Mathematical Analysis
& Convex Optimization

Vol. 2 (2021), No. 2, 93–104
https:// maco.lu.ac.ir
DOI: 10.52547/maco.2.2.10

Research Paper

STRONG CONVERGENCE THEOREMS FOR A GENERAL CLASS OF MAPPING VIA QUASI-IMPLICIT ITERATIVE SCHEMES

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Abstract. In this paper, we introduce a novel iterative scheme called quasi-implicit iterative scheme and study its stability as well as strong convergence for general class of maps in a normed linear space. Further, we proved rate of convergence and gave a numerical example to demonstrate that our iterative scheme is faster than semi-implicit iterative scheme and many more other iterative schemes in this direction.

MSC(2010): 47H09, 47H10, 47J05, 65J15.
Keywords: trong convergence, Quasi-implicit iterative scheme, Semi-implicit iterative scheme, Stability, Generalized contractive-like operators, fixed point, Real normed space.

1. Introduction

The most celebrated theorem for existence of fixed point for a self-mapping on a complete metric space (Z, d) was initiated by Banach.

Definition 1.1. A self map $\Gamma : Z \longrightarrow Z$ is called a contraction if for all $x, y \in Z$, there exists $\rho \in [0, 1)$ such that

$$d(\Gamma x, \Gamma y) \leq \rho d(x, y). \tag{1.1}$$

Dwelling on Definition 1.1 and using metric space setting, Banach, in 1922, gave the following well-know contraction principle.

Theorem 1.2. *Let $\Gamma : Z \longrightarrow Z$ be a contraction mapping defined on a complete metric space (Z, d). Then, Γ possesses a unique fixed point $x^\star \in Z$. Furthermore, we have*

$$\lim_{n\to\infty} \Gamma^n x = x^\star \tag{1.2}$$

with

$$d(\Gamma^n x, x^\star) \leq \frac{\rho^n}{1-\rho} d(x, \Gamma x). \tag{1.3}$$

Theorem 1.1, in the company of its direct generalizations, has been a vital tool in applications for solving nonlinear functional equations. Despite its indispensable position in real-life applications, Banach contraction principle suffers one major drawback (the contractive condition (1.1) requires that Γ be continuous throughout Z). Motivated by this challenge, Kannan

Date: Received: April 21, 2021, Accepted: October 25, 2021.
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[3] established a fixed point theorem for which the restriction on Γ (continuity condition) is not required. To be precise, he proved the following theorem:

Theorem 1.3. *Let (Z, d) be a complete metric space and $\Gamma : Z \longrightarrow Z$ a mapping with $\rho \in (0, \frac{1}{2})$ such that for all $x, y \in Z$,*

$$d(\Gamma x, \Gamma y) \leq \rho[d(x, \Gamma x) + d(y, \Gamma y)]. \tag{1.4}$$

Then, Γ has a unique fixed in Z.

Example 1.4. (see [3]) Let $Z = \mathbb{R}$ be the set of real numbers with the usual metric and $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\Gamma x = \begin{cases} 0. \ if \ x \in (-\infty, 2] \\ -\frac{1}{2}, \ if \ x \in (2, \infty). \end{cases} \tag{1.5}$$

Then, Γ satisfies (1.4) with $\rho = 0.2$. Also, Γ is not continuous and $F_\Gamma = \{0\}$.

Subsequent upon Kannan's theorem, so many interesting results respecting different classes of contractive-type conditions that do not require the continuity of Γ are now in literature, see for example [4]. Further, in 1972, Zamfirescu [35] gave an interesting theorem which extended both Banach's and Kannan's fixed point theorems, together with other similar results of this kind, as follows:

Theorem 1.5. *Let (Z, d) be a complete metric space and $\Gamma : Z \longrightarrow Z$ a mapping such that for each pair of points $x, y \in Z$, there exist the real numbers δ, ω and σ satisfying $0 < \delta < 1, 0 < \omega < \frac{1}{2}$ and $0 < \sigma < \frac{1}{2}$ such that at least one of the following statements holds*

$$\begin{cases} z_1 : d(\Gamma x, \Gamma y) \leq \delta d(x, y); \\ z_2 : d(\Gamma x, \Gamma y) \leq \omega[d(x, \Gamma x) + d(y, \Gamma y)]; \\ z_3 : d(\Gamma x, \Gamma y) \leq \sigma[d(x, \Gamma y) + d(y, \Gamma x)]. \end{cases} \tag{1.6}$$

Then, Γ has a unique fixed point in Z.

The class of mapping defined by (1.6) has been adjudged to be the most general contractive-like operators and has considerably attracted a lot of interest in recent times (see, for example, [8], [9], [11], [14], [36], [14], [15], [24], [37] and the references therein).

In [30], it was shown that (1.6) implies

$$d(\Gamma x, \Gamma y) \leq hmax\{d(x, y), \frac{1}{2}[d(x, \Gamma x) + d(y, \Gamma y)], \frac{1}{2}[[d(x, \Gamma y) + d(y, \Gamma x)]]\}, \tag{1.7}$$

and

$$\|\Gamma x - \Gamma y\| \leq \rho\|x - y\| + 2\rho\|x - \Gamma x\|, \tag{1.8}$$

where $h, \rho \in [0, 1)$ *and* $x, y \in Z$. Note that (1.8) reduces to (1.1) if x is the fixed point of Γ.

Let E be a normed space, Q a nonempty, closed and convex subset of E and $\Gamma : Q \longrightarrow Q$ a self map of Q. For each $x, y \in E$, there exist $L \geq 0$ *and* $\rho \in [0, 1)$ such that

$$\|\Gamma x - \Gamma y\| \leq \rho\|x - y\| + L\|x - \Gamma x\|, \tag{1.9}$$

In 1995, Osilike [24] used the contractive condition defined by (1.9) to prove several generalizations and extensions of most of the results contained in Rhoades [30]. Inspired by the results in [24], Imoru and Olatinwo [15] gave a generalisation of (1.9) as follows: For each $x, y \in E$, there exists a monotone increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *with* $\phi(0) = 0$ such that

$$\|\Gamma x - \Gamma y\| \leq \rho\|x - y\| + \phi(\|x - \Gamma x\|), \tag{1.10}$$

However, Chidume and Olaleru [11] proved with several examples that (1.1) remains the most general contractive condition and includes the likes of (1.8), (1.9) and (1.10).

Different studies have shown that virtually every physical, technical or biological process, from celestial motion to bridge design, to interaction between neurons can be modeled in the form

$$\Gamma(t) = x. \tag{1.11}$$

Equation such as (1.11), which is used to solve real-life problems, may not necessarily be directly solvable; that is, the close form solution may be impossible or practically difficult to attain. Fortunately, (1.11) can equivalently be transformed into a fixed point problem of the form

$$\Gamma(t) = t, \tag{1.12}$$

whose solution can be obtained using approximate fixed point theorem-which, among other things, could help to disclose some vital information on the existence or existence and uniqueness of solution of the original equation.

Let (Z, d) be a complete metric space and $\Gamma : Z \longrightarrow Z$ a selfmap of Z. We shall denote the set of fixed points of Γ by $F_\Gamma = \{q \in Z : \Gamma q = q\}$. In the past few years, various iteration schemes for which the fixed point of (1.12) could be approximately obtained have been extensively studied, see for example, [2], [1], [10], [5], [7], [6], [37], [12], [17], [23], [19], [25],[21],[16],[29], [27],[29], etc. In search of iterative scheme with better convergence rate, authors in [37] introduced a new iterative scheme called semi-implicit iterative (SII) scheme as follows: Let E be a normed space, Q a nonempty, closed and convex subset of E and $\Gamma : Q \longrightarrow Q$ a self map of Q. The semi-implicit iterative (SII) scheme $\{z_v\}_{n=1}^{\infty}$ is defined, for arbitrary $z_0 \in E$, by

$$\begin{cases} z_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Gamma z_n \\ x_n = (1 - \beta_n)y_n + \beta_n \Gamma x_n \\ y_n = (1 - \gamma)z_n + \gamma_n \Gamma y_n, \end{cases} \tag{1.13}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \subset [0, 1]$. They showed (through numerical example) that the scheme (1.13) so introduced converges faster than the iterative schemes studied in ([38] and [39]).

It is worthy to mention that in application, the stability of the iterative schemes studied above is quite invaluable. The first researcher to demonstrate this respecting the Banach contraction conditions is Ostrowski [26]. Afterwards, several authors have developed this subject basically because of its indispensable position in the current trend of computer programming. Some recent works in this direction could be seen in [6],[10],[13],[27], [28],[26], [37], [36] and the references therein.

Inspired and motivated by these developments, in this paper, we shall introduce a new iteration scheme and then prove that our scheme is more efficient in terms of convergence rate than the one studied in [37]. In addition, we establish strong convergence and stability results of our iterative scheme in the setting of an arbitrary Banach space.

2. Preliminary

The following definitions, lemmas and propositions will be needed to prove our main results.

Definition 2.1. (see [26], [36]) Let (Y, d) be a metric space and let $\Gamma : Y \longrightarrow Y$ be a self-map of Y. Let $\{x_n\}_{n=0}^{\infty} \subseteq Y$ be a sequence generated by an iteration scheme

$$x_{n+1} = g(\Gamma, x_n), \tag{2.1}$$

where $x_0 \in Y$ is the initial approximation and g is proper function. Suppeose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point q of Γ. Let $\{t_n\}_{n=0}^{\infty} \subseteq Y$ be an arbitrary sequence and set $\epsilon_n = d(t_n, g(\Gamma, t_n)), n = 1, 2, \cdots$ Then, the iteration scheme (2.1) is called Γ-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = q$.

Note that in practice, the sequence $\{t_n\}_{n=0}^{\infty}$ could be obtained in the following manner: let $x_0 \in Y$. Set $x_{n+1} = g(\Gamma, x_n)$ and let $t_0 = x_0$. Now, $x_1 = g(\Gamma, x_0)$ because of rounding in the function Γ, and a new value t_1 (approximately equal to x_1) might be calculated to give t_2, an approximate value of $g(\Gamma, t_1)$. The procedure is continued to yield the sequence $\{t_n\}_{n=0}^{\infty}$, an approximate sequence of $\{x_n\}_{n=0}^{\infty}$.

Lemma 2.2. *(see, e.g., [36]) Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\tau_n \to 0$ as $n \to \infty$. For $0 \leq \delta < 1$, let $\{w_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $w_{n+1} \leq \delta w_n + \tau_n, n = 0, 1, 2, \cdots$ Then, $w_n \to 0$ as $n \to \infty$.*

Lemma 2.3. *(see, e.g, [37]) Let $\{\omega_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$ be two non-negative real sequences satisfying the following inequality*

$$\omega_n \leq (1 - \lambda_n)\omega_n + \sigma_n,$$

such that $\lambda_n \in (0, 1)$ for all $n \geq n_0, \sum_{n=0}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \omega_n$=0.

Definition 2.4. (see, e.g, [37]) Let $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ be real convergent sequences with limit c *and* d respectively. Then, c_n is faster than d_n if

$$\lim_{n\to\infty} \left|\left|\frac{c_n - c}{d_n - d}\right|\right| = 0.$$

Definition 2.5. (see, e.g, [37]) Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be two fixed point iterations that converge to the same fixed point q on a nonlinear space Z such that the error estimates $|u_n - q| \leq c_n$ *and* $|v_n - q| \leq d_n$ are available such that $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are two sequences of positive integers that converge to zero. If c_n converges faster than d_n then we say that u_n converges faster to q than v_n.

3. Convergence Results

In this section, we introduce a new iterative scheme which we shall call quasi-implicit iterative scheme. Let $(X, \|.\|)$ be a normed linear space, Z be a nonempty closed convex subset of X and $\Gamma : Z \longrightarrow Z$ be a self map of Z. Pick an arbitrary element $z_0 \in Z$ and define the sequence $\{z_n\}_{n=0}^{\infty}$ iteratively as follows:

$$\begin{cases} z_{n+1} = (1 - \delta_n)s_n + \delta_n \Gamma s_n \\ s_n = (1 - \gamma_n)\Gamma t_n + \gamma_n \Gamma s_n \\ t_n = (1 - \alpha_n)\Gamma r_n + \alpha_n \Gamma t_n \\ r_n = (1 - \beta_n)z_n + \beta_n \Gamma r_n, \end{cases} \tag{3.1}$$

where $\{\delta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ with $\{\delta_n\}_{n=0}^{\infty}$ satisfying $\sum_{n=0}^{\infty}\delta_n = \infty$.

Remark 3.1. Note that:

(1) If $\beta_n = 0$ in (3.1), we have

$$\begin{cases} z_{n+1} = (1-\delta_n)s_n + \delta_n \Gamma s_n \\ s_n = (1-\gamma_n)\Gamma t_n + \gamma_n \Gamma s_n \\ t_n = (1-\alpha_n)\Gamma z_n + \alpha_n \Gamma t_n \end{cases} \tag{3.2}$$

(2) If $\alpha_n = \beta_n = 0$ in (3.1), we obtain

$$\begin{cases} z_{n+1} = (1-\delta_n)s_n + \delta_n \Gamma s_n \\ s_n = (1-\gamma_n)\Gamma t_n + \gamma \Gamma s_n \\ t_n = \Gamma z_n \end{cases} \tag{3.3}$$

(3) If $\delta_n = 1$ and $\alpha_n = \beta_n = 0$ in (3.1), we get

$$\begin{cases} z_{n+1} = \Gamma_n s_n \\ s_n = (1-\gamma_n)\Gamma t_n + \gamma_n \Gamma s_n \\ t_n = \Gamma z_n \end{cases} \tag{3.4}$$

Theorem 3.2. *Let D be a nonempty convex and closed subset of an arbitrary Banach space E and $\Gamma : D \longrightarrow D$ be a mapping satisfying the inequality*

$$\|\Gamma s - \Gamma r\| \le \rho \|s - r\|, \tag{3.5}$$

where $0 \le \rho < 1$. Pick a point $z_0 \in D$, then the sequence $\{z_n\}_{n=0}^{\infty}$ defined by (3.1) converges to the fixed point of Γ provided $\sum_{n=0}^{\infty}\delta_n = \infty$.

Proof. Theorem 1.1 above guarantees that Γ has a unique fixed point in D (say q). Using (3.1) and (3.5), we estimate as follows:

$$\begin{aligned} \|z_{n+1} - q\| &\le (1-\delta_n)\|z_n - q\| + \delta_n \|\Gamma s_n - q\| \\ &\le (1-\delta_n)\|z_n - q\| + \delta_n \rho \|s_n - q\|. \end{aligned} \tag{3.6}$$

Since from (3.1) and (3.5)

$$\begin{aligned} \|s_n - q\| &\le (1-\gamma_n)\|\Gamma t_n - q\| + \gamma_n \|\Gamma s_n - q\| \\ &\le (1-\gamma_n)\rho\|t_n - q\| + \gamma_n \rho \|s_n - q\|, \end{aligned}$$

it follows that

$$\|s_n - q\| \le \frac{(1-\gamma_n)\rho}{1-\gamma_n \rho}\|t_n - q\|. \tag{3.7}$$

Again, from (3.1) and (3.5), we obtain

$$\begin{aligned} \|t_n - q\| &\le (1-\alpha_n)\|\Gamma r_n - q\| + \alpha_n \|\Gamma t_n - q\| \\ &\le (1-\alpha_n)\rho\|r_n - q\| + \gamma_n \rho \|t_n - q\|, \\ &\le \frac{(1-\alpha_n)\rho}{1-\alpha_n \rho}\|r_n - q\|. \end{aligned} \tag{3.8}$$

Furthermore, using (3.1) and (3.5), we get

$$\begin{aligned}\|r_n - q\| &\leq (1-\beta_n)\|z_n - q\| + \beta_n\|\Gamma r_n - q\| \\ &\leq (1-\beta_n)\|z_n - q\| + \beta_n\rho\|r_n - q\| \\ &\leq \frac{(1-\beta_n)}{1-\beta_n\rho}\|z_n - q\|. \end{aligned} \tag{3.9}$$

Set

$$\frac{\sigma_n}{\epsilon_n} = \frac{(1-\gamma_n)\rho}{1-\gamma_n\rho}$$

so that

$$1 - \frac{\sigma_n}{\epsilon_n} = 1 - \frac{(1-\gamma_n)\rho}{1-\gamma_n\rho} = \frac{1-\rho}{1-\gamma_n\rho} \geq 1-\rho. \tag{3.10}$$

(3.10) implies that

$$\frac{\sigma_n}{\epsilon_n} \leq \rho. \tag{3.11}$$

By setting

$$\frac{\sigma_n^\star}{\epsilon_n^\star} = \frac{(1-\alpha_n)\rho}{1-\alpha_n\rho}, \quad \frac{\sigma_n^{\star\star}}{\epsilon_n^{\star\star}} = \frac{(1-\beta_n)}{1-\beta_n\rho},$$

and following similar approach as in (3.10), we obtain

$$\frac{\sigma_n^\star}{\epsilon_n^\star} \leq \rho. \tag{3.12}$$

and

$$\frac{\sigma_n^{\star\star}}{\epsilon_n^{\star\star}} \leq 1 - \beta_n(1-\rho) < 1. \tag{3.13}$$

Putting (3.7), (3.8) and (3.9) into (3.6) and simplifying, using (3.11), (3.12) and (3.13), we get

$$\begin{aligned}\|z_{n+1} - q\| &\leq (1-\delta_n)\rho^2\|z_n - q\| + \delta_n\rho^3\|z_n - q\| \\ &= [1-\delta_n(1-\rho)]\|z_n - q\|. \end{aligned} \tag{3.14}$$

By repeating similar process as in (3.14) for $\|z_n - q\|, \|z_{n-1} - q\|, \|z_{n-2} - q\|, \cdots, \|z_{n-n} - q\|$, substituting the results successively at the right hand side of (3.14) and simplifying, we obtain

$$\|z_{n+1} - q\| \leq \rho^{2(n+1)}\prod_{\ell=0}^{n}[1-\delta_\ell(1-\rho)]\|z_0 - q\|, n = 0, 1, \cdots \tag{3.15}$$

Now, since $0 \leq \rho < 1, \delta_n \in [0.1]$ *and* $\sum_{n=0}^{\infty}\delta_n = \infty$, we have

$$\rho^{2(n+1)}\prod_{\ell=0}^{n}[1-\delta_\ell(1-\rho)] = 0.$$

In view of the above information, we obtain from (3.15) that $z_n \to q$ *as* $n \to \infty$. Hence, $\{z_n\}_{n=0}^{\infty}$ converges strongly to q. □

The corollary below is a consequence of Theorem 3.1.

Corollary 3.3. *Let D be a nonempty convex and closed subset of an arbitrary Banach space E and $\Gamma : D \longrightarrow D$ be a mapping satisfying the inequality*

$$\|\Gamma s - \Gamma r\| \leq \rho \|s - r\|, \tag{3.16}$$

where $0 \leq \rho < 1$. Pick a point $z_0 \in D$, then

(a) (3.2) *converges strongly to the fixed point of* Γ*;*
(b) (3.3) *converges strongly to the fixed point of* Γ*;*
(c) (3.4) *converges strongly to the fixed point of* Γ*.*

4. Stability Results

In this section, we show that the quasi-implicit iterative scheme defined by (3.1) is Γ-stable.

Theorem 4.1. *Let E be a an arbitrary Banach space and $\Gamma : E \longrightarrow E$ be a self map of E with a fixed point satisfying the inequality*

$$\|\Gamma s - \Gamma r\| \leq \rho \|s - r\| \tag{4.1}$$

for each $s, r \in E$. where $0 \leq \rho < 1$. Pick a point $z_0 \in D$, then the sequence $\{z_n\}_{n=0}^{\infty}$ defined by (3.1) with $0 < \delta < \delta_n, 0 < \gamma < \gamma_n, 0 < \beta < \beta_n$ and $0 < \alpha < \alpha_n$ is Γ-stable.

Proof. Let $\{w_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ be arbitrary sequences, where

$$\begin{cases} v_n = \gamma_n \Gamma y_n + (1 - \gamma_n)\Gamma u_n \\ u_n = \alpha_n \Gamma u_n + (1 - \alpha_n)\Gamma w_n \\ w_n = \beta_n \Gamma w_n + (1 - \beta_n) y_n. \end{cases} \tag{4.2}$$

Define

$$\xi_n = \|y_{n+1} - (1 - \delta_n)v_n - \delta_n \Gamma v_n\|. \tag{4.3}$$

and suppose $\xi_n \to 0$ *as* $n \to \infty$. Using (5.1), we show that $y_n \to q$ as n$\to\infty$.

Now, from (4.3), we get

$$\begin{aligned} \|y_{n+1} - q\| &\leq \|y_{n+1} - (1 - \delta_n)v_n - \delta_n \Gamma v_n\| + \|(1 - \delta_n)v_n + \delta_n \Gamma v_n - q\| \\ &= \xi_n + \|(1 - \delta_n)v_n + \delta_n \Gamma v_n - q\| \\ &\leq \xi_n + (1 - \delta_n)\|v_n - q\| + \delta_n \|\Gamma v_n - q\| \\ &\leq \xi_n + (1 - \delta_n)\|v_n - q\| + \delta_n \rho \|v_n - q\| \end{aligned} \tag{4.4}$$

Also, from (4.2), we obtain

$$\begin{aligned} \|v_n - q\| &\leq \gamma_n \|\Gamma v_n - q\| + (1 - \gamma_n)\|\Gamma u_n - q\| \\ &\leq \gamma_n \rho \|v_n - q\| + (1 - \gamma_n)\rho \|u_n - q\| \\ &\leq \frac{(1 - \gamma_n)\rho}{1 - \gamma_n \rho}\|u_n - q\|. \end{aligned} \tag{4.5}$$

Furthermore, since

$$\begin{aligned} \|u_n - q\| &\leq \alpha_n \|\Gamma u_n - q\| + (1 - \alpha_n)\|\Gamma w_n - q\| \\ &\leq \gamma_n \rho \|u_n - q\| + (1 - \alpha_n)\rho \|w_n - q\| \\ &\leq \frac{(1 - \alpha_n)\rho}{1 - \alpha_n \rho}\|w_n - q\| \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \|w_n - q\| &\leq \beta_n\|\Gamma w_n - q\| + (1-\beta_n)\|y_n - q\| \\ &\leq \beta_n\rho\|w_n - q\| + (1-\beta_n)\|y_n - q\| \\ &\leq \frac{(1-\beta_n)}{1-\beta_n\rho}\|y_n - q\|, \end{aligned} \tag{4.7}$$

it follows from (4.4) and (4.5) that

$$\begin{aligned} \|y_{n+1} - q\| &\leq \xi_n + (1-\delta_n)\Big(\frac{(1-\gamma_n)\rho}{1-\gamma_n\rho}\Big)\Big(\frac{(1-\alpha_n)\rho}{1-\alpha_n\rho}\Big)\Big(\frac{(1-\beta_n)}{1-\beta_n\rho}\Big)\|y_n - q\| \\ &\quad + \delta_n\rho\Big(\frac{(1-\gamma_n)\rho}{1-\gamma_n\rho}\Big)\Big(\frac{(1-\alpha_n)\rho}{1-\alpha_n\rho}\Big)\Big(\frac{(1-\beta_n)}{1-\beta_n\rho}\Big)\|y_n - q\|. \end{aligned} \tag{4.8}$$

(3.11), (3.12), (3.13) and (4.6) imply

$$\begin{aligned} \|y_{n+1} - q\| &\leq \xi_n + (1-\delta_n)\rho^2\|y_n - q\| + \delta_n\rho^3\|y_n - q\| \\ &= [1-\delta_n(1-\rho)]\rho^2\|y_n - q\| + \xi_n. \end{aligned} \tag{4.9}$$

From our assumption that $\xi_n \to 0$ *as* $n \to \infty$, Lemma 2.1 and (4.9), we conclude that $y_n \to q$ *as* $n \to \infty$.

Conversely, let $y_n \to q$ *as* $n \to \infty$. We will now prove that $\xi_n \to 0$ *as* $n \to \infty$. Indeed, from (4.3), we have

$$\begin{aligned} \xi_n &= \|y_{n+1} - q - [(1-\delta_n)v_n + \delta_n\Gamma v_n - q]\| \\ &\leq \|y_{n+1} - q\| + \|(1-\delta_n)v_n + \delta_n\Gamma v_n - q\| \\ &\leq \|y_{n+1} - q\| + (1-\delta_n)\|v_n - q\| + \delta_n\|\Gamma v_n - q\| \\ &\leq \|y_{n+1} - q\| + (1-\delta_n)\|v_n - q\| + \delta_n\rho\|v_n - q\|. \end{aligned} \tag{4.10}$$

Using (4.5), (4.6), (4.7) and simplifying via (3.11), (3.12) and (3.13), we get

$$\xi_n \leq \|y_{n+1} - q\| + [1-\delta_n(1-\rho)]\rho^2\|.y_n - q\| \tag{4.11}$$

Since, $y_n \to q$ *as* $n \to \infty$, it follows from (4.11) that $\xi_n \to 0$ *as* $n \to \infty$. Consequently, the quasi-implicit iterative scheme defined by (3.1) is Γ-stable. □

The corollary below follows directly from Theorem 4.1.

Corollary 4.2. *Let E be a an arbitrary Banach space and $\Gamma : E \longrightarrow E$ be a self map of E with a fixed point satisfying the inequality*

$$\|\Gamma s - \Gamma r\| \leq \rho\|s - r\| \tag{4.12}$$

for each $s, r \in E$. where $0 \leq \rho < 1$. Pick a point $z_0 \in D$, then the sequence $\{z_n\}_{n=0}^{\infty}$ defined by (3.2), (3.3) and (3.4) with $0 < \delta < \delta_n, 0 < \gamma < \gamma_n$ and $0 < \alpha < \alpha_n$ is Γ-stable.

5. Rate of Convergence

In this section, we will show that the quasi-implicit iterative scheme (3.1) converges faster than semi-implicit iterative scheme (which in turn is faster than implicit Mann iterative scheme, implicit lshikawa iterative scheme and implicit S-iterative scheme, see [37] for details) for general class of contractive mappings.

TABLE 1. **Comparison of the convergent rate of SII and QII:** $\Gamma z = \dfrac{x}{2}$.

n	SEMI-IMPLICIT SCHEME (SII)	QUASI-IMPLICIT SCHEME (QII)	
1	1.000000	1.000000	
2	0.305556	0.111111	
3	0.093364	0.012346	
4	0.028528	0.001372	
5	0.008717	0.000152	
6	0.002663	0.000017	
7	0.000814	0.000002	
8	0.000249	0.000000	
9	0.000076	0.000000	
10	0.000023	0.000000	
12	0.000007	0.000000	
13	0.000002	0.000000	
14	0.000001	0.000000	
15	0.000000	0.000000	

Theorem 5.1. *Let D be a nonempty, closed and convex subset of normed linear space $(E, \|, \|)$ and $\Gamma : D \longrightarrow D$ be a self map of D with satisfying the inequality*

$$\|\Gamma s - \Gamma r\| \leq \rho \|s - r\| \tag{5.1}$$

for each $s, r \in E$. where $0 \leq \rho < 1$. Pick a point $z_0 \in D$. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence defined by (3.1) with $\{\delta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ such that $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\delta \leq \delta_i \leq \delta_n \forall n \in \mathbb{N}$. Then $\{z_n\}_{n=0}^{\infty}$ converges faster to q than the iterative scheme (1.13).

Proof. From (3.15), and the assumption $\delta \leq \delta_i \leq \delta_n$ for some $\delta > 0$, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} \|z_{n+1} - q\| &\leq \rho^{2(n+1)} \prod_{\ell=0}^{n} [1 - \delta_\ell (1 - \rho)] \|z_0 - q\| \\ &= \rho^{2(n+1)} [1 - \delta(1 - \rho)] \|z_0 - q\|. \end{aligned} \tag{5.2}$$

Similarly, in (Bosede et al [37], Theorem 3.1), the authors proved that the iterative scheme (1.13) is of the form

$$\|z_{n+1} - q\| \leq \prod_{\ell=0}^{n} [1 - \delta_\ell (1 - \rho)] \|z_0 - q\|. \tag{5.3}$$

Now, since $\delta \leq \delta_\ell \leq 1$ for $\delta > 0$ and for all $n \in \mathbb{N}$, then from (5.3), we have

$$\begin{aligned} \|z_{n+1} - q\| &\leq \prod_{\ell=0}^{n} [1 - \delta_\ell (1 - \rho)] \|z_0 - q\| \\ &= [1 - \delta(1 - \rho)] \|z_0 - q\|. \end{aligned} \tag{5.4}$$

Put

$$\begin{aligned} \lambda_n &= \rho^{2(n+1)} [1 - \delta(1 - \rho)] \|z_0 - q\|, \\ \omega_n &= [1 - \delta(1 - \rho)] \|z_0 - q\| \end{aligned}$$

and define

$$\begin{aligned}\theta_n &= \frac{\lambda_n}{\omega_n}\\ &= \frac{\rho^{2(n+1)}[1-\delta(1-\rho)]\|z_0-q\|}{[1-\delta(1-\rho)]\|z_0-q\|}\\ &= \rho^{2(n+1)},\end{aligned}$$

Then, $\theta_n \to 0 \; as \; n \to \infty$ since $\rho \in [0,1)$. Thus, the sequence $\{z_n\}_{n=0}^{\infty}$ defined (3.1) converges faster than the semi-implicit iterative scheme defined by (1.13). □

Now, we show the validity of the above proof with the following example.

Example 5.2. (see [2]) Let $E=[0,1]$ and $\Gamma : E \longrightarrow E$ be defined by $\Gamma x = \frac{1}{2}x, x \neq 0$. Taking $\delta_n = \gamma_n = \alpha_n = \beta_n = 1 - \frac{1}{n} \; for \; n \geq 2$, the comparison of the convergence of semi-implicit iterative (SII) scheme and quasi-implicit iterative (QII) scheme to the fixed point $q=0$ are as given (with the initial point $x=1$) above.

Conclusion

Using the above numerical example, it is observed that quasi-implicit iteration (QII) has faster convergence rate than semi-implicit iteration (SII), which in turn has been proven in [37] to be faster in convergence than implicit Mann iteration, implicit Ishikawa iteration and implicit S-iteration.

Abbreviations Used

Not applicable

Declaration:

Availability of Data and Material

Not applicable

Competing lnterest

The authors declare that there is no conflict of interest.

Funding

Not applicable

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