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Research Paper

ON FIXED POINT RESULTS FOR FINITE FAMILIES OF α-HEMICONTRACTIVE MAPPINGS, VARIATIONA LNEQUALITY PROBLEMS AND SPLIT EQUILIBRIUM PROBLEMS

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Abstract. ln this paper, we introduce an iterative scheme for approximating a common element of the fixed point sets of a finite family of a multivalued α-hemicontractive mappings, the set of solutions of a finite family of variational inequality problems and the set of solutions of a finite family of equilibrium problems. Using our scheme, we establish strong convergence theorems of the aforementioned problems in the framework of real Hilbert spaces. Our results improve, extend, generalise and unify many recent results in this direction.

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1. **Introduction**

Throughout this paper, we assume that H, H_1, H_2 are real Hilbert spaces, C, Q, A *and* I are nonempty, closed and convex subset of H_1 *and* H_2, a bounded linear operator from H_1 to H_2 and an identity operator on H, H_1 *or* H_2, respectively. Also, the following notations: $\mathtt{R}, \mathtt{N}, \rightarrow, \rightharpoonup$ *and* $\omega_\omega = \{x : \exists\{x_{n_j}\} \subset \{x_n\} \ni x_{n_j} \rightharpoonup x\}$ will be used to denote the set of all real numbers, the set of natural numbers, strong convergence, weak convergence and weak limit of the sequence $\{x_n\}$, respectively. Let $CB(C)$ denotes the family of nonempty, closed and bounded subsets of C, $K(C)$ denotes a family of nonempty and compact subsets of C, 2^H denotes the family of nonempty subset of H and D denotes the Hausdorff metric induced by the metric d on H. Then, the Hausdorff metric D is defined by

$$D(A, B) = \max\{(\sup d(x, B), x \in A), (\sup d(y, A), y \in B)\},$$

for all $A, B \in CB(C)$, where $d(x, B) = \inf\{\|x - b\| : b \in B\}$. A point $x \in H$ is called a fixed point of Γ if $x = \Gamma x$. If $\Gamma : H \longrightarrow 2^H$ is a multivalued map from H into the family of a nonempty subset of H, then an element $x \in C$ is called a fixed point of Γ if $x \in \Gamma x$. If $\Gamma x = \{x\}$, then x is called a strict fixed point of Γ. $F(\Gamma) = \{x \in C : x \in \Gamma x\}$ (respectively $F(\Gamma) = \{x \in C : x = \Gamma x\}$) is called fixed point set of a multivalued (respectively single-valued) map Γ whereas the set $F_s(\Gamma) = \{x \in C : \{x\} = \Gamma x\}$ is called the strict fixed point set of Γ. In an attempt to solve an equation of the form

$$Aw = 0, \tag{1.1}$$

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Browder[4] introduced an operator Γ defined by $\Gamma = I - A$. He called such an operator pseudocontractive. It is well-known that the solution of an equation of the type (1.1) now corresponds to fixed points of Γ. Generally speaking, pseudocontractive operators are not continuous, hence the reason for imposing the continuity condition (Lipschitz condition) in the course of studying the fixed point of such operators.

Definition 1.1. Let $\Gamma : C \longrightarrow CB(C)$ be a multivalued mapping. Then:

(1) Γ is called L-Lipshitizian (see, for example, [2]) if there exists $L > 0$ such that

$$D(\Gamma x, \Gamma y) \leq L\|x - y\|, \forall x, y \in C. \tag{1.2}$$

Γ is called nonexpansive if it is Lipschitz continuous with $L = 1$ in (1.2). Aside from being an obvious generalization of the contraction mapping, the class of nonexpansive mapping is one of the initial classes of mappings for which fixed point results were obtained using the geometric structure of the underlying Banach space rather than the compactness property. A nonexpansive multivalued mapping Γ with a nonempty fixed point set is called quasi-nonexpansive multivalued mapping(Recall that a mapping $\Gamma : C \longrightarrow CB(C)$ such that $D(\Gamma x, \Gamma p) \leq |x - p\|, \forall (x, p) \in C \times F(\Gamma)$ is called quasi-nonexpansive).

(2) Γ is called demicontractive (see, for example, [2]) if $F(\Gamma) = \{x \in C : x \in \Gamma x\} \neq \emptyset$ and for all $u \in \Gamma$ satisfying $\|u - p\| \leq D(\Gamma x, \Gamma p)$, there exists $k \in (0, 1)$ such that

$$D^2(\Gamma x, \Gamma p) \leq \|x - p\|^2 + k\|x - u\|^2, \forall x \in C \textit{ and } \forall p \in F(\Gamma). \tag{1.3}$$

The class of strictly pseudocontractive mappings is a subclass of the class of demicontractive mappings. In fact, a strictly pseudontractive mapping with a nonempty fixed point set is a demicontraction (Recall that a nonlinear mapping Γ is called strictly pseudocontraction if there exists $k \in (0, 1)$ such that for all $x, y \in C$, one has $D^2(\Gamma x, \Gamma y) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \forall u \in \Gamma x, v \in \Gamma y$. If $k = 1$, then one has a pseudocontraction). Note that if k in (1.3) is 1, then Γ is called hemicontractive multivalued mapping (see [19]). Thus, the class of demicontractive multivalued napping is a proper subclass of the class of hemicontractive multivalued mapping. While hemicontractive (single and multivalued) mappings are not generally continuous, the demicontractive mappings inherit Lipschitz property from the definition.

(3) Γ is said to be α-hemicontractive multivalued mapping (see, for example, [28]) if $F(\Gamma) = \{x \in C : x \in Sx\} \neq \emptyset$ and for all $u \in \Gamma$ satisfying $\|u - p\| \leq D(Sx, Sp)$, we have

$$D^2(\Gamma x, \Gamma \alpha p) \leq \|x - \alpha p\|^2 + \|x - u\|^2, \forall x \in C \textit{ and } \forall p \in F(\Gamma), \tag{1.4}$$

for some $\alpha \geq 1$. The class of mapping defined by (1.4) is a superclass of the class of α-demicontractive multivalued mapping(where a mapping $\Gamma : C \longrightarrow CB(C)$ is called α-demicontractive (see, for example, [28]) if $F(\Gamma) = \{x \in C : x \in \Gamma x\} \neq \emptyset$ and for all $u \in S$ satisfying $\|u - p\| \leq D(Sx, Sp)$, there exists $k \in (0, 1)$ such that $D^2(\Gamma x, \Gamma \alpha p) \leq |x - \alpha p\|^2 + k\|x - u\|^2$, $\forall x \in C$, $\forall p \in F(\Gamma)$ and for some $\alpha \geq 1$). It is easy to see, using Kato's Lemma (see, for example, [9]), that (1.4) is equivalent to

$$\langle x - u, x - \alpha p\rangle \geq 0, \forall x \in C\ , \forall p \in F(S), \forall u \in S \ \text{ and for some } \alpha \geq 1. \tag{1.5}$$

ln [12], Osilike and Onah introduced a new class of mapping called α-hemicontractive mapping in a closed convex subset of a real Hilbert space. They showed that the class of α-demicontractive mapping introduced by Maruster and Maruster in [18] is a subclass of the class of α-hemicontractive mapping. Also, it was shown in [12] that the class of hemicontractive mapping and the class of α-hemicontractive mapping are independent(see [12] and the reference therein for more details).

Definition 1.2. Let $A : C \longrightarrow H_1$ be a given map. Then :

(1) A is said to be monotone if for all $s, t \in C$, the inequality below holds:

$$\langle As - At, s - t\rangle \geq 0. \tag{1.6}$$

(2) A is called η-inverse strongly monotone if for all $s, t \in C$, there exists $\eta > 0$ such that

$$\langle As - At, s - t\rangle \geq \eta\|As - At\|^2. \tag{1.7}$$

It is easy to see that the class of monotone mapping properly contains the class of η-inverse strongly monotone mappings (see, for example, [2]). In addition, every η-inverse strongly monotone mapping is $\frac{1}{\eta}$-Lipschitizian mapping. Furthermore, if $A^i : C \longrightarrow H_1$ is a nonlinear mapping for each $i = 1, 2, \cdots, N$, then the finite family of variational inequality problem is to find $w \in C$ such that

$$\langle v - w, A^i w\rangle \geq 0, \forall v \in C, i = 1, 2, \cdots, N. \tag{1.8}$$

The solution set for problem (1.8) is represented with $VI(C, A^i)$ for each $i = 1, 2, \cdots, N$. Note that if $N = 1$, then (1.8) reduces to the classical variational inequality problem, which was first studied by Stampacchia [22] as a useful tool for solving a partial differential equation. It has been established that the problem of the type (1.8) is related with the convex minimization problem, the complementary problem, the problem of finding a point $s \in C$ such that $0 \in As$, fixed point problems, etc. Consequent upon the relationship between fixed point problems and variational inequality problems, several researchers have attempted to construct different iterative schemes for finding common element of solution set of variational inequality problem and fixed point problem for some nonlinear mappings in different spaces (see, for instance, [2],[5], [10], [11], [27], [24], [28],[29], [31], [36] and the references contained therein).

Definition 1.3. Let $F : C \times C \longrightarrow R$ be a bifunction. The equilibrium problem for F, which emanated from a study on variational inequality and optimization (see [41] for details), is to find a point $w \in C$ such that

$$F(w, z) \geq 0, \forall z \in C. \tag{1.9}$$

The set of all solutions of (1.9) is denoted by $EP(F)$; that is, $EP(F) = \{w \in C : F(w, z)\} \geq 0, \forall z \in C$. If $F^i : C^i \times C^i \longrightarrow R$ is a finite family of bifunctions, then the finite family of equilibrium problem is to find common elements for the set

$$EP(F^i) = \{w \in C^i : F^i(w, z), \forall z \in C^i, i = 1, 2, \cdots, N\} \geq 0, \tag{1.10}$$

which was considered in [42]. It is easy to see that (1.10) reduces to (1.9) if $N = 1$. The study has shown that several problems arising in engineering, transportation, economics, physics, optimization, etc can be reduced to finding solutions of equilibrium problems. Because of the interactions naturally evident in the mathematical formulation of physical problems, diverse means of analyzing algorithms for finding solutions of equilibrium and related problems have emerged. Subsequently, in consideration of the close relationship between equilibrium problems and fixed point problems, different iterative methods for finding the common element in the set of solutions of these problems have been investigated and studied by different researchers (see, for example, [1], [2], [3], [17], [20], [21], [23], [24], [28], [36], [41] and references contained therein).

Motivated by problem (1.9), He [43] studied the following problem consisting of two equilibrium problems: Let $F_1 : C \times C \longrightarrow R$ and $F_2 : Q \times Q \longrightarrow R$ be two bifunctions and $A : H_1 \longrightarrow H_2$ be a bounded linear operator. The split equilibrium problem (SEP, in short) is the problem of finding a point $w^\star \in C$ such that

$$F_1(w^\star, w) \geq 0, \forall w \in C, \tag{1.11}$$

and such that

$$z^\star = Aw^\star \in Q \ solves \ F_2(z^\star, z) \geq 0, \forall z \in Q. \tag{1.12}$$

The set of solution of problem (1.11) and (1.12) is denoted by

$$\Omega = \{q \in C : q \in EP(F_1) \ and \ Aq \in EP(F_2)\}.$$

Split equilibrium problem has the following problems as its consequence: split zero problem, split fixed point problem, classical equilibrium problem, split feasibility problem and split variational inequality problem, which have been investigated extensively (see, for example, [1], [2], [17], [20], [21], [24] and the references contained therein). If $F_1^i : C^i \times C^i \longrightarrow R$ and $F_2 : Q^i \times Q^i \longrightarrow R$ be two finite families of bifunctions and $A^i : H_1 \longrightarrow H_2$ a finite family of bounded linear operators, then the finite family of equilibrium problems is to find common elements for the following set:

$$\Omega^i = \{q \in C : q \in EP(F_1^i) \ and \ A^i q \in EP(F_2^i)\}, i = 1, 2, \cdots, N.$$

Remark 1.4. The problems below are some consequences of finite family of split equilibrium problem: For each $i = 1, 2, \cdots, N$,

(1) if
$$F_1^i(w, z) = \langle A^i z, w - z\rangle, \forall w, z \in C,$$
and
$$F_2^i(s, t) = \langle A^i t, s - t\rangle, \forall s, t \in Q, N$$
with some nonlinear mappings $A_1^i : C^i \longrightarrow H_1 \ and \ A_2^i : Q^i \longrightarrow H_1$, then the finite family of split equilibrium problems becomes finite family of split variational inequality problems;

(2) and for finite families of mappings $T_1^i : C^i \longrightarrow H_1 \ and \ S_2^i : Q^i \longrightarrow H_1$, if
$$F_1^i(w, z) = \langle (I - T^i)z, w - z\rangle, \forall w, z \in C,$$
and
$$F_2^i(s, t) = \langle (I - S^i)t, s - t\rangle, \forall s, t \in Q,$$
then the finite family of split equilibrium problems reduces to finite family of split fixed point problems and

(3) if $H_1 = H_2, A^i = I, Q^i = C^i \ and \ F_{2^i} \equiv 0$, then the finite family of split equilibrium problems reduces to classical finite family of equilibrium problems.

In recent times, several iterative techniques have been introduced by different researchers for approximating solutions of split equilibrium problems for finite family of bifunctions, finite family of variational inequality problems as well as a common element of the fixed point set of finite family of multivalued (or single-valued) mappings (with exemption of α-hemicontractive mappings) and the set of solutions of equilibrium problems (see, for example, [1], [21], [23], [24], [36], [43] and the references therein).

Very recently, Meche and Zegeye [24] proposed an iteration sequence (for finding common set of solutions of fixed point problem for multivalued hemicontractive-type mapping, finite family of split equilibrium and finite family of variational inequality problems) defined as follows:
Given $x_o, u \in C$, for each $i = 1, 2, \cdots, N$, let $\{x_n\}$ be a sequence in C generated by

$$\begin{cases} z_n^i = T_s^{F_1^i}(1 - \lambda^i B^{i\star}(1 - T_r^{F_2^i})B^i)x_n; \\ e_n^i = J_t^i z_n^i; \\ y_n = \sum_{i=1}^N \tau_n^i e_n^i; \\ u_n = (1 - \delta_n)y_n + \delta_n v_n; \\ x_{n+1} = a_n u + b_n w_n + c_n y_n, \end{cases} \tag{1.13}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0, \lambda \in (0, \frac{1}{d}), d = BB^\star$, where $B^\star$ is the adjoint of B, $a_n, \delta_n \subset (0, 1)$ and $b_n, c_n \subset [a, b]$ for some $a, b \in (0, 1)$.

It is our purpose in this paper to first introduce a new iterative sequence and then prove strong convergence theorems of our new iterative sequence to the common solutions of fixed point problems for finite family of α-hemicontractive mapping (which is a more general operator than the one considered by Meche and Zegeye [24], even when $N = 1$), split equilibrium problems for finite family of bifunctions and finite family of variational inequality problems.

2. **Preliminaries**

In this section, we state some concepts and results that play a crucial role in proving our main results.
Let $\Gamma : C \longrightarrow C$ be a nonexpansive mapping with $F(\Gamma) \neq \emptyset$. Then, for every $x \in C$ and $y \in F(\Gamma)$, we obtain that

$$\langle x - \Gamma x, y - \Gamma x\rangle \leq \frac{1}{2}\|\Gamma x - x\|^2. \tag{2.1}$$

(see, for example, [17], [24] and [28]). Since $\emptyset \neq C \subset H_1$ is closed and convex, then for every $x \in H_1$, there exists a unique nearest point $P_C x \in C$ such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}. \tag{2.2}$$

The mapping $P_C : H \longrightarrow 2^C$ is characterised by

$$z = P_C x \in C \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \forall x \in H_1, y \in C. \tag{2.3}$$

Consider a multivalued mapping $\Gamma : C \longrightarrow CB(C)$ and a sequence $\{x_n\} \subset C : x_n \rightharpoonup x$. Then, $(I - \Gamma)$ is said to be demiclosed at zero if $x \in \Gamma x$ whenever $\lim_{n\to\infty} d(x_n, \Gamma x_n) = 0$, where I denotes the identity mapping on C. It has been established that if $\Gamma : C \longrightarrow C$ is a single-valued nonexpansive mapping, then $(I - \Gamma)$ is demiclosed at zero (see, for example,

[23]).

Conversely, given a β-inverse strongly monotone ($A : C \longrightarrow H_1$) and $\lambda \in (0, 2\beta)$, then $I - \lambda A$ is a nonexpansive mapping from C into H_1 (see, for example, [29],[30]). Nevertheless, if $\Gamma : C \longrightarrow H_1$ is nonexpansive mapping, then $A = I - \Gamma$ is $\dfrac{1}{2}$-inverse strongly monotone mapping (interested reader should consult [39] for more details).

In what follows, we shall use the following common assumptions:

Assumption G : Let H be a Hilbert space and C a nonempty, closed and convex subset of H. Let $F : C \times C \longrightarrow R$ be any given bifunction satisfying the following conditions:
$G_1 : F(x, x) = 0, \forall x \in H$
G_2 :F is a monotone, i.e, $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$
$G_3 : \lim_{t \longrightarrow 0}(tz + (1 - t)y) \leq F(x, y), \forall x, y, z \in C$
G_4 : for each $x \in C, y \to F(x, y)$ is convex and lower semicontinous.
ln the proof of our main results, we shall make use of the following familiar lemmas:

Lemma 2.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\Gamma : C \longrightarrow CB(C)$ be an L-Lipschitz multivalued mapping with $F(\Gamma) \neq \emptyset$ and $\Gamma(\alpha q) = \alpha q$, for all $q \in F(\Gamma)$, for some $\alpha \geq 1$. Then, $F(\Gamma)$ is a closed subset of C.*

Proof. Using the technique as in the proof of Lemma 2.10 in [24], for some $\alpha \geq 1$, the proof follows immediately. □

Lemma 2.2. [24] *Let H be a Hilbert space. Then, for all $x_i \in H$ and $\alpha_i \in [0, 1]$, for $i = 1, 2, 3, \cdots, N$ such that $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_N = 1$, the following equality holds:*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_N x_N\|^2 = \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \sum_{1 \leq i,j \leq N} \alpha_i \alpha_j \|x_i - x_j\|.$$

Lemma 2.3. [24, 28] *Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers such that*

$$b_{n+1} \leq (1 - \alpha_n) b_n + \alpha_n \delta_n, \ for\ n \geq n_0,$$

where $n_0 \in N$ and the control sequences $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\{\delta_n\}_{n=0}^{\infty} \subset R$ satisfy the following restrictions :

$$\lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \lim_{n \to \infty} sup \delta_n \leq 0.$$

Then, $\lim_{n \to \infty} b_n = 0$.

Lemma 2.4. [27, 28] *Let H be a real Hilbert space. Then, for every $x, y \in H$, we have the following:*

i. $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle$;
ii. $\|x + y\| = \|x\|^2 + 2\langle x, x + y \rangle$.

Lemma 2.5. [40] *Let H be a real Hilbert space. Let $A, B \in CB(H)$ and $a \in A$. Then for any $\epsilon > 0$, there exists a point $b \in B$ such that $\|a - b\| \leq D(A, B) + \epsilon$. In particular, for every $a \in A$, there exists an element $b \in B$ such that $\|a - b\| \leq 2D(A, B) + \epsilon$.*

Lemma 2.6. [24, 28] *Let H be a Hilbert space and C a closed convex subset of H_1. Let $A : C \longrightarrow H$ be a continuous monotone mapping. Then, for $t > 0$ and for all $x \in H$, there exists $z \in C$ such that*

$$\langle Az, y - z \rangle + \frac{i}{t}\langle y - z, z - x \rangle \geq 0, \forall x, y \in C$$

Moreover, the mapping $J_t : H_1 \longrightarrow C$ *defined by*

$$J_t = \{z \in C : \langle Az, z - x\rangle + \frac{i}{t}\langle y - z, z - x\rangle \geq 0, \forall y \in C\},$$

is well-defined and satisfies the following properties:

(1) J_t *is nonempty and single valued;*
(2) J_t *is firmly nonexpansive; that is,*

$$\|J_t x - J_t y\|^2 \leq \langle J_t x - J_t y, x - y\rangle, \forall x, y \in H;$$

(3) $F(J_t) = VI(C, A)$*;*
(4) $VI(C, A)$ *is closed and convex.*

Lemma 2.7. [28] *Let* $F_1 : C \times C \longrightarrow R$ *be a bifunction satisfying assumption G. For* $s > 0$ *and for all* $x \in H_1$*, the mapping* $T_s^{F_1} : H_1 \longrightarrow C$ *defined by*

$$T_s^{F_1} x = \{x \in C : F_1(x, y) + \frac{1}{s}\langle y - z, x - y\rangle \geq 0, \forall y \in C\} \tag{2.4}$$

is well-defined and have the following properties:

(1) $T_s^{F_1}$ *is nonempty and single valued;*
(2) $T_s^{F_1}$ *is firmly nonexpansive, i.e,* $\|T_s^{F_1} x - T_s^{F_1} y\| \leq \langle T_s^{F_1} x - T_s^{F_1} y, x - y\rangle$*;*
(3) $F(T_s^{F_1}) = EP(T_s^{F_1})$*;*
(4) $EP(F_1)$ *is closed and convex.*

Furthermore, assume that $F_2 : Q \times Q \longrightarrow R$ *is another bifunction that satisfies assumption G. For* $r > 0$ *and for all* $x \in H_2$ *define the mapping* $T_s^{F_2} : H_2 \longrightarrow Q$ *as follows:*

$$T_s^{F_2} x = \{x \in Q : F_2(x, y) + \frac{1}{s}\langle y - z, x - y\rangle \geq 0, \forall y \in Q\} \tag{2.5}$$

Then, we have the following:

(1) $T_s^{F_2}$ *is nonempty and single valued;*
(2) $T_s^{F_2}$ *is firmly nonexpansive, i.e,* $\|T_s^{F_2} x - T_s^{F_2} y\| \leq \langle T_s^{F_2} x - T_s^{F_2} y, x - y\rangle$*;*
(3) $F(T_s^{F_2}) = EP(T_s^{F_2})$*;*
(4) $EP(F_2)$ *is closed and convex.*

Lemma 2.8. [26] *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} \leq a_{n_{i+1}}$ *for all* $i \in N$*. Then, there exists a nondecreasing sequence* $m_k \in N$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in N : a_{m_k} < a_{m_{k+1}}$ *and* $a_k < a_{m_{k+1}}$*. ln fact,* $m_k = max\{j \leq k : a_j < a_{j+1}\}$

Lemma 2.9. [37] *Let* $\{d_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative real numbers,* $\{\delta_n\}_{n=0}^{\infty}$ *be a sequence in* $(0, 1)$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{e_n\}_{n=0}^{\infty}$ *be a sequence of real numbers. Suppose that*

$$d_{n+} \leq (1 - \alpha_n)d_n + \alpha_n e_n, \forall n \geq 0.$$

If $\limsup_{k\to\infty} e_{n_k} \leq 0$ *for every subsequence* $\{d_{n_k}\}$ *of* $\{d_n\}$ *satisfying*

$$\liminf_{k\to\infty}(d_{n_k+1} - d_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} d_n = 0$.

3. **Main Results**

Now, we consider the following algorithm. Let H_1 and H_2 be real Hilbert spaces. Let $\{C^i\}_{i=1}^N$ and $\{Q^i\}_{i=1}^N$ be finite families of nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $\{A^i\}_{i=1}^N$ and $\{B^i\}_{i=1}^N$ be finite families of continuous monotone mappings and bounded linear operators (with their adjoints $\{B^{i\star}\}_{i=1}^N$) such that for all $i = 1, 2, \cdots, N$, $A^i : C^i \longrightarrow H_1$ and $B^i : H_1 \longrightarrow H_2$, respectively. Let $\{F_1^i\}_{i=1}^N$ and $\{F_2^i\}_{i=1}^N$ be finite families of bifunctions and $\{\Gamma^i\}_{i=1}^N$ a finite family of α-hemicontractive mappings such that $F_1^i : C^i \times C^i \longrightarrow R$, $F_2^i : Q^i \times Q^i \longrightarrow R$ and $\Gamma^i : C^i \longrightarrow CB(C^i)$ for some $\alpha \geq 1$, for all $i = 1, 2, \cdots, N$, respectively. Let $\{\{\delta_n^i\}_{i=1}^N\}_{n=0}^\infty$ and $\{\{\alpha_n^i\}_{i=1}^N\}_{n=0}^\infty$ be sequences in $(0, 1)$, $\{\{\tau_n^i\}_{i=1}^N\}_{n=0}^\infty$ a sequence in $(0, 1]$ and $\{\{\gamma_n^i\}_{i=1}^N\}_{n=0}^\infty$ and $\{\{\sigma_n^i\}_{i=1}^N\}_{n=0}^\infty$ two sequences in $[\delta, \xi]$ for some $\delta, \xi \in (0, 1)$, for all $i = 1, 2, \cdots, N$, then for arbitrary $x_0, u \in C^i$, we generate $\{x_n\}_{n=0}^\infty$ iteratively as follows:

Algorithm 3.1

$$
\begin{cases}
z_n^i = T_\sigma^{F_1^i}(I - \lambda^i B^{i\star}(I - T_\tau^{F_2^i})B^i)x_n; \\
t_n^i = J_\mu^i z_n^i; \\
s_n^i = \sum_{i=1}^N \tau_n^i t_n^i; \\
u_n^i = (1 - \delta_n^i)s_n^i + \delta_n^i v_n^i; \\
x_{n+1} = \alpha_n^i u + \beta_n^i[(1 - \gamma_n^i)w_n^i + \gamma_n^i x_n] + \sigma_n^i s_n^i,
\end{cases}
\tag{3.1}
$$

for $n \geq 0$, where $v_n^i \in \Gamma^i s_n^i$ and $w_n^i \in \Gamma^i u_n^i$ are such that $\|v_n^i - w_n^i\| \leq 2D(\Gamma^i s_n^i, \Gamma^i u_n^i)$, for each $i = 1, 2, \cdots, N$ and $\sigma, \tau, \mu > 0, \lambda^i \in \left(0, \dfrac{1}{\eta^i}\right)$, where $\eta^i = \|B^i\|^2$.

Theorem 3.1. *Let H_1, H_2, $\{\Gamma^i\}_{i=1}^N$, $\{C^i\}_{i=1}^N, \{Q^i\}_{i=1}^N, \{A^i\}_{i=1}^N, \{B^i\}_{i=1}^N, \{B^{i\star}\}_{i=1}^N, \{F_1^i\}_{i=1}^N$, $\{F_2^i\}_{i=1}^N$, $\{\{\delta_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\alpha_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\tau_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\gamma_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\sigma_n^i\}_{i=1}^N\}_{n=0}^\infty$ be as in Algorithm 3.1. Suppose F_1^i and F_2^i satisfying Assumption G, $\Theta = \cap_{i=1}^N(\Omega_i \cap VI(C^i, A^i)) \cap_{i=1}^N F(\Gamma^i) \neq \emptyset$ and $\Gamma^i \alpha q = \alpha q$ for all $q \in \Theta$, for all $i = 1, 2, \cdots, N$ and for some $\alpha \geq 1$, $\{x_n\}_{n=0}^\infty$ is bounded if for each $i = 1, 2, \cdots, N$ and for all $n \geq 0$,*

i. $\alpha_n^i + \beta_n^i + \sigma_n^i = 1$;
ii. $\sum_{n=1}^N \tau_n^i = 1$;
iii. $\alpha_n^i + \beta_n^i \leq \delta_n^i \leq \sigma < \frac{1}{\sqrt{1+4L^2}+1}$.

Proof. In view of condition (ii) of Lemma 2.7, it follows that $T_\sigma^{F_1^i}$ is firmly nonexpansive and hence nonexpansive for each $i = 1, 2, \cdots, N$. By virtue of the nonexpansiveness of $T_\tau^{F_2^i}$ which $I - T_\tau^{F_2^i}$ inherited as a consequence, it follows from the given hypothesis and Cauchy-Schwartz inequality that $B^{i\star}(I - T_\tau^{F_2^i})B^i$ is $\dfrac{1}{2\eta^i}$-inversely strongly monotone mappings for each $i = 1, 2, \cdots, N$. Since $\lambda^i \in \left(0, \dfrac{1}{\eta^i}\right)$ for each $i = 1, 2, \cdots, N$, we obtain that $I - B^{i\star}(I - T_\tau^{F_2^i})B^i$ is nonexpansive. Also, using condition (ii) of Lemma 2.7, $T_\sigma^{F_1^i}$ is nonexpansive for each $i = 1, 2, \cdots, N$.

From the above information, we have from (3.1) that

$$\|T_{\sigma}^{F_1^i}\Big(I - B^{i\star}(I - T_{\tau}^{F_2^i})B^i\Big)x - T_{\sigma}^{F_1^i}\Big(I - B^{i\star}(I - T_{\tau}^{F_2^i})B^i\Big)y\| \leq \|x - y\|. \tag{3.2}$$

Now, let $\alpha q \in \Theta$, for some $\alpha \geq 1$. Then, we obtain $\Gamma^i(\alpha q) = \alpha q, J_{\mu}^i(\alpha q) = \alpha q, \alpha q \in \Omega_i,$ *for each* $i = 1, 2, \cdots, N$ and as a consequence $T_{\sigma}^{F_1^i}(\alpha q) = \alpha q$ and $B^i(\alpha q) = \alpha q$, for each $i = 1, 2, \cdots, N$. This implies that $T_{\sigma}^{F_1^i}\Big(I - B^{i\star}(I - T_{\tau}^{F_2^i})B^i\Big)\alpha q = \alpha q$. Hence, from (3.1), we obtain

$$\|z_n^i - \alpha q\| = \|T_{\sigma}^{F_1^i}\Big(I - B^{i\star}(I - T_{\tau}^{F_2^i})B^i\Big)x_n - \alpha q\| \leq \|x_n - \alpha q\|. \tag{3.3}$$

Since, by condition (ii) of Lemma 2.6, J_{σ}^i is firmly nonexpansive and hence nonexpansive for each $i = 1, 2, \cdots, N$, it follows from (3.1) and (3.3) that

$$\|t_n^i - \alpha q\| = \|J_{\sigma}^i z_n^i - J_{\sigma}^i(\alpha q)\| \leq \|z_n^i - \alpha q\| \leq \|x_n - \alpha q\| \tag{3.4}$$

Using (3.1), (3.4), condition (ii) and triangular inequality, we get

$$\|s_n^i - \alpha q\| = \|\sum_{i=1}^{N} \tau_n^i t_n^i - \alpha q\| \leq \sum_{i=1}^{N} \tau_n^i \|t_n^i - \alpha q\| \leq \sum_{i=1}^{N} \tau_n^i \|x_n - \alpha q\| = \|x_n - \alpha q\|. \tag{3.5}$$

Since Γ^i is α-hemicontractive mappings for each $i = 1, 2, \cdots, N$ and $w_n^i \in \Gamma^i u_n^i$, it follows that

$$\|w_n^i - \alpha q\|^2 \leq D^2(\Gamma^i u_n^i, \Gamma^i \alpha q) \leq \|u_n^i - \alpha q\|^2 + \|u_n^i - w_n^i\|^2, \tag{3.6}$$

Again, since Γ^i is α-hemicontractive mappings for each $i = 1, 2, \cdots, N$, for some $\alpha \geq 1$ and $u_n^i \in \Gamma^i s_n^i$, we get, using (3.1), (3.5) and Lemma 2.2, that

$$\begin{aligned}
\|u_n^i - \alpha q\|^2 &= \|(1 - \delta_n^i)s_n^i + \delta_n^i v_n^i - \alpha q\|^2 \\
&= (1 - \delta_n^i)\|s_n^i - \alpha q\|^2 + \delta_n^i\|v_n^i - \alpha q\|^2 + \delta_n^i(1 - \delta_n^i)\|s_n^i - v_n^i\|^2 \\
&\leq (1 - \delta_n^i)\|s_n^i - \alpha q\|^2 + \delta_n^i D^2(\Gamma^i s_n^i, \Gamma^i(\alpha q)) + \delta_n^i\|s_n^i - v_n^i\|^2 + (\delta_n^i)^2\|s_n^i - v_n^i\|^2 \\
&\leq (1 - \delta_n^i)\|s_n^i - \alpha q\|^2 + \delta_n^i[\|s_n^i - \alpha q\|^2 + \|s_n^i - v_n^i\|^2] + \delta_n^i\|s_n^i - v_n^i\|^2 \\
&\quad +(\delta_n^i)^2\|s_n^i - v_n^i\|^2 \\
&= \|x_n - \alpha q\|^2 + (\delta_n^i)^2\|s_n^i - v_n^i\|^2 \qquad (3.7)
\end{aligned}$$

(3.6) and (3.7) imply that

$$\|w_n^i - \alpha q\|^2 \leq \|x_n - \alpha q\|^2 + (\delta_n^i)^2\|s_n^i - v_n^i\|^2 + \|u_n^i - w_n^i\|^2, \tag{3.8}$$

Since from (3.1)

$$\|s_n^i - u_n^i\|^2 = \|s_n^i - [(1 - \delta_n^i)s_n^i + \delta_n^i v_n^i]\|^2 = (\delta_n^i)^2\|s_n^i - v_n^i\|^2, \tag{3.9}$$

it follows from Lemma 2.2 (and the assumption $\|v_n^i - w_n^i\| \leq 2D(\Gamma^i s_n^i, \Gamma^i u_n^i)$) that

$$\begin{aligned}
\|u_n^i - w_n^i\|^2 &= \|(1 - \delta_n^i)(s_n^i - w_n^i) + \delta_n^i(v_n^i - w_n^i)\|^2 \\
&= (1 - \delta_n^i)\|s_n^i - w_n^i\|^2 + \delta_n^i\|v_n^i - w_n^i)\|^2 - \delta_n^i(1 - \delta_n^i)\|s_n^i - v_n^i\|^2 \\
&\leq (1 - \delta_n^i)\|s_n^i - w_n^i\|^2 + 4\delta_n^i D^2(\Gamma^i s_n^i, \Gamma^i u_n^i) \\
&\quad -\delta_n^i(1 - \delta_n^i)\|s_n^i - v_n^i\|^2. \qquad (3.10)
\end{aligned}$$

From (3.10) and the fact that Γ^i is L-Lipschitizian for each $i = 1, 2, \cdots, N$, we have

$$\begin{aligned} \|u_n^i - w_n^i\|^2 &\leq (1-\delta_n^i)\|s_n^i - w_n^i\|^2 + 4\delta_n^i L^2\|s_n^i - u_n^i\|^2 - \delta_n^i(1-\delta_n^i)\|s_n^i - v_n^i\|^2 \\ &= (1-\delta_n^i)\|s_n^i - w_n^i\|^2 + 4(\delta_n^i)^3 L^2\|s_n^i - v_n^i\|^2 - \delta_n^i\|s_n^i - v_n^i\|^2 \\ &\quad +(\delta_n^i)^2\|s_n^i - v_n^i\|^2 \\ &= (1-\delta_n^i)\|s_n^i - w_n^i\|^2 + \delta_n^i[4(\delta_n^i)^2 L^2 + \delta_n^i]\|s_n^i - v_n^i\|^2 \\ &\quad -\delta_n^i\|s_n^i - v_n^i\|^2. \end{aligned} \tag{3.11}$$

(3.8) and (3.11) imply that

$$\begin{aligned} \|w_n^i - \alpha q\|^2 &\leq \|x_n - \alpha q\|^2 + (\delta_n^i)^2\|s_n^i - v_n^i\|^2 + (1-\delta_n^i)\|s_n^i - w_n^i\|^2 \\ &\quad +\delta_n^i[4(\delta_n^i)^2 L^2 + \delta_n^i - 1]\|s_n^i - v_n^i\|^2 \\ &= \|x_n - \alpha q\|^2 + (1-\delta_n^i)\|s_n^i - w_n^i\|^2 - \delta_n^i[1 - 4(\delta_n^i)^2 L^2 - 2\delta_n^i] \\ &\quad \times\|s_n^i - v_n^i\|^2. \end{aligned} \tag{3.12}$$

Next, we estimate $\|x_{n+1} - \alpha q\|$:
Since from (3.1)

$$\|x_{n+1} - \alpha q\|^2 = \|\alpha_n^i(u - \alpha q) + \beta_n^i[(1-\gamma_n^i)(w_n^i - \alpha q) + \gamma_n^i(x_n - \alpha q)] + \sigma_n^i(s_n^i - \alpha q)\|^2,$$

it follows from Lemma 2.2 that

$$\begin{aligned} \|x_{n+1} - \alpha q\|^2 &= \alpha_n^i\|u - \alpha q\|^2 + \beta_n^i\|(1-\gamma_n^i)(w_n^i - \alpha q) + \gamma_n^i(x_n - \alpha q)\|^2 + \sigma_n^i\|s_n^i - \alpha q\|^2 \\ &\quad -\sigma_n^i\beta_n^i\|(1-\gamma_n^i)w_n^i + \gamma_n^i x_n - s_n^i\|^2 \\ &= \alpha_n^i\|u - \alpha q\|^2 + \beta_n^i\|(1-\gamma_n^i)(w_n^i - \alpha q) + \gamma_n^i(x_n - \alpha q)\|^2 + \sigma_n^i\|s_n^i - \alpha q\|^2 \\ &\quad -\sigma_n^i\beta_n^i\|(1-\gamma_n^i)(w_n^i - s_n^i) + \gamma_n^i(x_n - s_n^i)\|^2 \\ &= \alpha_n^i\|u - \alpha q\|^2 + \beta_n^i[(1-\gamma_n^i)\|w_n^i - \alpha q\|^2 + \gamma_n^i\|x_n - \alpha q\|^2 \\ &\quad -\gamma_n^i(1-\gamma_n^i)\|w_n^i - x_n\|^2] + \sigma_n^i\|s_n^i - \alpha q\|^2 - \sigma_n^i\beta_n^i[(1-\gamma_n^i)\|w_n^i - s_n^i\|^2 \\ &\quad +\gamma_n^i\|x_n - s_n^i\|^2 - \gamma_n^i(1-\gamma_n^i)\|w_n^i - x_n\|^2] \\ &= \alpha_n^i\|u - \alpha q\|^2 + \beta_n^i(1-\gamma_n^i)\|w_n^i - \alpha q\|^2 + \beta_n^i\gamma_n^i\|x_n - \alpha q\|^2 \\ &\quad -\beta_n^i\gamma_n^i(1-\gamma_n^i)\|w_n^i - x_n\|^2 + \sigma_n^i\|s_n^i - \alpha q\|^2 - \sigma_n^i\beta_n^i(1-\gamma_n^i)\|w_n^i - s_n^i\|^2 \\ &\quad -\sigma_n^i\beta_n^i\gamma_n^i\|x_n - s_n^i\|^2 + \sigma_n^i\beta_n^i\gamma_n^i(1-\gamma_n^i)\|w_n^i - x_n\|^2 \\ &\leq \alpha_n^i\|u - \alpha q\|^2 + \beta_n^i(1-\gamma_n^i)\|w_n^i - \alpha q\|^2 + \beta_n^i\gamma_n^i\|x_n - \alpha q\|^2 + \sigma_n^i\|s_n^i - \alpha q\|^2 \\ &\quad -\sigma_n^i\beta_n^i(1-\gamma_n^i)\|w_n^i - s_n^i\|^2 + \beta_n^i\gamma_n^i(1-\gamma_n^i)(\sigma_n^i - 1)\|w_n^i - x_n\|^2. \end{aligned} \tag{3.13}$$

Substituting (3.12) into (3.13) and simplifying, we have

$$\begin{aligned} \|x_{n+1} - \alpha q\|^2 &\leq \alpha_n^i\|u - \alpha q\|^2 + \beta_n^i(1-\gamma_n^i)\|x_n - \alpha q\|^2 + \beta_n^i(1-\gamma_n^i)(1-\delta_n^i)\|s_n^i - w_n^i\|^2 \\ &\quad -\beta_n^i(1-\gamma_n^i)\delta_n^i[1 - 4(\delta_n^i)^2 L^2 - 2\delta_n^i]\|s_n^i - v_n^i\|^2 + \beta_n^i\gamma_n^i\|x_n - \alpha q\|^2 \\ &\quad +\beta_n^i\gamma_n^i(1-\gamma_n^i)(\sigma_n^i - 1)\|w_n^i - x_n\|^2 + \sigma_n^i\|s_n^i - \alpha q\|^2 \\ &\quad -\sigma_n^i\beta_n^i(1-\gamma_n^i)\|w_n^i - s_n^i\|^2, \end{aligned}$$

which by (3.5) yields

$$\begin{aligned}
\|x_{n+1}-\alpha q\|^2 &\leq \alpha_n^i\|u-\alpha q\|^2+\beta_n^i(1-\gamma_n^i)\|x_n-\alpha q\|^2+\beta_n^i(1-\gamma_n^i)(1-\delta_n^i)\|s_n^i-w_n^i\|^2\\
&\quad -\beta_n^i(1-\gamma_n^i)\delta_n^i[1-4(\delta_n^i)^2L^2-2\delta_n^i]\|s_n^i-v_n^i\|^2+\beta_n^i\gamma_n^i\|x_n-\alpha q\|^2\\
&\quad +\beta_n^i\gamma_n^i(1-\gamma_n^i)(\sigma_n^i-1)\|w_n^i-x_n\|^2+\sigma_n^i\|x_n-\alpha q\|^2\\
&\quad -\sigma_n^i\beta_n^i(1-\gamma_n^i)\|w_n^i-s_n^i\|^2\\
&= \alpha_n^i\|u-\alpha q\|^2+[\beta_n^i(1-\gamma_n^i)+\beta_n^i\gamma_n^i+\sigma_n^i]\|x_n-\alpha q\|^2\\
&\quad +\beta_n^i(1-\gamma_n^i)(1-\delta_n^i-\sigma_n^i)\|s_n^i-w_n^i\|^2-\beta_n^i(1-\gamma_n^i)\delta_n^i[1-4(\delta_n^i)^2L^2-2\delta_n^i]\\
&\quad \times\|s_n^i-v_n^i\|^2+\beta_n^i\gamma_n^i(1-\gamma_n^i)(\sigma_n^i-1)\|w_n^i-x_n\|^2\\
&\leq \alpha_n^i\|u-\alpha q\|^2+(\beta_n^i+\sigma_n^i)\|x_n-\alpha q\|^2+\beta_n^i(1-\gamma_n^i)(1-\delta_n^i-\sigma_n^i)\|s_n^i-w_n^i\|^2\\
&\quad -\beta_n^i(1-\gamma_n^i)\delta_n^i[1-4(\delta_n^i)^2L^2-2\delta_n^i]\|s_n^i-v_n^i\|^2. \qquad (3.14)
\end{aligned}$$

It is easy to see from conditions [(i) and (ii)] that

$$\begin{cases}1-4(\delta_n^i)^2L^2-2\delta_n^i\geq 1-4(\sigma_n^i)^2L^2-2\sigma_n^i>0;\\ \beta_n^i\gamma_n^i(1-\delta_n^i-\sigma_n^i)=\beta_n^i\gamma_n^i(\alpha_n^i+\beta_n^in^i-\sigma_n^i)\leq 0\\ \alpha_n^i+\beta_n^i+\sigma_n^i=1.\end{cases} \qquad (3.15)$$

(3.14) and (3.15) imply that

$$\begin{aligned}\|x_{n+1}-\alpha q\|^2 &\leq \alpha_n^i\|u-\alpha q\|^2+(\beta_n^i+\sigma_n^i)\|x_n-\alpha q\|^2\\ &\leq max\{\|u-\alpha q\|^2,\|x_n-\alpha q\|^2\}.\end{aligned}$$

Using mathematical induction priciple, it follows from the last inequality that

$$\|x_{n+1}-\alpha q\|^2 \leq max\{\|u-\alpha q\|^2,\|x_n-\alpha q\|^2\}.$$

Hence, the sequence $\{x_n\}_{n=0}^\infty$ is bounded as required. As a consequence, the following sequences are also bounded:$\{\{s_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{z_n^i\}_{i=1}^N\}_{n=0}^\infty$ *and* $\{\{u_n^i\}_{i=1}^N\}_{n=0}^\infty$. This completes the proof. □

Theorem 3.2. *Let H_1, H_2, $\{C^i\}_{i=1}^N, \{Q^i\}_{i=1}^N, \{A^i\}_{i=1}^N, \{B^i\}_{i=1}^N, \{B^{i\star}\}_{i=1}^N, \{F_1^i\}_{i=1}^N, \{F_2^i\}_{i=1}^N$, $\{\{\delta_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\alpha_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\tau_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\gamma_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\sigma_n^i\}_{i=1}^N\}_{n=0}^\infty$ be as in Algorithm 3.1. Let $\Gamma^i: C^i\longrightarrow CB(C^i)$ be an L-Lipschitz α-hemicontractive multivalued mappings such that $(I-\Gamma^i)$ is demiclosed at zero for each $i=1,2,\cdots,N$ and for some $\alpha\geq 1$. Suppose F_1^i and F_2^i satisfying Assumption G, $\Theta=\cap_{i=1}^N(\Omega_i\cap VI(C^i,A^i))\cap_{i=1}^N F(\Gamma^i)\neq\emptyset$ and $\Gamma^i\alpha q=\alpha q$ for all $q\in\Theta$, for all $i=1,2,\cdots,N$ and for some $\alpha\geq 1$. If for each $i=1,2,\cdots,N$ and for all $n\geq 0$,*

i. $\alpha_n^i+\beta_n^i+\sigma_n^i=1$ *and* $0<\alpha\leq\gamma_n^i,\sigma_n^i\leq\beta<1$;
ii. $\lim_{n\to\infty}\alpha_n^i=0, \sum_{n=0}^\infty=\infty$
iii. $\sum_{n=1}^N\tau_n^i=1$ *and* $0<\delta\leq\tau_n^i\leq 1$;
iv. $\alpha_n^i+\beta_n^i\leq\delta_n^i\leq\sigma<\frac{1}{\sqrt{1+4L^2}+1}$.

Let $x_0,u\in C^i$ be arbitrary for each $i=1,2,\cdots,N$. Then, the sequence $\{x_n\}_{n=0}^\infty$ given by (3.1) strongly converges to $\alpha q=P_\Theta(u)$.

Proof. We note that P_Θ is well defined since Θ is nonempty, closed subset of C^i, for each $i=1,2,\cdots,N$, and from Theorem 3.1, the sequence $\{x_n\}_{n=0}^\infty$ is bounded and so are the sequences $\{\{s_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{z_n^i\}_{i=1}^N\}_{n=0}^\infty$ *and* $\{\{u_n^i\}_{i=1}^N\}_{n=0}^\infty$ for each $i = 1, 2, \cdots, N$. Let $\alpha q \in \Theta$. Then, by the nonexpansivity of $T_\sigma^{F_1^i}$, for each $i = 1, 2, \cdots, N$, we obtain

$$\begin{aligned} \|z_n^i - \alpha q\|^2 &= \|T_\sigma^{F_1^i}\Big(I - B^{i\star}(I - T_\tau^{F_2^i})B^i\Big)x_n - T_\sigma^{F_1^i}\Big(I - B^{i\star}(I - T_\tau^{F_2^i})B^i\Big)(\alpha q)\|^2 \\ &\leq \|\Big(I - B^{i\star}(I - T_\tau^{F_2^i})B^i\Big)x_n - \Big(I - B^{i\star}(I - T_\tau^{F_2^i})B^i\Big)(\alpha q)\|^2 \\ &= \|(x_n - \alpha q) - \lambda^i\Big[B^{i\star}(I - T_\tau^{F_2^i})B^i x_n - B^{i\star}(I - T_\tau^{F_2^i})B^i(\alpha q)\Big]\|^2 \\ &= \|x_n - \alpha q\|^2 - 2\lambda^i\langle x_n - \alpha q, B^{i\star}(I - T_\tau^{F_2^i})B^i x_n - B^{i\star}(I - T_\tau^{F_2^i})B^i(\alpha q)\rangle \\ &\quad +(\lambda^i)^2\|B^{i\star}(I - T_\tau^{F_2^i})B^i x_n - B^{i\star}(I - T_\tau^{F_2^i})B^i(\alpha q)\|^2. \end{aligned} \tag{3.16}$$

Since $B^{i\star}(I - T_\tau^{F_2^i})B^i$ is $\dfrac{1}{2\eta^i}$-inverse strongly monotone and $B^i(\alpha q) = T_\tau^{F_2^i}B^i(\alpha q)$, for each $i = 1, 2, \cdots, N$, it follows from (3.16) that

$$\begin{aligned} \|z_n^i - \alpha q\|^2 &\leq \|x_n - \alpha q\|^2 - \frac{\lambda^i}{\eta^i}\|B^{i\star}(I - T_\tau^{F_2^i})B^i x_n - B^{i\star}(I - T_\tau^{F_2^i})B^i(\alpha q)\|^2 \\ &\quad +(\lambda^i)^2\|B^{i\star}(I - T_\tau^{F_2^i})B^i x_n - B^{i\star}(I - T_\tau^{F_2^i})B^i(\alpha q)\|^2 \\ &= \|x_n - \alpha q\|^2 \\ &\quad +\lambda^i(\lambda^i - \frac{\lambda^i}{\eta^i})\|B^{i\star}(I - T_\tau^{F_2^i})B^i x_n - B^{i\star}(I - T_\tau^{F_2^i})B^i(\alpha q)\|^2. \end{aligned} \tag{3.17}$$

Further, from (3.1), (3.3), Lemma 2.2 and (3.17), we obtain

$$\begin{aligned} \|s_n^i - \alpha q\|^2 &\leq \sum_{i=1}^N \|t_n^i - \alpha q\|^2 \\ &\leq \sum_{i=1}^N \|z_n^i - \alpha q\|^2 \\ &\leq \|x_n - \alpha q\|^2 + \sum_{i=1}^N \lambda^i\Big(\lambda^i - \frac{\lambda^i}{\eta^i}\Big) \\ &\quad \times\|B^{i\star}(I - T_\tau^{F_2^i})B^i x_n - B^{i\star}(I - T_\tau^{F_2^i})B^i(\alpha q)\|^2. \end{aligned} \tag{3.18}$$

Again, from (3.1) and condition (ii) of Lemma 2.4, we get

$$\begin{aligned} \|x_{n+1} - \alpha q\|^2 &= \|\alpha_n^i u + \beta_n^i[(1 - \gamma_n^i)w_n^i + \gamma_n^i x_n] + \sigma_n^i s_n^i - \alpha q\|^2 \\ &= \|\beta_n^i[((1 - \gamma_n^i)w_n^i + \gamma_n^i x_n) - \alpha q] + \sigma_n^i(s_n^i - \alpha q) + \alpha_n^i(u - \alpha q)\|^2 \\ &\leq \|\beta_n^i[((1 - \gamma_n^i)w_n^i + \gamma_n^i x_n) - \alpha q] + \sigma_n^i(s_n^i - \alpha q)\|^2 + 2\alpha_n^i\langle u - \alpha q, x_{n+1} - \alpha q\rangle, \end{aligned}$$

which by Lemma 2.2 yields

$$\begin{aligned}
\|x_{n+1}-\alpha q\|^2 &\leq \beta_n^i\|(1-\gamma_n^i)w_n^i+\gamma_n^i x_n-\alpha q\|^2+\sigma_n^i\|s_n^i-\alpha q)\|^2\\
&\quad -\sigma_n^i\beta_n^i\|(1-\gamma_n^i)w_n^i+\gamma_n^i x_n-s_n^i\|^2+2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle\\
&= \beta_n^i\|(1-\gamma_n^i)(w_n^i-\alpha q)+\gamma_n^i(x_n-\alpha q)\|^2+\sigma_n^i\|s_n^i-\alpha q)\|^2\\
&\quad -\sigma_n^i\beta_n^i\|(1-\gamma_n^i)(w_n^i-s_n^i)+\gamma_n^i(x_n-s_n^i)\|^2+2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle\\
&= \beta_n^i(1-\gamma_n^i)\|w_n^i-\alpha q\|^2+\beta_n^i\gamma_n^i\|x_n-\alpha q\|^2+\sigma_n^i\|s_n^i-\alpha q)\|^2\\
&\quad +\beta_n^i\gamma_n^i(1-\gamma_n^i)(\sigma_n^i-1)\|w_n^i-x_n\|^2-\sigma_n^i\beta_n^i 1-\gamma_n^i)\|w_n^i-s_n^i\|^2\\
&\quad -\sigma_n^i\beta_n^i\gamma_n^i\|x_n-s_n^i)\|^2+2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle. \qquad (3.19)
\end{aligned}$$

Using (3.12) into (3.19) and simplifying, we get

$$\begin{aligned}
\|x_{n+1}-\alpha q\|^2 &\leq \beta_n^i(1-\gamma_n^i)\|x_n-\alpha q\|^2+\beta_n^i(1-\gamma_n^i)(1-\delta_n^i)\|s_n^i-w_n^i\|^2\\
&\quad -\delta_n^i\beta_n^i(1-\gamma_n^i)[1-4(\delta_n^i)^2L^2-2\delta_n^i]\|s_n^i-v_n^i\|^2+\beta_n^i\gamma_n^i\|x_n-\alpha q\|^2\\
&\quad +\sigma_n^i\|s_n^i-\alpha q\|^2+\beta_n^i\gamma_n^i(1-\gamma_n^i)(\sigma_n^i-1)\|w_n^i-x_n\|^2-\sigma_n^i\beta_n^i(1-\gamma_n^i)\\
&\quad \times\|w_n^i-s_n^i\|^2-\sigma_n^i\beta_n^i\gamma_n^i\|x_n-s_n^i)\|^2+2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle. \qquad (3.20)
\end{aligned}$$

(3.18) and (3.20) imply that

$$\begin{aligned}
\|x_{n+1}-\alpha q\|^2 &\leq (\beta_n^i+\sigma_n^i)\|x_n-\alpha q\|^2+\beta_n^i(1-\gamma_n^i)(1-\sigma_n^i-\delta_n^i)\|s_n^i-w_n^i\|^2\\
&\quad -\delta_n^i\beta_n^i(1-\gamma_n^i)[1-4(\delta_n^i)^2L^2-2\delta_n^i]\|s_n^i-v_n^i\|^2\\
&\quad +\sum_{i=1}^N\sigma_n^i\lambda^i\Big(\lambda^i-\frac{\lambda^i}{\eta^i}\Big)\|B^{i\star}(I-T_\tau^{F_2^i})B^i x_n\|^2\\
&\quad +\beta_n^i\gamma_n^i(1-\gamma_n^i)(\sigma_n^i-1)\|w_n^i-x_n\|^2-\sigma_n^i\beta_n^i\gamma_n^i\|x_n-s_n^i)\|^2\\
&\quad +2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle\\
&\leq (\beta_n^i+\sigma_n^i)\|x_n-\alpha q\|^2+\beta_n^i(1-\gamma_n^i)(1-\sigma_n^i-\delta_n^i)\|s_n^i-w_n^i\|^2\\
&\quad -\delta_n^i\beta_n^i(1-\gamma_n^i)[1-4(\delta_n^i)^2L^2-2\delta_n^i]\|s_n^i-v_n^i\|^2\\
&\quad +\sum_{i=1}^N\sigma_n^i\lambda^i\Big(\lambda^i-\frac{\lambda^i}{\eta^i}\Big)\|B^{i\star}(I-T_\tau^{F_2^i})B^i x_n\|^2-\sigma_n^i\beta_n^i\gamma_n^i\|x_n-s_n^i)\|^2\\
&\quad +2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle.
\end{aligned}$$

Using (3.15), conditions [(i) and (iii)] and the fact that $\lambda^i\in\Big(0.\dfrac{1}{\eta^i}\Big)$ for each $i=1,2,\cdots,N$, the last inequality becomes

$$\begin{aligned}
\|x_{n+1}-\alpha q\|^2 &\leq (1-\alpha_n^i)\|x_n-\alpha q\|^2-\sum_{i=1}^N\sigma_n^i\lambda^i\Big(\frac{\lambda^i}{\eta^i}-\lambda^i\Big)\|B^{i\star}(I-T_\tau^{F_2^i})B^i x_n\|^2\\
&\quad +2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle. \qquad (3.21)
\end{aligned}$$

Now, we complete the proof by the next two cases.

Case A:

Assume that there exists a positive integer n_0 such that $\{\|x_n-\alpha q\|\}$ is decreasing for all $n\geq n_0$. Then, the sequence $\{\|x_n-\alpha q\|\}$ is convergent and $\|x_n-\alpha q\|-\|x_{n+1}-\alpha q\|\to 0\ as\ n\to\infty$.

From (3.21), we get

$$\sum_{i=1}^{N}\sigma_n^i\lambda^i\Big(\frac{\lambda^i}{\eta^i}-\lambda^i\Big)\|B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\|^2 \leq (1-\alpha_n^i)\|x_n-\alpha q\|^2-\|x_{n+1}-\alpha q\|^2 +2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle.$$

Since $\alpha_n^i \to 0$ *as* $n\to\infty, \forall i=1,2,\cdots,N$, we have

$$\lim_{n\to\infty}\|B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\|=0, \tag{3.22}$$

and hence

$$\lim_{n\to\infty}\|x_n-(x_n-\lambda^iB^{i\star}(I-T_\tau^{F_2^i})B^i)x_n\|=0. \tag{3.23}$$

Using condition (i) of Lemma 2.4, (3.1) and the fact that $T_{\sigma_n}^{F_1^i}$ and $(I-\lambda^iB^{i\star}(I-T_\tau^{F_2^i})B^i)$ are fiemly nonexpansive and nonexpansive, respectively, we get

$$\begin{aligned}
\|z_n^i-\alpha q\|^2 &= \|T_\sigma^{F_1^i}(I-\lambda^iB^{i\star}(I-T_\tau^{F_2^i})B^i)x_n-T_\sigma^{F_1^i}(\alpha q)\|^2\\
&\leq \langle z_n^i-\alpha q,(I-\lambda^iB^{i\star}(I-T_\tau^{F_2^i})B^i)x_n-\alpha q\rangle\\
&= \frac{1}{2}\Big\{\|z_n^i-\alpha q\|^2+\|(I-\lambda^iB^{i\star}(I-T_\tau^{F_2^i})B^i)x_n-\alpha q\|^2\\
&\quad -\|z_n^i-(I-\lambda^iB^{i\star}(I-T_\tau^{F_2^i})B^i)x_n\|^2\Big\}\\
&\leq \frac{1}{2}\Big\{\|z_n^i-\alpha q\|^2+\|x_n-\alpha q\|^2-\|z_n^i-x_n\|^2-2\lambda^i\langle z_n^i-x_n,B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\rangle\\
&\quad -(\lambda^i)^2\|B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\|^2\Big\}
\end{aligned} \tag{3.24}$$

(3.24) implies

$$\|z_n^i-\alpha q\|^2 \leq \|x_n-\alpha q\|^2-\|z_n^i-x_n\|^2+2\lambda^i\langle x_n-z_n^i,B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\rangle. \tag{3.25}$$

Since

$$\|s_n^i-\alpha q\|^2\leq\sum_{i=1}^{N}\tau_n^i\|z_n^i-\alpha q\|^2 \quad (\textit{ see } (3.16)),$$

it follows from (3.25) that

$$\|s_n^i-\alpha q\|^2\leq\|x_n-\alpha q\|^2-\sum_{i=1}^{N}\tau_n^i\|z_n^i-x_n\|^2+2\sum_{i=1}^{N}\tau_n^i\lambda^i\langle x_n-z_n^i,B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\rangle. \tag{3.26}$$

Putting (3.12) and (3.26) into (3.19), we obtain

$$\begin{aligned}
\|x_{n+1}-\alpha q\|^2 &\leq (\beta_n^i+\sigma_n^i)\|x_n-\alpha q\|^2+\beta_n^i(1-\gamma_n^i)(1-\sigma_n^i-\delta_n^i)\|w_n^i-s_n^i\|^2\\
&\quad -\beta_n^i\delta_n^i(1-\gamma_n^i)[1-4(\delta_n^i)^2L^2-2\delta_n^i]\|s_n^i-v_n^i\|^2-\sum_{i=1}^{N}\sigma_n^i\tau_n^i\|z_n^i-x_n\|^2\\
&\quad +2\sum_{i=1}^{N}\sigma_n^i\tau_n^i\lambda^i\langle x_n-z_n^i,B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\rangle\\
&\quad +2\alpha_n^i\langle u-\alpha q,x_{n+1}-\alpha q\rangle.
\end{aligned} \tag{3.27}$$

Thus, from (3.15) and (3.27), we have

$$\begin{aligned}\sum_{i=1}^{N}\sigma_n^i\tau_n^i\|z_n^i-x_n\|^2 \;&\leq\; (1-\alpha_n^i)\|x_n-\alpha q\|^2-\|x_{n+1}-\alpha q\|^2\\ &\quad +2\sum_{i=1}^{N}\sigma_n^i\tau_n^i\lambda^i\|x_n-z_n^i\|\|B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\|\\ &\quad +2\alpha_n^i\langle u-\alpha q,x_{n+1}-\alpha q\rangle\end{aligned}$$

Hence, since $\{x_n\}$ *and* $\{z_n\}$ are bounded and $\alpha_n^i\to 0$ *as* $n\to\infty$ for each $i=1,2,\cdots,N$, we obtain (using (3.22)) that

$$\lim_{n\to\infty}\|z_n^i-x_n\|=0,\;\; for\ each\ i=1,2,\cdots,N. \tag{3.28}$$

Again, since from (3.15) and (3.27)

$$\begin{aligned}\beta_n^i\delta_n^i(1-\gamma_n^i)[1-4(\delta_n^i)^2L^2-2\delta_n^i]\|s_n^i-v_n^i\|^2 \;&\leq\; (1-\alpha_n^i)\|x_n-\alpha q\|^2-\|x_{n+1}-\alpha q\|^2\\ &\quad +2\sum_{i=1}^{N}\sigma_n^i\tau_n^i\lambda^i\|x_n-z_n^i\|\|B^{i\star}(I-T_\tau^{F_2^i})B^ix_n\|\\ &\quad +2\alpha_n^i\langle u-\alpha q,x_{n+1}-\alpha q\rangle,\end{aligned}$$

we obtain from (3.15), condition (i) and the fact that$\alpha_n^i\to 0$ *as* $n\to\infty$ for each $i=1,2,\cdots,N$, that

$$\lim_{n\to\infty}\|s_n^i-v_n^i\|=0,\;\; for\ each\ i=1,2,\cdots,N. \tag{3.29}$$

Since $v_n^i\in\Gamma^i$ for each $i=1,2,\cdots,N$, using (3.29), we obtain

$$\lim_{n\to\infty}d(s_n^i,\Gamma^is_n^i)=0,\;\; for\ each\ i=1,2,\cdots,N. \tag{3.30}$$

Conversely, since $J_\mu^i(\alpha q)=\alpha q$ and is firmly nonexpansive for each $i=1,2,\cdots,N$, it follows from condition (i) of Lemma 2.4 that

$$\begin{aligned}\|t_n^i-\alpha q\|^2 &= \|J_\mu^iz_n^i-\alpha q\|^2\\ &\leq \langle t_n^i-\alpha q,z_n^i-\alpha q\rangle\\ &= \frac{1}{2}\Big(\|t_n^i-\alpha q\|^2+\|z_n^i-\alpha q\|^2-\|z_n^i-t_n^i\|^2\Big).\end{aligned} \tag{3.31}$$

From (3.3) and (3.31), we get

$$\|t_n^i-\alpha q\|^2 \;\leq\; \|x_n-\alpha q\|^2-\|z_n^i-t_n^i\|^2. \tag{3.32}$$

Since

$$\|s_n^i-\alpha q\|^2 \;\leq\; \sum_{i=1}^{N}\tau_n^i\|t_n^i-\alpha q\|^2 \quad (see \quad (3.5)),$$

it follows from (3.32) that

$$\|s_n^i-\alpha q\|^2 \;\leq\; \|x_n-\alpha q\|^2-\sum_{i=1}^{N}\tau_n^i\|z_n^i-t_n^i\|^2. \tag{3.33}$$

Substituting (3.12) and (3.33) into (3.19) and simplifying, we get

$$\begin{aligned}\|x_{n+1}-\alpha q\|^2 \leq\ & (\sigma_n^i+\beta_n^i)\|x_n-\alpha q\|^2+\beta_n^i(1-\gamma_n^i-\sigma_n^i)\|s_n^i-w_n^i\|^2\\ & +\delta_n^i\beta_n^i(1-\gamma_n^i)[1-4(\delta_n^i)^2L^2-2\delta_n^i]\|s_n^i-v_n^i\|^2-\sum_{i=1}^{N}\sigma_n^i\tau_n^i\|z_n^i-t_n^i\|^2\\ & +\sigma_n^i\beta_n^i(1-\gamma_n^i)(1-\sigma_n^i)\|w_n^i-x_n\|^2-\sigma_n^i\beta_n^i\gamma_n^i\|x_n-s_n^i\|^2\\ & +2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle, \end{aligned} \tag{3.34}$$

which by (3.15) yields

$$\begin{aligned}\|x_{n+1}-\alpha q\|^2 \leq\ & (1-\alpha_n^i)\|x_n-\alpha q\|^2-\sum_{i=1}^{N}\sigma_n^i\tau_n^i\|z_n^i-t_n^i\|^2\\ & +2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle, \end{aligned} \tag{3.35}$$

(3.35) implies

$$\begin{aligned}\sum_{i=1}^{N}\sigma_n^i\tau_n^i\|z_n^i-t_n^i\|^2 \leq\ & (1-\alpha_n^i)\|x_n-\alpha q\|^2-\|x_{n+1}-\alpha q\|^2\\ & +2\alpha_n^i\langle u-\alpha q, x_{n+1}-\alpha q\rangle.\end{aligned}$$

Since $\alpha_n^i\to 0$ as $n\to\infty$ for each $i=1,2,\cdots,N$, it follows from the last inequality that

$$\lim_{n\to\infty}\|z_n^i-t_n^i\|=0, \text{for each} \quad i=1,2,\cdots,N. \tag{3.36}$$

Observe that

$$\|t_n^i-x_n\|\leq\|t_n^i-z_n^i\|+\|z_n^i-x_n\|. \tag{3.37}$$

(3.28), (3.36) and (3.37) imply

$$\lim_{n\to\infty}\|t_n^i-x_n\|=0, \text{for each} \quad i=1,2,\cdots,N. \tag{3.38}$$

Consequent upon (3.38), we have

$$\begin{aligned}\|s_n^i-x_n\| &= \|\sum_{i=1}^{N}\tau_n^i t_n^i-x_n\|\\ &\leq \sum_{i=1}^{N}\tau_n^i\|t_n^i-x_n\|\to 0, \text{for each} \quad i=1,2,\cdots,N.\end{aligned} \tag{3.39}$$

Since, for each $i=1,2,\cdots,N, \Gamma^i$ is an L-Lipchitizian mapping, using (3.9), (3.30) and the fact that $\|v_n^i-w_n^i\|\leq 2D(\Gamma^i s_n^i,\Gamma^i u_n^i)$ for each $i=1,2,\cdots,N$, we have

$$\begin{aligned}\|s_n^i-w_n^i\| &\leq \|s_n^i-v_n^i\|+\|v_n^i-w_n^i\|\\ &\leq \|s_n^i-v_n^i\|+2L\|s_n^i-u_n^i\|\\ &= \|s_n^i-v_n^i\|+2L\delta_n^i\|s_n^i-v_n^i\|\to 0, \text{for each} \quad i=1,2,\cdots,N.\end{aligned} \tag{3.40}$$

Now, since $\{s_n^i\}$ is bounded and $\alpha_n^i \to 0$, for each $i = 1, 2, \dots, N$, it follows from (3.39) and (3.40) that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - s_n^i\| + \|s_n^i - x_n\| \\
&= \|\alpha_n^i u + \beta_n^i[(1-\gamma_n^i)w_n^i + \gamma_n^i x_n] + \sigma_n^i s_n^i - s_n^i\| + \|s_n^i - x_n\| \\
&= \|\alpha_n^i(u - s_n^i) - \beta_n^i(1-\gamma_n^i)(s_n^i - w_n^i) - \beta_n^i\gamma_n^i(s_n^i - x_n)\| + \|s_n^i - x_n\| \\
&\leq \alpha_n^i\|u - s_n^i\| + \beta_n^i(1-\gamma_n^i)\|s_n^i - w_n^i\| \\
&\quad +(1+\beta_n^i\gamma_n^i)\|s_n^i - x_n\| \to 0, \text{for each} \quad i = 1, 2, \cdots, N.
\end{aligned} \tag{3.41}
$$

In addition, from (3.15) and (3.34), we get

$$
\|x_{n+1} - \alpha q\|^2 \leq (1-\alpha_n^i)\|x_n - \alpha q\|^2 + 2\alpha_n^i\langle u - \alpha q, x_{n+1} - \alpha q\rangle, \tag{3.42}
$$

Now, let $\alpha q = P_\Theta(u)$. Then, we show that

$$
\limsup_{n\to\infty}\langle u - \alpha q, x_{n+1} - \alpha q\rangle \leq 0. \tag{3.43}
$$

Since the sequence $\{x_{n+1}\}$ is bounded (by Theorem 3.1), we can choose a subsequence $\{x_{n_k+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_k+1} \rightharpoonup \omega \quad \text{as} \quad k \to \infty$ so that

$$
\limsup_{n\to\infty}\langle u - \alpha q, x_{n+1} - \alpha q\rangle = \lim_{k\to\infty}\langle u - \alpha q, x_{n_k+1} - \alpha q\rangle. \tag{3.44}
$$

Since C^i is closed and convex, it follows that that C^i is weakly closed for each $i = 1, 2, \cdots, N$. Hence, $x_{n_k+1} \rightharpoonup \omega \quad \text{as} \quad k \to \infty$. Therefore, we obtain from (3.28), (3.39) and (3.41) that $x_{n_k} \rightharpoonup \omega, z_{n_k}^i \rightharpoonup \omega \quad \text{and} \quad s_{n_k}^i \rightharpoonup \omega \quad \text{as} \quad k \to \infty$ for each $i = 1, 2, \cdots, N$.
Consequently, the demiclosedness of $(I - \Gamma^i)$ at zero (for each $i = 1, 2, \cdots, n$) and (3.30) guarantees that the weak limit of $s_{n_k}^i$ is a fixed point of the multivalued mapping Γ^i; that is, $\omega \in F(\Gamma)$.

Again, since $(I - \lambda^i B^{i\star}(I - T_\tau^{F_2^i}))B^i$ is nonexpansive fro each $i = 1, 2, \cdots, N$ and $x_{n_k} \rightharpoonup \omega$, the demiclosedness principle for nonexpansive mappings and (3.23) imply that

$$
\omega = (I - \lambda^i B^{i\star}(I - T_\tau^{F_2^i}))\omega, \text{for each} \quad i = 1, 2, \cdots, N. \tag{3.45}
$$

(3.45), with the condition that $\lambda^i > 0$ for each $i = 1, 2, \cdots, N$, implies that

$$
B^{i\star}(I - T_\tau^{F_2^i})B^i\omega = 0, \text{for each} \quad i = 1, 2, \cdots, N.
$$

Therefore, using (2.1), we have

$$
T_\tau^{F_2^i}B^i\omega = B^i\omega, \text{for each} \quad i = 1, 2, \cdots, N.
$$

so that

$$
B^i\omega \in EP(F_2^i), \text{for each} \quad i = 1, 2, \cdots, N.
$$

Also, since $T_\sigma^{F_1^i}(I - \lambda^i B^{i\star}(I - T_\tau^{F_2^i}))B^i$ is nonexpansive for each $i = 1, 2, \cdots, N$, we obtain from the demiclosedness principle of nonexpansive mapping and (3.28) that

$$
\omega = T_\sigma^{F_1^i}(I - \lambda^i B^{i\star}(I - T_\tau^{F_2^i}))B^i\omega.
$$

Since $B^i\omega = T_\tau^{F_2^i}B^i\omega$, we have that $\omega = T_\sigma^{F_1^i}\omega$ for each $i = 1, 2, \cdots, N$, Therefore,

$$
\omega \in \cap_{i=1}^N \Omega_i.\omega \in \cap_{i=1}^N \Omega_i.
$$

On the other hand, (3.28) and the fact that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ imply that $z^i_{n_k} \rightharpoonup \omega$ as $k \to \infty$ for each $i = 1, 2, \cdots, N$. Also, the demiclosedness principle for nonexpansive mapping and

$$\lim_{n\to\infty} \|z^i_n - J^i_\mu z^i_n\| = \lim_{n\to\infty} \|z^i_n - t^i_n\| = 0$$

ensure that the weak limit ω of the sequence $\{z^i_n\}$ is a fixed point of the mapping J^i_μ for each $i = 1, 2, \cdots, N$; that is, $\omega = J^i_\mu \omega$. This fact together with Lemma 2.6 yields

$$\omega \in \cap_{i=1}^N VI(C^i, A^i).$$

Therefore,

$$\omega \in \cap_{i=1}^N (\Omega_i \cap VI(C^i, A^i)) \cap_{i=1}^N F(\Gamma^i).$$

Now, using the fact that $\alpha q = P_\Theta(u), x_{n_k+} \rightharpoonup \omega$ as $k \to \infty$ and (2.2), we get

$$\begin{aligned} \limsup_{n\to\infty} \langle u - \alpha q, x_{n+1} - \alpha q\rangle &= \lim_{k\to\infty} \langle u - \alpha q, x_{n_k+1} - \alpha q\rangle \\ &= \langle u - \alpha q, \omega - \alpha q\rangle \\ &\le 0, \end{aligned} \tag{3.46}$$

which is as required by (3.43). In addition, since αq was arbitrary and $\alpha q \in \Theta$, it follows from (3.35) and Lemma 2.3 that

$$\lim_{n\to\infty} \|x_n - \alpha q\| = 0.$$

That is, $x_n \to \alpha q = P_\Theta(u)$ as $n \to \infty$.

Case B:

Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - \alpha q\| \le \|x_{n_i+1} - \alpha q\|, \forall i \in N.$$

Then, in view of Lemma 2.8, there exists a nondecreasing sequence $\{\nu_k\} \in N$ such that $\nu_k \to \infty$ and

$$\|x_{\nu_k} - \alpha q\| \le \|x_{\nu_k+1} - \alpha q\|, \|x_k - \alpha q\| \le \|x_{\nu_k+1} - \alpha q\|, \forall k \in N. \tag{3.47}$$

Thus, from (3.28), (3.29), (3.39) and the fact that $\alpha^i_n \to 0$ as $n \to \infty$ *for each* $i = 1, 2, \cdots, N$, we get

$$\lim_{k\to\infty} \|s^i_{\nu_k} - v^i_{\nu_k}\| = 0, \lim_{k\to\infty} \|z^i_{\nu_k} - x_{\nu_k}\| = 0, \lim_{k\to\infty} \|s^i_n - x_{\nu_k}\| = 0 \ for\ each\ i = 1, 2, \cdots, N.$$

Then, since $\alpha q = P_\Theta(u)$, using the similar argument as in Case A, we have

$$\limsup_{k\to\infty} \langle u - \alpha q, x_{\nu_k+1} - \alpha q\rangle \le 0. \tag{3.48}$$

Next, since $\alpha q \in \Theta$, we obtain from (3.42) that

$$\|x_{\nu_k+1} - \alpha q\|^2 \le (1 - \alpha^i_{\nu_k})\|x_{\nu_k} - \alpha q\|^2 + 2\alpha^i_{\nu_k}\langle u - \alpha q, x_{\nu_k+1} - \alpha q\rangle, \tag{3.49}$$

(3.47), (3.49) and the fact that $\alpha^i_n > 0$ yield

$$\|x_{\nu_k} - \alpha q\|^2 \le 2\langle u - \alpha q, x_{\nu_k+1} - \alpha q\rangle, \tag{3.50}$$

It follows from (3.48) and (3.50) that $\|x_{\nu_k} - \alpha q\| \to 0\ as\ k \to \infty$. This together with (3.47) implies that $\|x_{\nu_k+1} - \alpha q\| \to 0\ as\ k \to \infty$. Since from (3.47), we have $\|x_k - \alpha q\| \le \|x_{\nu_k+1} - \alpha q\|, \forall k \in N$, it follows that

$$x_k \to \alpha q\ as\ k \to \infty.$$

Therefore, from the above two cases, we deduce that the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\alpha q = P_{\Theta}(u)$. This completes the proof. $\square$

If, in Theorem 3.2, we assume that Γ^i is a finite family of single-valued α-hemicontractive mapping, then we get the following result:

Corollary 3.3. *Let H_1, H_2, $\{C^i\}_{i=1}^N, \{Q^i\}_{i=1}^N, \{A^i\}_{i=1}^N, \{B^i\}_{i=1}^N, \{B^{i\star}\}_{i=1}^N, \{F_1^i\}_{i=1}^N, \{F_2^i\}_{i=1}^N$, $\{\{\delta_n^i\}_{i=1}^N\}_{n=0}^{\infty}, \{\{\alpha_n^i\}_{i=1}^N\}_{n=0}^{\infty}, \{\{\tau_n^i\}_{i=1}^N\}_{n=0}^{\infty}, \{\{\gamma_n^i\}_{i=1}^N\}_{n=0}^{\infty}, \{\{\sigma_n^i\}_{i=1}^N\}_{n=0}^{\infty}$ be as in Algorithm 3.1. Let $\Gamma^i C^i \longrightarrow CB(C^i)$ be an L-Lipschitz α-hemicontractive mapping such that $(I-\Gamma^i)$ is demiclosed at zero for each $i = 1, 2, \cdots, N$ and for some $\alpha \geq 1$. Suppose F_1^i and F_2^i satisfying Assumption G, $\Theta = \cap_{i=1}^N(\Omega_i \cap VI(C^i, A^i)) \cap_{i=1}^N F(\Gamma^i) \neq \emptyset$ and $\Gamma^i \alpha q = \alpha q$ for all $q \in \Theta$, for all $i = 1, 2, \cdots, N$ and for some $\alpha \geq 1$. If for each $i = 1, 2, \cdots, N$ and for all $n \geq 0$,*

i. $\alpha_n^i + \beta_n^i + \sigma_n^i = 1$ *and* $0 < \alpha \leq \gamma_n^i, \sigma_n^i \leq \beta < 1$;
ii. $\lim_{n\to\infty} \alpha_n^i = 0, \sum_{n=0}^{\infty} = \infty$
iii. $\sum_{n=1}^N \tau_n^i = 1$ *and* $0 < \delta \leq \tau_n^i \leq 1$;
iv. $\alpha_n^i + \beta_n^i \leq \delta_n^i \leq \sigma < \frac{1}{\sqrt{1+4L^2}+1}$.

Let $x_0, u \in C^i$ be arbitrary for each $i = 1, 2, \cdots, N$. Then, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$\begin{cases} z_n^i = T_\sigma^{F_1^i}(I - \lambda^i B^{i\star}(I - T_\tau^{F_2^i})B^i)x_n; \\ t_n^i = J_\mu^i z_n^i; \\ s_n^i = \sum_{i=1}^N \tau_n^i t_n^i; \\ u_n^i = (1-\delta_n^i)s_n^i + \delta_n^i v_n^i; \\ x_{n+1} = \alpha_n^i u + \beta_n^i[(1-\gamma_n^i)w_n^i + \gamma_n^i x_n] + \sigma_n^i s_n^i, \end{cases} \tag{3.51}$$

for $n \geq 0$, where $v_n^i \in \Gamma^i s_n^i$ and $w_n^i \in \Gamma^i u_n^i$ are such that $\|v_n^i - w_n^i\| \leq 2D(\Gamma^i s_n^i, \Gamma^i u_n^i)$, for each $i = 1, 2, \cdots, N$ and $\sigma, \tau, \mu > 0, \lambda^i \in \left(0, \dfrac{1}{\eta^i}\right)$, where $\eta^i = \|B^i\|^2$, strongly converges to $\alpha q = P_{\Theta}(u)$.

If, in Theorem 3.2, we assume that $A^i \equiv 0$ for each $i = 1, 2, \cdots, N$, then we obtain the followinf results on finite families of fixed point problem for α-hemicontaractive mapping and split equilibrium problems.

Corollary 3.4. *Let H_1, H_2, $\{C^i\}_{i=1}^N, \{Q^i\}_{i=1}^N, \{B^i\}_{i=1}^N, \{B^{i\star}\}_{i=1}^N, \{F_1^i\}_{i=1}^N, \{F_2^i\}_{i=1}^N$, $\{\{\delta_n^i\}_{i=1}^N\}_{n=0}^{\infty}, \{\{\alpha_n^i\}_{i=1}^N\}_{n=0}^{\infty}, \{\{\tau_n^i\}_{i=1}^N\}_{n=0}^{\infty}, \{\{\gamma_n^i\}_{i=1}^N\}_{n=0}^{\infty}, \{\{\sigma_n^i\}_{i=1}^N\}_{n=0}^{\infty}$ be as in Algorithm 3.1. Let $\Gamma^i C^i \longrightarrow CB(C^i)$ be an L-Lipschitz α-hemicontractive mapping such that $(I - \Gamma^i)$ is demiclosed at zero for each $i = 1, 2, \cdots, N$ and for some $\alpha \geq 1$. Suppose F_1^i and F_2^i satisfying Assumption G, $\Theta = \cap_{i=1}^N(\Omega_i) \cap \cap_{i=1}^N F(\Gamma^i) \neq \emptyset$ and $\Gamma^i \alpha q = \alpha q$ for all $q \in \Theta$, for all $i = 1, 2, \cdots, N$ and for some $\alpha \geq 1$. If for each $i = 1, 2, \cdots, N$ and for all $n \geq 0$,*

i. $\alpha_n^i + \beta_n^i + \sigma_n^i = 1$ *and* $0 < \alpha \leq \gamma_n^i, \sigma_n^i \leq \beta < 1$;
ii. $\lim_{n\to\infty} \alpha_n^i = 0, \sum_{n=0}^{\infty} = \infty$
iii. $\sum_{n=1}^N \tau_n^i = 1$ *and* $0 < \delta \leq \tau_n^i \leq 1$;
iv. $\alpha_n^i + \beta_n^i \leq \delta_n^i \leq \sigma < \frac{1}{\sqrt{1+4L^2}+1}$.

Let $x_0, u \in C^i$ be arbitrary for each $i = 1, 2, \cdots, N$. Then, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$\begin{cases} z_n^i = T_\sigma^{F_1^i}(I - \lambda^i B^{i\star}(I - T_\tau^{F_2^i})B^i)x_n; \\ t_n^i = J_\mu^i z_n^i; \\ s_n^i = \sum_{i=1}^N \tau_n^i t_n^i; \\ u_n^i = (1-\delta_n^i)s_n^i + \delta_n^i v_n^i; \\ x_{n+1} = \alpha_n^i u + \beta_n^i[(1-\gamma_n^i)w_n^i + \gamma_n^i x_n] + \sigma_n^i s_n^i, \end{cases} \tag{3.52}$$

for $n \geq 0$, where $v_n^i \in \Gamma^i s_n^i$ and $w_n^i \in \Gamma^i u_n^i$ are such that $\|v_n^i - w_n^i\| \leq 2D(\Gamma^i s_n^i, \Gamma^i u_n^i)$, for each $i = 1, 2, \cdots, N$ and $\sigma, \tau, \mu > 0, \lambda^i \in \left(0, \dfrac{1}{\eta^i}\right)$, where $\eta^i = \|B^i\|^2$, strongly converges to $\alpha q = P_\Theta(u)$.

If, in Theorem 3.2, we assume that $H_1 = H_2, C^i = Q^i, B^i \equiv 1$ *and* $F_{2^i} \equiv 0$, then we obtain the following corollary:

Corollary 3.5. *Let H_1, $\{C^i\}_{i=1}^N, \{A^i\}_{i=1}^N, \{F_1^i\}_{i=1}^N, \{\{\delta_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\alpha_n^i\}_{i=1}^N\}_{n=0}^\infty$, $\{\{\tau_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\gamma_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\sigma_n^i\}_{i=1}^N\}_{n=0}^\infty$ be as in Algorithm 3.1. Let $\Gamma^i C^i \longrightarrow CB(C^i)$ be an L-Lipschitz α-hemicontractive mapping such that $(I - \Gamma^i)$ is demiclosed at zero for each $i = 1, 2, \cdots, N$ and for some $\alpha \geq 1$. Suppose F_1^i satisfying Assumption G, $\Theta = \cap_{i=1}^N(\Omega_i \cap VI(C^i, A^i)) \cap_{i=1}^N F(\Gamma^i) \neq \emptyset$ and $\Gamma^i \alpha q = \alpha q$ for all $q \in \Theta$, for all $i = 1, 2, \cdots, N$ and for some $\alpha \geq 1$. If for each $i = 1, 2, \cdots, N$ and for all $n \geq 0$,*

i. $\alpha_n^i + \beta_n^i + \sigma_n^i = 1$ *and* $0 < \alpha \leq \gamma_n^i, \sigma_n^i \leq \beta < 1$;
ii. $\lim_{n\to\infty} \alpha_n^i = 0, \sum_{n=0}^\infty = \infty$
iii. $\sum_{n=1}^N \tau_n^i = 1$ *and* $0 < \delta \leq \tau_n^i \leq 1$;
iv. $\alpha_n^i + \beta_n^i \leq \delta_n^i \leq \sigma < \frac{1}{\sqrt{1+4L^2}+1}$.

Let $x_0, u \in C^i$ be arbitrary for each $i = 1, 2, \cdots, N$. Then, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$\begin{cases} z_n^i = T_\sigma^{F_1^i}; \\ t_n^i = J_\mu^i z_n^i; \\ s_n^i = \sum_{i=1}^N \tau_n^i t_n^i; \\ u_n^i = (1-\delta_n^i)s_n^i + \delta_n^i v_n^i; \\ x_{n+1} = \alpha_n^i u + \beta_n^i[(1-\gamma_n^i)w_n^i + \gamma_n^i x_n] + \sigma_n^i s_n^i, \end{cases} \tag{3.53}$$

for $n \geq 0$, where $v_n^i \in \Gamma^i s_n^i$ and $w_n^i \in \Gamma^i u_n^i$ are such that $\|v_n^i - w_n^i\| \leq 2D(\Gamma^i s_n^i, \Gamma^i u_n^i)$, for each $i = 1, 2, \cdots, N$ and $\sigma, \tau, \mu > 0, \lambda^i \in \left(0, \dfrac{1}{\eta^i}\right)$, where $\eta^i = \|B^i\|^2$, strongly converges to $\alpha q = P_\Theta(u)$.

If, in Corollary 3.3, we assume that Γ^i is an identity mapping, $\alpha_n^i = \delta_n^i \equiv 0$, for each $i = 1, 2, \cdots, N$, we obtain the following results on finite families of variational inequality problems and equilibrium problems:

Corollary 3.6. *Let H_1, H_2, $\{C^i\}_{i=1}^N, \{Q^i\}_{i=1}^N, \{A^i\}_{i=1}^N, \{B^i\}_{i=1}^N, \{B^{i\star}\}_{i=1}^N, \{F_1^i\}_{i=1}^N, \{F_2^i\}_{i=1}^N$, $\{\{\delta_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\alpha_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\tau_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\gamma_n^i\}_{i=1}^N\}_{n=0}^\infty, \{\{\sigma_n^i\}_{i=1}^N\}_{n=0}^\infty$ be as in Algorithm 3.1. Suppose F_1^i and F_2^i satisfying Assumption G, $\Theta = \cap_{i=1}^N(\Omega_i \cap VI(C^i, A^i)) \neq \emptyset$ and $\Gamma^i \alpha q = \alpha q$ for all $q \in \Theta$, for all $i = 1, 2, \cdots, N$ and for some $\alpha \geq 1$. If for each $i = 1, 2, \cdots, N$ and for all $n \geq 0$,*

i. $\alpha_n^i + \beta_n^i + \sigma_n^i = 1$ *and* $0 < \alpha \leq \gamma_n^i, \sigma_n^i \leq \beta < 1$;
ii. $\lim_{n\to\infty} \alpha_n^i = 0, \sum_{n=0}^{\infty} = \infty$
iii. $\sum_{n=1}^{N} \tau_n^i = 1$ *and* $0 < \delta \leq \tau_n^i \leq 1$;
iv. $\alpha_n^i + \beta_n^i \leq \delta_n^i \leq \sigma < \frac{1}{\sqrt{1+4L^2}+1}$.

Let $x_0, u \in C^i$ *be arbitrary for each* $i = 1, 2, \cdots, N$. *Then, the sequence* $\{x_n\}_{n=0}^{\infty}$ *given by*

$$\begin{cases} z_n^i = T_\sigma^{F_1^i}(I - \lambda^i B^{i\star}(I - T_\tau^{F_2^i})B^i)x_n; \\ t_n^i = J_\mu^i z_n^i; \\ s_n^i = \sum_{i=1}^{N} \tau_n^i t_n^i; \\ x_{n+1} = \alpha_n^i u + \beta_n^i x_n + \sigma_n^i s_n^i, \end{cases} \tag{3.54}$$

for $n \geq 0$, *where* $v_n^i \in \Gamma^i s_n^i$ *and* $w_n^i \in \Gamma^i u_n^i$ *are such that* $\|v_n^i - w_n^i\| \leq 2D(\Gamma^i s_n^i, \Gamma^i u_n^i)$, *for each* $i = 1, 2, \cdots, N$ *and* $\sigma, \tau, \mu > 0, \lambda^i \in \left(0, \dfrac{1}{\eta^i}\right)$, *where* $\eta^i = \|B^i\|^2$, *strongly converges to* $\alpha q = P_\Theta(u)$.

Remark 3.7. We note that since every α-demicontractive mappings is α-hemicontractive mappings, the results obtained in this paper for finite family of α-hemicontractive (single and multivalued) mappings also hold for finite families of α-demicontractive mappings provided that the indicated conditions are satisfied. Our results extend, improve, generalize and unify several recent results in the current literature (e.g.,[1, 2, 3, 12, 17,18] etc) on approximation of common solutions of finite families of fixed point problems for nonlinear mappings, variational inequality problems and split equilibrium problems. Theorem 3.2 extends the results of Meche and Zegeye [24] from Lipshitz hemicontractive-type multivalued mappings to the more general class of Lipshitz α-hemicontractive multivalued mappings. Also, in our results, a restriction of upper semicontinuity on the bifunctions is not required.

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