in P_{ab} and using (2.5) implies that

$$M_1(g)=\frac{1}{\eta(b,a)}\int_a^c g(x)dx=a+\frac{1}{2}\eta(b,a). \tag{2.6}$$

Using (1.4) and (2.5) deduce that

$$\begin{aligned}
&\frac{1}{\eta(b,a)}\int_a^c \eta(g(x),M_1(g))dx\\
&=\frac{1}{\eta(b,a)}\int_a^c \eta\Big(a+\frac{1}{2}\eta(b,a)+t\eta(x,a+\frac{1}{2}\eta(b,a)),a+\frac{1}{2}\eta(b,a)\Big)dx \qquad (2.7)\\
&=\frac{t}{\eta(b,a)}\int_a^c \eta(x,a+\frac{1}{2}\eta(b,a))dx=0.
\end{aligned}$$

It is easy to see that $\eta(g(x),M_1(g))\neq 0$, when $g(x)\neq M_1(g)$. Therefore, by using Jenson's inequality (1.7) and inequalities (2.5) and (2.7), for every $t\in[0,1]$ we obtain

$$\begin{aligned}
\mathsf{H}(t)\geq\ & f\Big(\frac{1}{\eta(b,a)}\int_a^c \big(a+\frac{1}{2}\eta(b,a)+t\eta(y,a+\frac{1}{2}\eta(b,a))\big)dy\\
&+K\frac{1}{\eta(b,a)}\int_a^c \eta(g(x),M_1(g))dx\\
&=f\Big(\frac{1}{\eta(b,a)}\int_a^c \big(a+\frac{1}{2}\eta(b,a)\big)+t\int_a^c \eta(y,a+\frac{1}{2}\eta(b,a))\Big)dy\\
&=f\big(a+\frac{1}{2}\eta(b,a)\big)=H(0),
\end{aligned}$$

for some $K\in\mathbb{R}$. Therefore, $\inf_{t\in[0,1]}\mathsf{H}(t)\geq\ \mathsf{H}(0)$ and the equality (2.17) is proved.

(*iii*) By convexity of H and using part (*ii*), for every $1\geq s>t>0$ we have

$$\frac{\mathsf{H}(s)-\mathsf{H}(t)}{s-t}\geq\frac{\mathsf{H}(t)-\mathsf{H}(0)}{t-0}\geq 0,$$

hence $\mathsf{H}(s)\geq\ \mathsf{H}(t)$.

(*iv*) By using (1.4) and putting $y:=a+s\eta(b,a), s\in[0,1]$ then,

$$\begin{aligned}
\mathsf{H}(\frac{1}{2})=&\frac{1}{\eta(b,a)}\\
&\times\int_a^{a+\eta(b,a)} f\Big(a+\frac{1}{2}\eta(b,a)+\frac{1}{2}\eta(y,a+\frac{1}{2}\eta(b,a))\Big)dy\\
&=\int_0^1 f\Big(a+\frac{1}{2}(s+\frac{1}{2})\eta(b,a)\Big)ds \qquad (2.8)\\
&=\frac{1}{\eta(b,a)}\int_a^c f\Big(\frac{c+y}{2}\Big)dy\\
&=\frac{2}{\eta(b,a)}\int_{a+\frac{1}{4}\eta(b,a)}^{a+\frac{3}{4}\eta(b,a)} f(u)du.
\end{aligned}$$

On the other hand, since H is convex on $[0,1]$, using the Hermite-Hadamard's inequality (1.1) and part (iii) yield

$$\begin{aligned} f\big(a+\frac{1}{2}\eta(b,a)\big) &= \mathsf{H}(0) \le \mathsf{H}(\frac{1}{2}) \\ &\le \int_0^1 \mathsf{H}(t)dt \le \frac{\mathsf{H}(0)+\mathsf{H}(1)}{2} \\ &= \frac{1}{2}\left(f\big(a+\frac{1}{2}\eta(b,a)\big)+\frac{1}{\eta(b,a)}\int_a^c f(x)dx\right). \end{aligned} \tag{2.9}$$

Now, combining (2.8) and (2.9) get us the required result in (iv) and proof is completed.

Let S be an invex set S with respect to $\eta : S \times S \to \mathbb{R}$, and f be a real valued function defined on S. Motivated by [2] we define the double integral mapping $\mathsf{F} : [0,1] \to \mathbb{R}$ as follow

$$\mathsf{F}(t) := \frac{1}{\eta(b,a)^2}\int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)} f(x+t\eta(y,x))dxdy.$$

where $a, b \in S$.
Note that, when S is an interval in $\mathbb{R}$ and $\eta(y,x) = y - x$, for every $x, y \in S$, then $\mathsf{F} = F$.

Theorem 2.2. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S\times S \to [0,+\infty)$ and $a, b \in S$ with $\eta(b,a) \neq 0$. If $c := a+\eta(b,a)$, $R := P_{ab}\times P_{ab}$ and $f : S \to \mathbb{R}$ is a preinvex function then*
(i) $\mathsf{F}(\frac{1}{2}+\tau) = \mathsf{F}(\frac{1}{2}-\tau)$ *for every* $\tau \in [0,\frac{1}{2}]$ *and* $\mathsf{F}(t) = \mathsf{F}(1-t)$, *for every* $t \in [0,1]$
(ii) If η satisfies condition C then, F is a convex function on $[0,1]$.
(iii) If η satisfies condition C then, we have the bounds

$$\sup_{t\in[0,1]} \mathsf{F}(t) = \mathsf{F}(0) = \mathsf{F}(1) = \frac{1}{\eta(b,a)}\int_a^c f(x)dx.$$

and

$$\begin{aligned} \inf_{t\in[0,1]} \mathsf{F}(t) &= \mathsf{F}(\frac{1}{2}) \\ &= \frac{1}{\eta(b,a)^2}\int_a^c\int_a^c f\big(x+\frac{1}{2}\eta(y,x)\big)dxdy. \end{aligned}$$

(iv) If η satisfies condition C then the following inequality holds

$$f\big(a+\frac{1}{2}\eta(b,a)\big) \le \mathsf{F}(\frac{1}{2}).$$

(v) If η satisfies condition C then, F decreases monotonically on $[0,\frac{1}{2}]$ and increases monotonically on $[0,\frac{1}{2}]$.
(vi) *For every $t \in [0,1]$ the following inequality holds*

$$\mathsf{H}(t) \le \mathsf{F}(t).$$

Proof. (i) This part is obvious.
(ii) Fix $t_1, t_2 \in [0,1]$. By using (1.4), for every $\lambda \in [0,1]$ we get

$$\begin{aligned}
&x + ((1-\lambda)t_1 + \lambda t_2)\eta(y,x) \\
&\quad = x + t_1\eta(y,x) + \lambda(t_2 - t_1)\eta(y,x) \\
&\quad = x + t_1\eta(y,x) + \lambda\eta(x + t_2\eta(y,x), x + t_1\eta(y,x)).
\end{aligned}$$

Hence by preinvexity of f we have

$$\begin{aligned}
&f(x + ((1-\lambda)t_1 + \lambda t_2)\eta(y,x)) \\
&\quad \le (1-\lambda)f(x + t_1\eta(y,x)) + \lambda f(x + t_2\eta(y,x)).
\end{aligned}$$

Integrating the above inequality on R implies that

$$\mathsf{F}((1-\lambda)t_1 + \lambda t_2) \le (1-\lambda)\mathsf{F}(t_1) + \lambda\mathsf{F}(t_2),$$

which shows the convexity of F.
(iii) For every $x, y \in P_{ab}$ and $t \in [0,1]$ we have

$$f(x + t\eta(y,x)) \le (1-t)f(x) + tf(y).$$

By integrating the above inequality in R we get

$$\int_a^c \int_a^c f(x + t\eta(y,x))dxdy \le \eta(b,a)\int_a^c f(x)dx.$$

This shows that for every $t \in [0,1]$,

$$\mathsf{F}(t) \le \frac{1}{\eta(b,a)}\int_a^c f(x)dx = \mathsf{F}(0). \tag{2.10}$$

On the other hand, if we use the change of variables $x := a + s\eta(b,a), y := a + t\eta(b,a)$ then by simple computation we get

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{pmatrix} \eta(b,a) & 0 \\ 0 & \eta(b,a) \end{pmatrix},$$

hence $\det \frac{\partial(x,y)}{\partial(s,t)} = \eta(b,a)^2$. By using this equality and (1.4) we obtain

$$\begin{aligned}
\mathsf{F}(1) &= \frac{1}{\eta(b,a)^2}\int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)} f(x + \eta(y,x))dxdy \\
&= \frac{1}{\eta(b,a)^2}\int_0^1\int_0^1 f(a + t\eta(b,a))\eta(b,a)^2 dtds \\
&= \int_0^1 f(a + t\eta(b,a))dt \\
&= \frac{1}{\eta(b,a)}\int_a^c f(x)dx.
\end{aligned} \tag{2.11}$$

Combining (2.10) and(2.11) deduce that,

$$\sup \mathsf{F}(t)_{t\in[0,1]} = \mathsf{F}(0) = \mathsf{F}(1),$$

hence the proof of the first part of (iii) is completed. For second part, since f is preinvex and η satisfies condition C, for every $t \in [0,1]$ we have

$$\begin{aligned} f(x+\frac{1}{2}\eta(y,x)) &= f(x+t\eta(y,x)+\frac{1}{2}(1-2t)\eta(y,x)) \\ &= f(x+t\eta(y,x)+\frac{1}{2}\eta\Big(x+(1-t)\eta(y,x), x+t\eta(y,x)\Big)\Big) \\ &\leq \frac{1}{2}\Big[f(x+t\eta(y,x))+f(x+(1-t)\eta(y,x))\Big]. \end{aligned}$$

Integrating this inequality in R we have

$$\begin{aligned} &\int_a^c\int_a^c f(x+\frac{1}{2}\eta(y,x))dxdy \\ &\leq \frac{1}{2}\int_a^c\int_a^c \Big(f(x+t\eta(y,x))+f(x+(1-t)\eta(y,x))\Big)dxdy \\ &= \int_a^c\int_a^c f(x+t\eta(y,x))dxdy, \end{aligned}$$

which implies that $\mathsf{F}(t) \geq \mathsf{F}(\frac{1}{2})$ for all $t \in [0,1]$, hence the statement is thus proved.

(iv) For every $x, y \in P_{ab}$, with putting

$$x := a+s\eta(b,a), y := a+t\eta(b,a), \ \ s,t \in [0,1],$$

and (1.4) we have

$$\begin{aligned} \mathsf{F}(\frac{1}{2}) &= \frac{1}{\eta(b,a)^2}\int_a^c\int_a^c f(x+\frac{1}{2}\eta(y,x))dxdy \\ &= \int_0^1\int_0^1 f\big(a+\frac{1}{2}(s+t)\eta(b,a)\big)dsdt. \end{aligned} \tag{2.12}$$

Now, we consider two cases:

Case1. If $s+t=1$ then, a simple computation shows that the right hand side of (2.12) is equal to $f(a+\eta(b,a))$, hence $\mathsf{F}(\frac{1}{2}) = f(a+\frac{1}{2}\eta(b,a))$.

Case2. If $s+t \neq 1$ then, we define the function $g : R_1 \to P_{ab}$ as follow,

$$g(x,y) := x+\frac{1}{2}\eta(y,x),$$

where, $R_1 := P_{ab} \times P_{ab} - A$, and

$$A := \{(x,y)|\ x = a+s\eta(b,a), y = a+t\eta(b,a),\ s+t=1\}.$$

Note that, if we want to integrate over the R, removing the set A doesn't make any difference because it form a null set. Hence, integrating in R

implies that

$$\begin{aligned} M_1(g) &= \frac{1}{\eta(b,a)^2}\int_a^c\int_a^c g(x,y)dxdy \\ &= \frac{1}{\eta(b,a)^2}\left[\int_a^c\int_a^c xdxdy + \frac{1}{2}\int_a^c\int_a^c \eta(y,x)dxdy\right] \\ &= a + \frac{1}{2}\eta(b,a). \end{aligned} \tag{2.13}$$

Indeed, if we put

$$m := \frac{1}{2\eta(b,a)^2}\int_a^c\int_a^c \eta(y,x)dxdy, \tag{2.14}$$

then, by using (1.4) we obtain

$$\begin{aligned} m &= \frac{1}{2\eta(b,a)^2} \\ &\times \int_0^1\int_0^1 \eta\big(a+s\eta(b,a), a+t\eta(b,a)\big)\eta(b,a)^2 dsdt \\ &= \frac{\eta(b,a)}{2}\int_0^1\int_0^1 (s-t)dsdt = 0. \end{aligned} \tag{2.15}$$

On the other hand, it is easy to see that

$$\eta\big(g(x,y), M_1(g)\big) = \frac{1}{2}(s+t-1)\eta(b,a) \neq 0.$$

Now, by Jensen's inequality (1.7), there exits $K \in \mathbb{R}$ such that

$$\begin{aligned} \mathsf{F}(\frac{1}{2}) =& \frac{1}{\eta(b,a)^2}\int_a^c\int_a^c f(x+\frac{1}{2}\eta(y,x))dxdy \\ &\geq f\Big(\frac{1}{\eta(b,a)^2}\int_a^c\int_a^c (x+\frac{1}{2}\eta(y,x))dxdy\Big) \\ &+K\frac{1}{\eta(b,a)^2}\int_a^c\int_a^c \eta\Big(x+\frac{1}{2}\eta(y,x), a+\frac{1}{2}\eta(b,a)\Big)dxdy \\ &= f\Big(\frac{1}{\eta(b,a)^2}\int_a^c\int_a^c xdxdy + \frac{1}{2\eta(b,a)^2}\int_a^c\int_a^c \eta(y,x)dxdy\Big) \\ &+K\frac{1}{\eta(b,a)^2}\int_a^c\int_a^c \eta\Big(x+\frac{1}{2}\eta(y,x), a+\frac{1}{2}\eta(b,a)\Big)dxdy \\ &= f\big(a+\frac{1}{2}\eta(b,a)+m\big)+n, \end{aligned}$$

in which m defied in (2.14) and

$$n := K\frac{1}{\eta(b,a)^2}\int_a^c\int_a^c \eta\Big(x+\frac{1}{2}\eta(y,x), a+\frac{1}{2}\eta(b,a)\Big)dxdy.$$

Note that, $m = 0$ by (2.15) and it is easy to see that

$$\frac{1}{\eta(b,a)^2}\int_a^c\int_a^c xdxdy = a + \frac{1}{2}\eta(b,a).$$

Now, by a simple computation we have

$$\begin{aligned} n &= \frac{1}{2}K\int_0^1\int_0^1 \eta\Big(a+\frac{1}{2}(t+s)\eta(b,a), a+\frac{1}{2}\eta(b,a)\Big)dsdt \\ &= \frac{1}{4}K\eta(b,a)\int_0^1\int_0^1 (s+t-1)dsdt = 0, \end{aligned} \tag{2.16}$$

this completes the proof of part (iv).
(v) By statement (iii), for every $t \in [0,1]$, $\mathsf{F}(t) \geq \mathsf{F}(\frac{1}{2})$ so, by convexity of F, for every $1 \geq s > t > \frac{1}{2}$ we have

$$\frac{\mathsf{F}(s)-\mathsf{F}(t)}{s-t} \geq \frac{\mathsf{F}(t)-\mathsf{F}(\frac{1}{2})}{t-\frac{1}{2}} \geq 0,$$

hence $\mathsf{F}(s) \geq \mathsf{F}(t)$. The fact that F decreases monotonically on $[0,\frac{1}{2}]$ follows from the above conclusion and using statement (i).
(vi) Let $y = a + s_0\eta(b,a)$ for some $s_0 \in [0,1]$. If we put $x := a + s\eta(b,a)$ and use (1.4) then, we have

$$\begin{aligned} &\int_a^c (x+t\eta(y,x))dx \\ &= \int_a^c xdx + t\int_a^c \eta(y,x)dx \\ &= \eta(b,a)\left(\int_0^1 (a+s\eta(b,a))ds + t\eta(y,x)\int_0^1 (s_0-s)ds\right) \\ &= \eta(b,a)\Big(a+\frac{1}{2}\eta(b,a)+t\eta(b,a)(s_0-\frac{1}{2})\Big) \\ &= \eta(b,a)\Big(a+\frac{1}{2}\eta(b,a)+t\eta\big(y,a+\frac{1}{2}\eta(b,a)\big)\Big). \end{aligned}$$

This shows that

$$\frac{1}{\eta(b,a)}\int_a^c (x+t\eta(y,x))dx = a+\frac{1}{2}\eta(b,a)+t\eta\big(y,a+\frac{1}{2}\eta(b,a)\big),$$

hence

$$\mathsf{H}(t) = \frac{1}{\eta(b,a)}\int_a^c f\left(\frac{1}{\eta(b,a)}\int_a^c (x+t\eta(y,x))dx\right)dy. \tag{2.17}$$

Now, we consider three cases:
Case1. If $x = a+\frac{1}{2}\eta(b,a)$ then, by using (2.17) and a simple computation we see that $\mathsf{F}(t) = \mathsf{H}(t)$, for all $t \in [0,1]$.
Case2. If $t = 1$ then, it is easy to see that, $\mathsf{F}(1) = \mathsf{H}(1)$.
Case3. If $t \neq 1$, $x \neq a+\frac{1}{2}\eta(b,a)$ then, we define the function $g : Q_{ab} \to P_{ab}$ as follow

$$g(x) := x + t\eta(y,x),$$

where, $Q_{ab} := P_{ab} - \{a+\frac{1}{2}\eta(b,a)\}$. Note that, if we want to integrate over the P_{ab}, then removing the $a+\frac{1}{2}\eta(b,a)$ doesn't make any difference because it form a null set. It is easy to see that

$$M_1(g) = a + \frac{1}{2}\eta(b,a) + t\eta\big(y, a + \frac{1}{2}\eta(b,a)\big),$$

and for every $x \in Q_{ab}$,

$$\eta(g(x), M_1(g)) = (1-t)(s - \frac{1}{2})\eta(b,a) \neq 0.$$

Hence,

$$\begin{aligned} &\frac{1}{\eta(b,a)}\int_a^c \eta(g(x), M_1(g))dx \\ &= (1-t)\eta(b,a)\int_0^1 \big(s - \frac{1}{2}\big)ds = 0. \end{aligned} \tag{2.18}$$

Now, by using (2.17), (2.18) and Jenson's inequality (1.7) we find $K \in \mathbb{R}$ such that

$$\begin{aligned} \mathsf{H}(t) \leq &\frac{1}{\eta(b,a)}\int_a^c \Big[\frac{1}{\eta(b,a)}\int_a^c f\big(x + t\eta(y,x)\big)dx \\ &- K\frac{1}{\eta(b,a)}\int_a^c \eta(g(x), M_1(g))dx\Big]dy \\ &= \frac{1}{\eta(b,a)^2}\int_a^c\int_a^c f\big(x + t\eta(y,x)\big)dxdy \\ &= \mathsf{F}(t), \end{aligned}$$

for all $t \in [0,1)$ and the proof is completed.

Note that, in the special case if we take $\eta(y,x) := y - x$ then, η satisfies conditions C and also S and f will be a convex set and a convex function, respectively. Therefore, theorems 2.1 and 2.2 gives us Dragomir's results introduced in Theorems 1.1 and 1.2, respectively.

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Availability of data and materials
All the results are new in this research article. However some basic definitions and results are included. There is no other source of data except the given references.

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