Mathematical Analysis
& Convex Optimization
Vol. 2 (2021), No. 2, 105–117
https:// maco.lu.ac.ir
DOI: 10.52547/maco.2.2.11
Research Paper

GENERALIZED SCHUR-CONVEX SUMS AND CO-ORDINATED CONVEX FUNCTIONS IN PLANE

NOZAR SAFAEI

Abstract. In the paper, we investigate Schur-convexity of differences which are obtained from the Hermite-Hadamard type inequality for co-ordinated convex functions on a square in plane. A generated Schur-convex sums by co-ordinated convex functions also is given.

MSC(2010): 26A51; 26D15.
Keywords: Schur-convexity, convex functions on the co-ordinates.

1. Introduction

The notion of Schur-convexity was first introduced by Issai Schur in 1923. Since then numerous papers have been published about it, see for example [4, 6, 7, 12]. It has many important applications in analytic inequality, geometric inequality, combinatorial analysis, numerical analysis, matrix theory, statistics, information theory, quantum physics and so on. Let us recall the definition of Schur-convexity.

Definition 1.1. [1] Suppose that I is an interval of real numbers. A function $f : I^n \to \mathbb{R}$, is said to be Schur-convex on I^n if

$$f(x_1, x_2, \ldots, x_n) \le f(y_1, y_2, \ldots, y_n)$$

for all $x = (x_1, x_2, \ldots, x_n)$, $\quad y = (y_1, y_2, \ldots, y_n) \in I^n$ with $x \prec y$, that is

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad k = 1, 2, \ldots, n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$, denotes the $i-$th largest component in x. A function f is said to be Schur-concave on I if $-f$ is Schur-convex.

Recall that a $n \times n$ square matrix P is said to be a permutation matrix if each row and column has a single unite entry, and all other entries are zero. Also the function $f : I^n \to \mathbb{R}$ is said to be a symmetric function if $f(Px) = f(x)$, for every permutation matrix P, and for every $x \in I^n$, see [1, 9]. In order to prove our result, we shall need the following theorem which gives a useful characterization of Schur-convexity, see [1].

Date: Received: April 11, 2021, Accepted: October 25, 2021.

Theorem 1.2. *Let $f : I^n \to \mathbb{R}$ be a continuous symmetric function. If f is differentiable on I^n, then f is Schur-convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0,$$

for all $x_i, x_j \in I, i, j = 1, 2, \ldots, n$. The function f is Schur-concave if and only if the reverse inequality holds.

In [6] Elezović and Pečarić proved a theorem which gives relationship between convexity and Schur-convexity.

Theorem 1.3. *Let f be a continuous function on an interval $I \subset \mathbb{R}$, and*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x, y \in I,\ x \neq y, \\ f(x), & x = y \in I. \end{cases}$$

Then $F(x,y)$ is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

The next theorem contains two result which established in [4] by Y. Chu et al.

Theorem 1.4. *Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a continuous function.*

(i) *The function*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - f(\frac{x+y}{2}), & x \neq y, \\ 0, & x = y, \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

(ii) *The function*

$$F(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)dt, & x \neq y, \\ 0, & x = y, \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I, the following double inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(t)dt \leq \frac{f(x)+f(y)}{2},\ x, y \in I,\ x < y, \tag{1.1}$$

is known in the literature as Hermite-Hadamard inequality for convex functions. In [3] it was shwon that if $f : I \to \mathbb{R}$ is twice differentiable function, then for all $x, y \in I$ with $x < y$,

$$\begin{aligned} &\frac{2}{y-x}\int_x^y f(t)dt - (f(x)+f(y)) + \frac{y-x}{4}\big(f'(y) - f'(x)\big) \\ &= \frac{1}{y-x}\int_x^y \Big(t - \frac{x+y}{2}\Big)^2 f''(t)dt. \end{aligned} \tag{1.2}$$

In [5], S.S. Dragomir defined convex function on the co-ordinates (or co-ordinated convex functions) on the set $[a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ as follows.

Definition 1.5. A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is said to be convex on the co-ordinates on $[a, b] \times [c, d]$ if for every $y \in [c, d]$ and $x \in [a, b]$, the partial mappings,

$$f_y \colon [a, b] \to \mathbb{R}, \quad f_y(u) = f(u, y),$$

and

$$f_x \colon [c, d] \to \mathbb{R}, \quad f_x(v) = f(x, v),$$

are convex. This means that for every $(x, y), (z, w) \in [a, b] \times [c, d]$ and $t, s \in [0, 1]$,

$$\begin{aligned} f\left(tx + (1-t)z, sy + (1-s)w\right) &\le tsf(x, y) + s(1-t)f(z, y) \\ &\quad + t(1-s)f(x, w) + (1-t)(1-s)f(z, w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex functions which are not convex. The following Hermite-Hadamard type inequality for co-ordinated convex functions was also proved in [5].

Theorem 1.6. *Suppose that $f : [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates. Then,*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)dydx \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x, c)dx + \frac{1}{b-a}\int_a^b f(x, d)dx\right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a, y)dy + \frac{1}{d-c}\int_c^d f(b, y)dy\right] \\ &\le \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \tag{1.3}$$

The above inequalities are sharp.

Remark 1.7. If $f : [x, y] \times [x, y] \to \mathbb{R}$, is convex on the co-ordinates then inequality (1.3) reduces to the following inequality

$$\begin{aligned} f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) &\le \frac{1}{2(y-x)}\left[\int_x^y f\left(t, \frac{x+y}{2}\right)dt + \int_x^y f\left(\frac{x+y}{2}, s\right)ds\right] \\ &\le \frac{1}{(y-x)^2}\int_x^y\int_x^y f(t, s)dtds \\ &\le \frac{1}{4(y-x)}\left[\int_x^y (f(t, x)dt + f(t, y))dt\right. \\ &\quad \left. + \int_x^y (f(x, s)ds + f(y, s))ds\right] \\ &\le \frac{f(x, x) + f(x, y) + f(y, x) + f(y, y)}{4}. \end{aligned} \tag{1.4}$$

In [10] by using the inequality (1.3) we proved the following results.

Theorem 1.8. *Let $D := [a_1, b_1] \times [a_1, b_1]$ be a square in $\mathbb{R}^2$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on D°. Choose*

$a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a, b] \times [a, b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $F : \Delta \to \mathbb{R}$ defined by

$$F(x,y) := \begin{cases} \frac{1}{(y-x)^2}\int_x^y \int_x^y f(t,s)dtds, & x \neq y, \\ f(x,x), & x = y, \end{cases} \tag{1.5}$$

is Schur-convex on Δ.

Theorem 1.9. *Let $D := [a_1, b_1] \times [a_1, b_1]$ be a square in $\mathbb{R}^2$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous third order partial derivatives on D°. Choose $a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a, b] \times [a, b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $G : \Delta \to \mathbb{R}$ defined by*

$$G(x,y) := \begin{cases} \frac{1}{(y-x)^2}\int_x^y \int_x^y f(t,s)dtds - f(\frac{x+y}{2}, \frac{x+y}{2}), & x \neq y, \\ 0, & x = y, \end{cases} \tag{1.6}$$

is Schur-convex on Δ.

We recall the following lemma from [2], which is known as Leibniz's Formula.

Lemma 1.10. *Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ and $\frac{\partial f}{\partial t} : [a, b] \times [c, d] \to \mathbb{R}$ are continuous and $\alpha_1, \alpha_2 : [c, d] \to [a, b]$ are differentiable functions. Then, the function $\varphi : [c, d] \to \mathbb{R}$ defined by*

$$\varphi(t) = \int_{\alpha_1(t)}^{\alpha_2(t)} f(x,t)dx,$$

has a derivative for each $t \in [c, d]$, which is given by

$$\varphi'(t) = f(\alpha_2(t), t)\alpha_2'(t) - f(\alpha_1(t), t)\alpha_1'(t) + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{\partial f}{\partial t}(x,t)dx.$$

Moreover, in [10] we proved the following lemma which will be useful in the sequal. A version of the following lemma proved in [11].

Lemma 1.11. *Let $F(u, v) = \int_u^v \int_u^v f(x, y)dxdy$, where $f(x, y)$ is continuous on the rectangle $[a, p] \times [a, q]$, $u = u(b)$ and $v = v(b)$ are differentiable with $a \leq u(b) \leq p$ and $a \leq v(b) \leq q$. Then,*

$$\begin{aligned} \frac{\partial F}{\partial b} &= \left(\int_u^v f(x,v)dx + \int_u^v f(v,y)dy\right) v'(b) \\ &\quad - \left(\int_u^v f(x,u)dx + \int_u^v f(u,y)dy\right) u'(b). \end{aligned} \tag{1.7}$$

The main purpose of this article is to establish the Schur-convexity of symmetric functions which we obtain from the inequality (1.3).

2. Main Results

In this section we prove theorems like those Theorems 1.8, 1.9 for all differences related to inequalities in (1.3). To reach our goal, we need the following two lemmas, see [10].

Lemma 2.1. *Let $D := [a_1, b_1] \times [a_1, b_1]$ be a square in $\mathbb{R}^2$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on D° (the interior of D). Choose $a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a, b] \times [a, b]$. Suppose that the function $F_1 : \Delta \to \mathbb{R}$ is defined by*

$$F_1(x,y) := \begin{cases} \frac{1}{2(y-x)}\left(\int_x^y f(t, \frac{x+y}{2})dt + \int_x^y f(\frac{x+y}{2}, s)ds\right) & \\ -f(\frac{x+y}{2}, \frac{x+y}{2}), & x \neq y, \\ 0, & x = y. \end{cases}$$

Then, for all $t_0 \in [a, b]$,

$$\begin{aligned} \frac{\partial F_1}{\partial x}\Big|_{(t_0,t_0)} &= \frac{\partial F_1}{\partial y}\Big|_{(t_0,t_0)} \\ &= \frac{\frac{\partial f}{\partial t}(t,s)\big|_{(t_0,t_0)} + \frac{\partial f}{\partial s}(t,s)\big|_{(t_0,t_0)} - \frac{\partial f}{\partial t}(t,t)\big|_{t_0}}{2}. \end{aligned}$$

The proof of the following lemma is similar to once in lemma 2.1 hence we omit it.

Lemma 2.2. *Let $D := [a_1, b_1] \times [a_1, b_1]$ be a square in $\mathbb{R}^2$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous third order partial derivatives on D°. Choose $a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a, b] \times [a, b]$. Suppose that the function $G_1 : \Delta \to \mathbb{R}$ is defined by*

$$G_1(x,y) := \begin{cases} \frac{1}{4(y-x)}\left(\int_x^y (f(t,x) + f(t,y))dt + \int_x^y (f(x,s) + f(y,s))ds\right) & \\ -\frac{1}{(y-x)^2}\int_x^y \int_x^y f(t,s)dtds, & x \neq y, \\ 0, & x = y. \end{cases}$$

Then, for all $t_0 \in [a, b]$,

$$\begin{aligned} \frac{\partial G_1}{\partial x}\Big|_{(t_0,t_0)} &= \frac{\partial G_1}{\partial y}\Big|_{(t_0,t_0)} \\ &= \frac{\frac{\partial f}{\partial t}(t,s)\big|_{(t_0,t_0)} + \frac{\partial f}{\partial s}(t,s)\big|_{(t_0,t_0)} - \frac{\partial f}{\partial t}(t,t)\big|_{t_0}}{12}. \end{aligned}$$

In the following result we establish the Schur-convexity of the difference between the right-hand side and the left hand side of the first inequality in (1.4).

Theorem 2.3. *Let $D := [a_1, b_1] \times [a_1, b_1]$ be a square in $\mathbb{R}^2$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on D°. Choose $a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a, b] \times [a, b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $F_1 : \Delta \to \mathbb{R}$ defined by*

$$F_1(x,y) := \begin{cases} \frac{1}{2(y-x)}\left(\int_x^y f(t, \frac{x+y}{2})dt + \int_x^y f(\frac{x+y}{2}, s)ds\right) & \\ -f(\frac{x+y}{2}, \frac{x+y}{2}), & x \neq y, \\ 0, & x = y, \end{cases} \tag{2.1}$$

is Schur-convex on Δ.

Proof. Case 1: If $x, y \in [a, b]$, with $x = y$. Then Lemma 2.1 implies that

$$(y-x)\left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial x}\right) = 0.$$

Case 2: If $x, y \in [a,b]$, with $x \neq y$. Since f is convex on the co-ordinates on Δ, the mapping $g_{\frac{x+y}{2}}(t) := f(t, \frac{x+y}{2})$ is convex on $[a,b]$ for every $x, y \in [a,b]$. Then by using the second inequality in (1.1), for the mapping g we have

$$\frac{1}{y-x}\int_x^y g_{\frac{x+y}{2}}(t)dt \le \frac{g_{\frac{x+y}{2}}(x)+g_{\frac{x+y}{2}}(y)}{2}.$$

That is,

$$\frac{1}{y-x}\int_x^y f\left(t, \frac{x+y}{2}\right)dt \le \frac{f\left(x, \frac{x+y}{2}\right)+f\left(y, \frac{x+y}{2}\right)}{2}, \tag{2.2}$$

for every $x, y \in [a,b]$ with $x \neq y$. Similar way for the mapping $g_{\frac{x+y}{2}}(s) := f(\frac{x+y}{2}, s)$ we have

$$\frac{1}{y-x}\int_x^y f\left(\frac{x+y}{2}, s\right)ds \le \frac{f\left(\frac{x+y}{2}, x\right)+f(\frac{x+y}{2}, y)}{2}. \tag{2.3}$$

Summing inequalities (2.2) and (2.3) we have

$$\begin{aligned}
&\frac{1}{y-x}\int_x^y f\left(t, \frac{x+y}{2}\right)dt + \frac{1}{y-x}\int_x^y f\left(\frac{x+y}{2}, s\right)ds \\
&\le \frac{1}{2}\left[f\left(x, \frac{x+y}{2}\right)+f\left(y, \frac{x+y}{2}\right)+f\left(\frac{x+y}{2}, x\right)+f\left(\frac{x+y}{2}, y\right)\right].
\end{aligned} \tag{2.4}$$

By using Lemma 1.10 for the function F_1 in (2.1) we get

$$\begin{aligned}
\frac{\partial F_1}{\partial y} =& \frac{-1}{2(y-x)^2}\left[\int_x^y f\left(t, \frac{x+y}{2}\right)dt + \int_x^y f\left(\frac{x+y}{2}, s\right)ds\right] \\
&+ \frac{1}{2(y-x)}\left[f\left(y, \frac{x+y}{2}\right)dt + \int_x^y \frac{\partial f}{\partial y}\left(t, \frac{x+y}{2}\right)dt\right. \\
&\left.+ f\left(\frac{x+y}{2}, y\right) + \int_x^y \frac{\partial f}{\partial y}\left(\frac{x+y}{2}, s\right)ds\right] - \frac{\partial f}{\partial y}\left(\frac{x+y}{2}, \frac{x+y}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F_1}{\partial x} =& \frac{1}{2(y-x)^2}\left[\int_x^y f\left(t, \frac{x+y}{2}\right)dt + \int_x^y f\left(\frac{x+y}{2}, s\right)ds\right] \\
&+ \frac{1}{2(y-x)}\left[-f\left(x, \frac{x+y}{2}\right)dt + \int_x^y \frac{\partial f}{\partial x}\left(t, \frac{x+y}{2}\right)dt\right. \\
&\left.- f\left(\frac{x+y}{2}, x\right) + \int_x^y \frac{\partial f}{\partial x}\left(\frac{x+y}{2}, s\right)ds\right] - \frac{\partial f}{\partial x}\left(\frac{x+y}{2}, \frac{x+y}{2}\right).
\end{aligned}$$

Since $\frac{\partial f}{\partial y}(t, \frac{x+y}{2}) = \frac{\partial f}{\partial x}(t, \frac{x+y}{2})$, $\frac{\partial f}{\partial y}(\frac{x+y}{2}, s) = \frac{\partial f}{\partial x}(\frac{x+y}{2}, s)$, $\frac{\partial f}{\partial y}(\frac{x+y}{2}, \frac{x+y}{2}) = \frac{\partial f}{\partial x}(\frac{x+y}{2}, \frac{x+y}{2})$ it follows that

$$\begin{aligned}
(y-x)\left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial x}\right) =& \frac{-1}{y-x}\left(\int_x^y f\left(t, \frac{x+y}{2}\right)dt + \int_x^y f\left(\frac{x+y}{2}, s\right)ds\right) \\
&+ \frac{1}{2}\left[f\left(y, \frac{x+y}{2}\right) + f\left(\frac{x+y}{2}, y\right)\right. \\
&\left.+ f\left(x, \frac{x+y}{2}\right) + f\left(\frac{x+y}{2}, x\right)\right].
\end{aligned}$$

Then, from inequality (2.4) we have $(y-x)\left(\frac{\partial F_1}{\partial y}-\frac{\partial F_1}{\partial x}\right)\geq 0$. Therefore, by Theorem 1.2 the function F_1 is Schur-convex. $\square$

In the following result we establish the Schur-convexity of the difference between the right-hand side and the left hand hand side of the third inequality in (1.4).

Theorem 2.4. *Let $D:=[a_1,b_1]\times[a_1,b_1]$ be a square in $\mathbb{R}^2$ with $a_1<b_1$, and the function $f:D\to\mathbb{R}$ is continuous, and has continuous third order partial derivatives on D°. Choose $a,b\in(a_1,b_1)$, with $a<b$, and let $\Delta:=[a,b]\times[a,b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $G_1:\Delta\to\mathbb{R}$ defined by*

$$G_1(x,y):=\begin{cases}\frac{1}{4(y-x)}\Big(\int_x^y\big(f(t,x)+f(t,y)\big)dt+\int_x^y\big(f(x,s)+f(y,s)\big)ds\Big)\\ -\frac{1}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds, & x\neq y,\\ 0, & x=y,\end{cases}\tag{2.5}$$

is Schur-convex on Δ.

Proof. Case 1: If $x,y\in[a,b]$, with $x=y$. Then Lemma 2.2 implies that

$$(y-x)\left(\frac{\partial G_1}{\partial y}-\frac{\partial G_1}{\partial x}\right)=0.$$

Case 2: If $x,y\in[a,b]$, with $x\neq y$. Since f is convex on the co-ordinates on Δ, the mapping $g_s(t):=f(t,s)$ is convex on $[a,b]$ for every $s\in[a,b]$. By convexity of g, $g_s''(t)\geq 0$ for every $t\in[a,b]$. Then by equality (1.2) we have

$$\begin{aligned}&\frac{2}{y-x}\int_x^y g_s(t)dt-\big(g_s(x)+g_s(y)\big)\\&+\frac{1}{4}(y-x)\Big(\frac{\partial g_s(t)}{\partial t}\Big|_{t=y}-\frac{\partial g_s(t)}{\partial t}\Big|_{t=x}\Big)\geq 0.\end{aligned}$$

That is,

$$\begin{aligned}&\frac{2}{y-x}\int_x^y f(t,s)dt-\big(f(x,s)+f(y,s)\big)\\&+\frac{1}{4}(y-x)\Big(\frac{\partial f}{\partial y}(y,s)-\frac{\partial f}{\partial x}(x,s)\Big)\geq 0,\end{aligned}$$

for every $x,y\in[a,b]$, with $x\neq y$. Integrating this inequality on $[x,y]$, and multiple by $\frac{2}{y-x}$ we have

$$\begin{aligned}&\frac{4}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds-\frac{2}{y-x}\int_x^y\big(f(x,s)+f(y,s)\big)ds\\&+\frac{1}{2}\Big[\int_x^y\Big(\frac{\partial f}{\partial y}(y,s)-\frac{\partial f}{\partial x}(x,s)\Big)ds\Big]\geq 0.\end{aligned}\tag{2.6}$$

for every $x,y\in[a,b]$, with $x<y$. Similar way for the mapping $g_t(s):=f(t,s)$ we have

$$\begin{aligned}&\frac{4}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds-\frac{2}{y-x}\int_x^y\big(f(t,x)+f(t,y)\big)dt\\&+\frac{1}{2}\Big[\int_x^y\Big(\frac{\partial f}{\partial y}(y,s)-\frac{\partial f}{\partial x}(x,s)\Big)dt\Big]\geq 0.\end{aligned}\tag{2.7}$$

Summing inequalities (2.6) and (2.7), and dividing by 2 we have

$$
\begin{aligned}
&\frac{4}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds\\
&-\frac{1}{y-x}\Big(\int_x^y\big(f(t,x)+f(t,y)\big)dt+\int_x^y\big(f(x,s)+f(y,s)\big)ds\Big)\\
&+\frac{1}{4}\Big[\int_x^y\Big(\frac{\partial f}{\partial y}(y,s)-\frac{\partial f}{\partial x}(x,s)\Big)dt\\
&+\int_x^y\Big(\frac{\partial f}{\partial y}(y,s)-\frac{\partial f}{\partial x}(x,s)\Big)ds\Big]\geq 0,
\end{aligned}
\tag{2.8}
$$

for every $x,y\in[a,b]$, with $x\neq y$.

For the function G_1 was defined in (2.5) we get

$$
\begin{aligned}
\frac{\partial G_1}{\partial y}=&\frac{1}{4}\Big[\frac{-1}{(y-x)^2}\Big(\int_x^y(f(t,x)+f(t,y))dt+\int_x^y(f(x,s)+f(y,s))ds\Big)\\
&+\frac{1}{y-x}\Big(f(y,x)+f(y,y)+\int_x^y\frac{\partial f}{\partial y}(t,y)dt+f(x,y)+f(y,y)\\
&+\int_x^y\frac{\partial f}{\partial y}(y,s)ds\Big)\Big]+\frac{2}{(y-x)^3}\int_x^y\int_x^y f(t,s)dtds\\
&-\frac{1}{(y-x)^2}\Big(\int_x^y f(t,y)dt+\int_x^y f(y,s)ds\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial G_1}{\partial x}=&\frac{1}{4}\Big[\frac{1}{(y-x)^2}\Big(\int_x^y(f(t,x)+f(t,y))dt+\int_x^y(f(x,s)+f(y,s))ds\Big)\\
&+\frac{1}{y-x}\Big(-f(x,x)-f(x,y)+\int_x^y\frac{\partial f}{\partial x}(t,x)dt-f(y,x)-f(x,x)\\
&+\int_x^y\frac{\partial f}{\partial x}(x,s)ds\Big)\Big]-\frac{2}{(y-x)^3}\int_x^y\int_x^y f(t,s)dtds\\
&+\frac{1}{(y-x)^2}\Big(\int_x^y f(t,x)dt+\int_x^y f(x,s)ds\Big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(y-x)\left(\frac{\partial G_1}{\partial y}-\frac{\partial G_1}{\partial x}\right)\\
=&\frac{-1}{2(y-x)}\Big(\int_x^y(f(t,x)+f(t,y))dt+\int_x^y(f(x,s)+f(y,s))ds\Big)\\
&+\frac{1}{2}\Big(f(x,x)+f(y,y)+f(x,y)+f(y,x)\Big)+\frac{4}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds\\
&-\frac{1}{y-x}\Big(\int_x^y(f(t,x)+f(t,y))dt+\int_x^y(f(x,s)+f(y,s))ds\Big)\\
&+\frac{1}{4}\Big[\int_x^y\Big(\frac{\partial f}{\partial y}(t,y)-\frac{\partial f}{\partial x}(t,x)\Big)dt+\int_x^y\Big(\frac{\partial f}{\partial y}(y,s)-\frac{\partial f}{\partial x}(t,x)\Big)ds\Big]\\
=&A+B,
\end{aligned}
$$

where

$$\begin{aligned}A =&\frac{-1}{2(y-x)}\Big[\int_x^y (f(t,x)+f(t,y))dt + (f(x,s)+f(y,s))ds\Big]\\ &+\frac{1}{2}\Big(f(x,x)+f(x,y)+f(y,x)+f(y,y)\Big),\end{aligned}$$

$$\begin{aligned}B =&\frac{4}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds\\ &-\frac{1}{y-x}\Big[\int_x^y (f(t,x)+f(t,y))dt+\int_x^y (f(x,s)+f(y,s))ds\Big]\\ &+\frac{1}{4}\Big[\int_x^y \Big(\frac{\partial f}{\partial y}(t,y)-\frac{\partial f}{\partial x}(t,x)\Big)dt+\int_x^y \Big(\frac{\partial f}{\partial y}(y,s)-\frac{\partial f}{\partial x}(t,x)\Big)ds\Big].\end{aligned}$$

Since f is convex on the co-ordinates, the last inequality in (1.4) shows that $A \geq 0$, and from inequality (2.8) we have $B \geq 0$. Thus we get $(y-x)\big(\frac{\partial G_1}{\partial y}-\frac{\partial G_1}{\partial x}\big) \geq 0$. Therefore, by Theorem 1.2 the function G_1 is Schur-convex. □

In the following result we establish the Schur-convexity of the difference between the right-hand side and the left hand hand side of the last inequality in (1.4).

Theorem 2.5. *Let $D := [a_1, b_1] \times [a_1, b_1]$ be a square in $\mathbb{R}^2$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on D°. Choose $a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a, b] \times [a, b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $F_2 : \Delta \to \mathbb{R}$ defined by*

$$F_2(x,y) := \begin{cases}\frac{1}{4}\Big(f(x,x)+f(x,y)+f(y,x)+f(y,y)\Big) & \\ -\frac{1}{4(y-x)}\Big(\int_x^y (f(t,x)+f(t,x))dt+(f(x,s)+f(y,s))ds\Big), & x\neq y,\\ 0, & x=y,\end{cases} \tag{2.9}$$

is Schur-convex on Δ.

Proof. Since f is convex on the co-ordinates on Δ, the function $g_x(t) = f(t, x)$ is convex for every $x \in [a, b]$. By using the part (ii) of theorem 1.4 for the function $g_x(t)$, the function H_1 defined by

$$H_1(x,y) = \begin{cases}\frac{g_x(x)+g_x(y)}{2}-\frac{1}{y-x}\int_x^y g_x(t)dt, & x\neq y,\\ 0, & x=y,\end{cases}$$

is Schur-convex. That is

$$H_1(x,y) = \begin{cases}\frac{f(x,x)+f(y,x)}{2}-\frac{1}{y-x}\int_x^y f(x,t)dt, & x\neq y,\\ 0, & x=y,\end{cases}$$

is Schur-convex, for every $x, y \in [a, b]$. Similarly way for the convex function $g_y(t) = f(t, y)$, the function H_2 defined by

$$H_2(x,y) = \begin{cases}\frac{f(x,y)+f(y,y)}{2}-\frac{1}{y-x}\int_x^y f(t,y)dt, & x\neq y,\\ 0, & x=y,\end{cases}$$

is Schur-convex. Also for the convex functions $g_x(s) = f(x, s)$ and $g_y(s) = f(y, s)$ the functions H_3 and H_4 which are defined by

$$H_3(x,y) = \begin{cases} \frac{f(x,x)+f(x,y)}{2} - \frac{1}{y-x}\int_x^y f(x,s)ds, & x \neq y, \\ 0, & x = y, \end{cases}$$

and

$$H_4(x,y) = \begin{cases} \frac{f(y,x)+f(y,y)}{2} - \frac{1}{y-x}\int_x^y f(y,s)ds, & x \neq y, \\ 0, & x = y, \end{cases}$$

are Schur-convex, for every $x, y \in [a, b]$. Since the sum of Schur-convex functions is Schur-convex, we see that the function

$$F_2 = \frac{H_1 + H_2 + H_3 + H_4}{4}$$

is Schur-convex on Δ. □

The following corollaries establish the Schur-convexity of other differences of inequalities in (1.4).

Corollary 2.6. *With the assumptions of Theorem 2.4, the function* $F_3 : [a, b] \times [a, b] \to \mathbb{R}$ *defined by*

$$F_3(x,y) := \begin{cases} \frac{1}{4}\Big(f(x,x) + f(x,y) + f(y,x) + f(y,y)\Big) & \\ -\frac{1}{(y-x)^2}\Big(\int_x^y \int_x^y f(t,s)dtds\Big), & x \neq y, \\ 0, & x = y, \end{cases} \tag{2.10}$$

is Schur-convex.

Proof. Since the functions G_1, F_2 in (2.5), and (2.9) are Schur-convex, it follows that $F_3 = G_1 + F_2$ is Schur-convex. □

Corollary 2.7. *With the assumptions of Theorem 2.4, the function* $F_4 : [a, b] \times [a, b] \to \mathbb{R}$ *defined by*

$$F_4(x,y) := \frac{1}{4}\Big(f(x,x) + f(x,y) + f(y,x) + f(y,y)\Big) - f\Big(\frac{x+y}{2}, \frac{x+y}{2}\Big),$$

is Schur-convex.

Proof. Since the functions G, F_3 in (1.6), and (2.10) are Schur-convex, it follows that $F_4 = G + F_3$ is Schur-convex. □

It is known that if a function $f : I \to \mathbb{R}$ is convex then it generates Schur-convex sums, that is the function $F : I^n \to \mathbb{R}$ defined by

$$F(x) = F(x_1, x_2, \ldots, x_n) = f(x_1) + f(x_2) + \ldots + f(x_n)$$

is Schur-convex. Convexity of f is a sufficient but not necessary condition under which F is Schur-convex(F is convex and symmetric), see[1]. In [8] Ng proved that a function $f : I \to \mathbb{R}$ generates Schur-convex sums if and only if it is Wright-convex. The following corollary shows that a co-ordinated convex function generates Schur-convex sums.

Corollary 2.8. *With the assumptions of Theorem 2.4, the symmetric function* $F_5 : [a, b] \times [a, b] \to \mathbb{R}$ *defined by*

$$F_5(x, y) := f(x, x) + f(x, y) + f(y, x) + f(y, y) \tag{2.11}$$

is Schur-convex.

Proof. Since the functions F, F_3 in (1.5), and (2.10) are Schur-convex, it follows that $F_5 = G + F_3$ is Schur-convex. □

The converse of corollary 2.8 is not true in general (consider $f(t, s) := 2t^2 - s^2$ on $[0, 1] \times [0, 1]$). Also the following example shows that the function F_5 in (2.11) need not to be convex on the co-ordinates.

Example 2.9. Consider the co-ordinated convex function $f(t, s) := t^3 - ts$ on $[0, 1] \times [0, 1]$, for the function F_5 in (2.11) we have $F_5(x, y) = 2x^3 + 2y^3 - x^2 - y^2 - 2xy$, where $x, y \in [0, 1]$. It is easy to see that F_5 is not convex on the co-ordinates on $[0, 1] \times [0, 1]$. Moreover F_5 is not convex on $[0, 1] \times [0, 1]$.

Corollary 2.10. *With the assumptions of Theorem 2.4, the symmetric function* $F_6 : [a, b] \times [a, b] \to \mathbb{R}$ *defined by*

$$F_6(x, y) := \begin{cases} \frac{1}{4(y-x)}\Big(\int_x^y (f(t, x) + f(t, y))dt + \int_x^y (f(x, s) + f(y, s))ds\Big) & \\ -f(\frac{x+y}{2}, \frac{x+y}{2}), & x \neq y, \\ 0, & x = y, \end{cases}$$

is Schur-convex.

Proof. Since the functions G, G_1 in (1.6), and (2.5) are Schur-convex, it follows that $F_6 = G + G_1$ is Schur-convex. □

Corollary 2.11. *With the assumptions of Theorem 2.4, the symmetric function* $F_7 : [a, b] \times [a, b] \to \mathbb{R}$ *defined by*

$$F_7(x, y) := \begin{cases} \frac{1}{4(y-x)}\Big(\int_x^y (f(t, x) + f(t, y))dt + \int_x^y (f(x, s) + f(y, s))ds\Big), & x \neq y, \\ f(x, x), & x = y, \end{cases}$$

is Schur-convex.

Proof. Since the functions F, G_1 in (1.5), and (2.5) are Schur-convex, it follows that $F_7 = F + G_1$ is Schur-convex. □

In the following examples we show that the converses of theorems 2.3 , 2.4, 2.5 are not true in general.

Example 2.12. Consider the non co-ordinated convex function :

$$f(t, s) := 3t^2 - 2s^2, \quad t, s \in [0, 1].$$

It is easy to see that for the function F_1 in (2.1), $F_1(x, x) = 0$, for every $x \in [0, 1]$. Moreover for every $x, y \in [0, 1]$, with $x \neq y$ we have

$$\begin{aligned} F_1(x, y) =& \frac{1}{2(y - x)} \int_x^y \left(t^2 + \left(\frac{x + y}{2}\right)^2\right) dt - \left(\frac{x + y}{2}\right)^2 \\ =& \frac{1}{24}(x - y)^2. \end{aligned}$$

Thus,

$$F_1(x,y) = \frac{1}{24}(x-y)^2,$$

for every $x, y \in [0,1]$. Clearly F_1 is symmetric, continuous and differentiable on $[0,1] \times [0,1]$. Thus, for every $x, y \in [0,1]$, we have

$$(y-x)\left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial x}\right) = \frac{1}{6}(y-x)^2 \geq 0.$$

Therefore, by Theorem 1.2 the function F_1 is Schur-convex.

Example 2.13. Consider the non co-ordinated convex function :

$$f(t,s) := t^2 - \frac{1}{4}s^2, \quad t, s \in [0,1].$$

It is easy to see that for the function G_1 in (2.5), $G_1(x,x) = 0$, for every $x \in [0,1]$. Moreover for every $x, y \in [0,1]$, with $x \neq y$ we have

$$\begin{aligned} G_1(x,y) =& \frac{1}{4(y-x)} \int_x^y \left(\frac{3}{2}t^2 + \frac{3}{4}(x^2+y^2)\right) dt - \frac{1}{(y-x)^2} \int_x^y \int_x^y \left(t^2 - \frac{1}{4}s^2\right) dtds \\ =& \frac{1}{16}(x-y)^2. \end{aligned}$$

Thus,

$$G_1(x,y) = \frac{1}{16}(x-y)^2,$$

for every $x, y \in [0,1]$. Clearly G_1 is symmetric, continuous and differentiable on $[0,1] \times [0,1]$. Thus, for every $x, y \in [0,1]$, we have

$$(y-x)\left(\frac{\partial G_1}{\partial y} - \frac{\partial G_1}{\partial x}\right) = \frac{1}{4}(y-x)^2 \geq 0.$$

Therefore, by Theorem 1.2 the function G_1 is Schur-convex.

Example 2.14. Consider the non co-ordinated convex function :

$$f(t,s) := 2t^2 - \frac{1}{2}s^2, \quad t, s \in [0,1].$$

It is easy to see that for the function F_2 in (2.9), $F_2(x,x) = 0$, for every $x \in [0,1]$. Moreover for every $x, y \in [0,1]$, with $x \neq y$ we have

$$\begin{aligned} F_2(x,y) =& \frac{3}{4}(x^2+y^2) - \frac{1}{4(y-x)} \int_x^y \left(3t^2 + \frac{3}{2}(x^2+y^2)\right) dt \\ =& \frac{1}{8}(x-y)^2. \end{aligned}$$

Thus,

$$F_2(x,y) = \frac{1}{8}(x-y)^2,$$

for every $x, y \in [0,1]$. Clearly F_2 is symmetric, continuous and differentiable on $[0,1] \times [0,1]$.

Thus we have

$$(y-x)\left(\frac{\partial F_2}{\partial y} - \frac{\partial F_2}{\partial x}\right) = \frac{1}{2}(y-x)^2 \geq 0,$$

for every $x, y \in [0,1]$. Therefore, by Theorem 1.2 the function F_2 is Schur-convex.

References

[1] B. C. Arnold, A. W. Marshal and I. Olkin, *Inequalities : Theory of Majorization and It's Applications*, Springer Series in Statistics, New York-Dordrecht-Heidelberg-London, 2011.
[2] R. G. Bartel, *The Elements of Real Analysis*, Jon Wiley and Sons, 2st edition, 1976.
[3] X.-L. Cheng and J. Sun, A note on the perturbed trapezoid inequality. *Journal of Inequalities in Pure and Applied Mathematics*, **3**(2): 1–7, 2002.
[4] Y. Chu, G. Wang, and X. Zhang, Schur convexity and Hadamard's inequality. *Mathematical Inequalities and Applications*, **13**(4): 725–731, 2010.
[5] S. S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plan. *Taiwanese Journal of Mathematics*, **5**: 775–778, 2001.
[6] N. Elezović and J. Pečarić, A note on Schur-convex functions. *Rocky Mountain Journal of Mathematics*, **30**(3): 853–856, 2000.
[7] I. Franjić and J. Pečarić, Schur-convexity and the Simpson formula. *Applied Mathematics Letters*, **24**: 1565–1568, 2011.
[8] C. T. Ng, Functions generating Schur-convex sums. *International Series of Numerical Mathematics*, **80**: 433–438, Birkhäuser, Boston, 1987.
[9] A. W. Roberts and D. E. Varberg, *Convex functions*, Academic Press, New York, 1973.
[10] N. Safaei and A. Barani, Schur-convexity related to co-ordinated convex functions in plane. *Journal of Mathematical Inequalities*, **15**(2): 467–476, 2021.
[11] H.-N. Shi, Schur-convex functions related to Hadamard-type inequalities. *Journal of Mathematical Inequalities*, **1**(1): 127–136, 2007.
[12] X. Zhang and Y. Chu, Convexity of the integral arithmetic mean of a convex function. *Rocky mountain Journal of Mathematics*, **40**(3): 1061–1068, 2010.

(Nozar Safaei) Department of Mathematics, Khorramabad Branch, Islamic Azad University, Khorramabad, Iran.
Email address: nouzarsafaei@yahoo.com