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Research Paper

COEFFICIENT BOUNDS FOR A NEW CLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SUBORDINATION

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Abstract. The main purpose of this article is to introduce and investigate the subcategory $\mathcal{H}_\Sigma(n,\beta;\phi)$ of bi-univalent functions in the open unit disk $\mathbb{U}$ related to subordination. Moreover, estimates on coefficient $|a_n|$ for functions belong to this subcategory are given applying different a technique. In addition, smaller upper bound and more accurate estimation than the previous outcomes are obtained.

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1. Introduction

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ be the category of functions f analytic in $\mathbb{U}$ that has the following representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ z \in \mathbb{U} \tag{1.1}$$

and denote by $\mathcal{S}$ the subclass of all functions of $\mathcal{A}$ which are univalent in $\mathbb{U}$.

If the functions f and g are analytic in $\mathbb{U}$, the function f is called to be *subordinate* to the function g, written $f(z) \prec g(z)$, if there exists a function w analytic in $\mathbb{U}$ with $|w(z)| < 1$, $z \in \mathbb{U}$, and $w(0) = 0$, such that $f = g \circ w$. In particular, if g is univalent in $\mathbb{U}$ then the following equivalence relationship holds

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Ma and Minda [21] defined the subcategories of starlike and convex functions utilizing the concept of subordination, where we make here the assumptions that the function ϕ has positive real part in $\mathbb{U}$, $\phi(\mathbb{U})$ is symmetric with respect to the real axis with $\phi(0) = 1$, $\phi'(0) = J_1 > 0$ and the power series expansion of the form

$$\phi(z) = 1 + J_1 z + J_2 z^2 + J_3 z^3 + \dots, \ z \in \mathbb{U}. \tag{1.2}$$

They introduced the categories as follows:

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{z f'(z)}{f(z)} \prec \phi(z), \ z \in \mathbb{U} \right\}$$

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and

$$\mathcal{K}(\phi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z),\ z \in \mathbb{U} \right\}.$$

The categories $\mathcal{S}^*(\phi)$ and $\mathcal{K}(\phi)$ for $\phi(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$ reduce to the categories $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$ of Janowski starlike and Janowski convex functions, respectively. Note that if $0 \le \alpha < 1$, then $\mathcal{S}^*[1 - 2\alpha, -1] =: \mathcal{S}^*(\alpha)$, the category of starlike functions of order α and $\mathcal{K}[1 - 2\alpha, -1] =: \mathcal{K}(\alpha)$ the category of convex functions of order α. In particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ are the popular categories of starlike and convex functions in $\mathbb{U}$, respectively. Moreover, the features of the category $\mathcal{S}^*_{\mathrm{e}} := \mathcal{S}^*(\mathrm{e}^z)$ was studied by Mendiratta et al. in [22].

The Koebe one-quarter theorem [13] ensures that every function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z\ (z \in \mathbb{U}) \quad \text{and} \quad f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \ge \frac{1}{4}\right),$$

with the power series

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots. \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the category of bi-univalent functions in $\mathbb{U}$. Lewin [20] studied the bi-univalent function category Σ and reported the bound for the second Taylor-Maclaurin coefficient $|a_2|$. In fact, a brief background overview of functions in the category Σ with interesting examples can be seen in the article of Srivastava et al. [28]. Deriving from the research [28], bounds for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of different categories of bi-univalent functions were reported, for example [16, 23, 24, 27]. Indeed, the study of bi-univalent functions was successfully revived by the pioneering work of Srivastava et al. [28] recently.

According to [28], many researchers try to study various subclasses of the category Σ of bi-univalent functions with different issues such as coefficient bounds and Fekete-Szegö inequalities in recent years, for example [10, 16, 24, 27]. In this area, some authors applied the Faber polynomial expansions to find the general bounds of $|a_n|$ for the bi-univalent functions [6, 7, 8, 9, 11, 17, 18, 26, 29, 30]. Faber [14] studied Faber polynomials that these polynomials play a major role in geometric function theory.

Utilizing the technique of convolution, Ruscheweyh [25] (see also [4]) defined the operator R^λ on the category of analytic functions $\mathcal{A}$ as

$$\mathrm{R}^\lambda f(z) = f(z) * \frac{z}{(1 - z)^{\lambda+1}}, \quad z \in \mathbb{U},\ \lambda \in \mathbb{R},\ \lambda > -1.$$

For $\lambda = n \in \mathbb{N} \cup \{0\}$ we have

$$\mathrm{R}^n f(z) = \frac{z\left(z^{n-1}f(z)\right)^{(n)}}{n!}.$$

The expression $\mathrm{R}^n f(z)$ is called an nth-order Ruscheweyh derivative of $f(z)$ and the symbol $*$ stands for Hadamard product (or convolution). We see that (see [19])

$$\mathrm{R}^n f(z) = z + \sum_{k=2}^{\infty} \sigma(n, k)a_kz^k,$$

where

$$\sigma(n,k)=\frac{\Gamma(n+k)}{(k-1)!\Gamma(n+1)}$$

The object of the present paper is to introduce a new subclass of Σ and derive bounds for the general Taylor-Maclaurin coefficients $|a_n|$ applying the Faber polynomial expansion techniques for the functions belong to this subclass where the results are not sharp. Further, estimates for the first coefficient $|a_2|$ of these functions are obtained.

2. **Main Results**

First, we introduce the category $\mathcal{H}_\Sigma(n,\beta;\phi)$ as follows:

Definition 2.1. A function $f\in\Sigma$ given by (1.1) is said to be in the category $\mathcal{H}_\Sigma(n,\beta;\phi)$ if the following conditions are satisfied

$$(2.1)\qquad (\mathrm{R}^n f(z))'+\beta z\,(\mathrm{R}^n f(z))''\prec\phi(z),\quad z\in\mathbb{U}$$

and

$$(2.2)\qquad (\mathrm{R}^n g(w))'+\beta z\,(\mathrm{R}^n g(w))''\prec\phi(w),\quad w\in\mathbb{U},$$

where $\beta\geq 0,\ g=f^{-1}$ and ϕ is the function given by (1.2)

Remark 2.2. For choices of n,β and ϕ, special cases of this category are obtained below:

(1) For $\beta=0$ and $n=0$, the category $\mathcal{H}_\Sigma(n,\beta;\phi)$ reduce to category $\mathcal{H}_\Sigma(\phi)$ [3].
(2) For $\phi(z)=\frac{1+(1-2\delta)z}{1-z}$ $(0\leq\delta<1)$, $\beta=0$ and $n=0$, the category $\mathcal{H}_\Sigma(n,\beta;\phi)$ reduce to category $\mathcal{H}_\Sigma(\delta)$ [28].
(3) For $\phi(z)=\frac{1+(1-2\delta)z}{1-z}$ $(0\leq\delta<1)$ and $n=0$, the category $\mathcal{H}_\Sigma(n,\beta;\phi)$ reduce to category $\mathcal{H}_\Sigma(\delta,\beta)$ [29] (see also [15]).

To establish the results, the following outcomes are needed.

Lemma 2.3. [31] *Let* $\sum\limits_{i=1}^{\infty}x_iz^i$ *be a polynomial. Then for any* $j\in\mathbb{N}$, *there are the polynomials* D_n^j *such that*

$$\left(\sum_{i=1}^{\infty}x_iz^i\right)^j=\sum_{n=j}^{\infty}D_n^jz^n$$

where

$$D_n^j=D_n^j(x_1,x_2,\dots x_{n-j+1})=\sum\frac{j!(x_1)^{i_1}\dots(x_{n-j+1})^{i_{n-j+1}}}{i_1!\dots i_{n-j+1}!},$$

where the sum is taken over all nonnegative integers $i_1,\dots,i_{n-j+1}$ *satisfying*

$$\begin{cases} i_1+i_2+\cdots+i_{n-j+1}=j,\\ i_1+2i_2+\cdots+(n-j+1)i_{n-j+1}=n.\end{cases}$$

It is clear that

$$D_n^1=x_n,\quad D_n^n=x_1^n,\quad n\geq 1$$

Lemma 2.4. [1, 2] *Let $f \in \mathcal{S}$ be given by (1.1). Then the coefficients of its inverse map $g = f^{-1}$ are given in terms of the Faber polynomials of f with*

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} =& \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] + \sum_{j \geq 7} a_2^{n-j} V_j \end{aligned}$$

such that V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$, and the expressions such as (for example) $(-m)!$ are to be interpreted symbolically *by*

$$(-m)! \equiv \Gamma(1-m) := (-m)(-m-1)(-m-2)\cdots, \; m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \; \mathbb{N} := \{1, 2, \ldots\}.$$

We remark that the first three terms of K_{n-1}^{-n} are given by

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3\left(2a_2^2 - a_3\right) \quad \text{and} \quad K_3^{-4} = -4\left(5a_2^3 - 5a_2 a_3 + a_4\right).$$

In general, for any real number p the expansion of K_n^p is given below (see for details, [1]; see also [2, p. 349])

$$K_n^p = p a_{n+1} + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \ldots + \frac{p!}{(p-n)!n!} D_n^n,$$

Lemma 2.5. [5] *Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$; $(n \geq 2)$ be a univalent function in $\mathbb{U}$ and*

$$f^{-1}(w) = w + \sum_{k=n}^{\infty} b_k w^k \quad (|w| < r_0(f);\; r_0(f) \geq 1/4).$$

Then

$$b_{2n-1} = n a_n^2 - a_{2n-1} \qquad \text{and} \qquad b_k = -a_k \quad \text{for} \quad (n \leq k \leq 2n-2).$$

Let $\mathcal{B}$ be the class of Schwarz functions, that is, $w \in \mathcal{B}$ if and only if w is an analytic function with $w(0) = 0$ and $|w(z)| < 1$ on $\mathbb{U}$.

Lemma 2.6. [13, p. 190] *Let the function $u \in \mathcal{B}$ with the power series expansion $u(z) = \sum_{n=1}^{\infty} u_n z^n$, $z \in \mathbb{U}$. Then, $|u_n| \leq 1$ for all $n = 1, 2, 3, \ldots$. Furthermore, $|u_n| = 1$ for some n $(n = 1, 2, 3, \ldots)$ if and only if $u(z) = e^{i\theta} z^n$, $\theta \in \mathbb{R}$.*

Lemma 2.7. [18, Corollary 2.3] *Let the function $u \in \mathcal{B}$ with the power series expansion given by $u(z) = \sum_{n=1}^{\infty} u_n z^n$, $z \in \mathbb{U}$. If $\gamma \geq 0$ then*

$$\left|u_2 + \gamma u_1^2\right| \leq 1 + (\gamma - 1)\left|u_1^2\right|.$$

Lemma 2.8. *Let $u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \in \mathcal{B}$ and $\gamma \geq 0$. Then, for all $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ the next inequality holds:*

$$\left|u_{2n} + \gamma u_n^2\right| \leq 1 + (\gamma - 1)\left|u_n^2\right|.$$

Proof. For $u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \in \mathcal{B}$ and a fixed $n \in \mathbb{N}$, let denote by $\varepsilon_k := e^{2k\pi i/n}$, $k \in \{1, 2, \dots, n\}$ the n-th order complex roots of the unity. If we define the function $v : \mathbb{U} \to \mathbb{C}$ by

$$v(z) := \frac{1}{n}\sum_{k=1}^{n} u\left(\varepsilon_k z\right),\ z \in \mathbb{U}, \tag{2.3}$$

using the fact that

$$\sum_{k=1}^{n} \varepsilon_k^m = \begin{cases} 0, & \text{if} \quad m \in \mathbb{N} \quad \text{is not a multiple of } n, \\ n, & \text{if} \quad m \in \mathbb{N} \quad \text{is a multiple of } n, \end{cases}$$

it follows

$$v(z) = u_n z^n + u_{2n} z^{2n} + \dots,\ z \in \mathbb{U}. \tag{2.4}$$

Since u is an analytic function in $\mathbb{U}$, from the definition (2.3) it follows that v ia also analytic in $\mathbb{U}$, and $v(0) = 0$. Moreover, since $u \in \mathcal{B}$, we have

$$|v(z)| \leq \frac{1}{n}\sum_{k=1}^{n} |u(\mathrm{e}^{-2\mathrm{i}k\pi/n} z)| < \frac{n}{n} = 1,\ z \in \mathbb{U},$$

therefore $v \in \mathcal{B}$.

Since the function $\chi(z) := z^n$ is a surjective endomorphism of the unit disc $\mathbb{U}$, setting $\zeta := z^n$ in (2.4) and using the fact that $v \in \mathcal{B}$ we deduce that the function $\varphi : \mathbb{U} \to \mathbb{C}$ by

$$\varphi(z) := u_n \zeta + u_{2n} \zeta^2 + u_{3n} \zeta^3 + \dots,\ \zeta \in \mathbb{U},$$

belongs to the class $\mathcal{B}$. Now, using Lemma 2.7 for the function $\varphi \in \mathcal{B}$ given by the above power series expansion we obtain the required result. □

Theorem 2.9. *Let the function $f(z) = z + \sum\limits_{k=p}^{\infty} a_k z^k \in \mathcal{H}_\Sigma(n, \beta; \phi)$; $(p \geq 2)$ with $0 \leq J_2 \leq J_1$. Then*

$$|a_p| \leq \min\left\{\frac{J_1}{p[1 + \beta(p-1)]\sigma(n,p)}, \sqrt{\frac{2J_1}{p(2p-1)[1 + \beta(2p-2)]\sigma(n,2p-1)}}\right\} \tag{2.5}$$

and

$$|pa_p^2 - a_{2p-1}| \leq \frac{J_1}{(2p-1)[1 + \beta(2p-2)]\sigma(n,2p-1)}.$$

Proof. For the function f of the form (1.1), we have

$$(\mathrm{R}^n f(z))' + \beta z\left(\mathrm{R}^n f(z)\right)'' = 1 + \sum_{k=2}^{\infty} k[1 + \beta(k-1)]\sigma(n,k) a_k z^{k-1} \tag{2.6}$$

and for its inverse map, $g = f^{-1}$, by Lemma 2.4 we obtain

$$(\mathrm{R}^n g(w))' + \beta z\left(\mathrm{R}^n g(w)\right)'' = 1 + \sum_{k=2}^{\infty} k[1 + \beta(k-1)]\sigma(n,k) b_k w^{k-1}. \tag{2.7}$$

where

$$b_k = \frac{1}{k} K_{k-1}^{-k}(a_2, a_3, \ldots, a_k)$$

On the other hand, since $f \in \mathcal{H}_\Sigma(n, \beta; \phi)$, then by the definition of subordination there are two functions $u, v \in \mathcal{B}$ with $u(z) = \sum_{k=1}^{\infty} u_k z^n$ and $v(z) = \sum_{k=1}^{\infty} q_k z^k$, respectively, so that

$$(\mathrm{R}^n f(z))' + \beta z \left(\mathrm{R}^n f(z)\right)'' = \phi(u(z)),$$

and

$$(\mathrm{R}^n g(w))' + \beta z \left(\mathrm{R}^n g(w)\right)'' = \phi(v(w)),$$

where applying (1.2) and Lemma 2.3, it follows that

(2.8)

$$\phi(u(z)) = 1 + J_1 u_1 z + (J_1 u_2 + J_2 u_1^2) z^2 + \ldots = 1 + \sum_{k=1}^{\infty} \sum_{s=1}^{k} J_s D_k^s(u_1, u_2, \cdots, u_{k-s+1}) z^k$$

and

$$\phi(v(w)) = 1 + \sum_{k=1}^{\infty} \sum_{s=1}^{k} J_s D_k^s(q_1, q_2, \ldots, q_{k-s+1}) w^k. \tag{2.9}$$

Comparing the corresponding coefficients of (2.6) and (2.8) we obtain

$$k[1 + \beta(k-1)]\sigma(n,k) a_k = \sum_{s=1}^{k-1} J_s D_{k-1}^s(u_1, u_2, \ldots, u_{k-s}). \tag{2.10}$$

Similarly, from (2.7) and (2.9) we get

$$k[1 + \beta(k-1)]\sigma(n,k) \frac{1}{k} K_{k-1}^{-k}(a_2, a_3, \ldots, a_k) = \sum_{s=1}^{k-1} J_s D_{k-1}^s(q_1, q_2, \ldots, q_{k-s}). \tag{2.11}$$

Since $a_2 = \cdots = a_{p-1} = 0$, we obtain $b_p = -a_p$ and since $J_1 > 0$ we have

$$u_1 = \cdots = u_{p-2} = 0, \quad q_1 = \cdots = q_{p-2} = 0. \tag{2.12}$$

Hence, for $k = p$ from (2.10) and (2.11) using (2.12) it follows that

$$p[1 + \beta(p-1)]\sigma(n,p) a_p = \sum_{s=1}^{p-1} J_s D_{p-1}^s(u_1, u_2, \ldots, u_{p-s}) = J_1 u_{p-1}, \tag{2.13}$$

and

$$-p[1 + \beta(p-1)]\sigma(n,p) a_p = \sum_{s=1}^{p-1} J_s D_{p-1}^s(q_1, q_2, \ldots, q_{p-s}) = J_1 q_{p-1}. \tag{2.14}$$

Now by solving the equations (2.13) and (2.14) and applying Lemma 2.6 we get

$$|a_p| \leq \frac{J_1}{p[1 + \beta(p-1)]\sigma(n,p)}. \tag{2.15}$$

Also, for $k = 2p-1$ from (2.10) using Lemma 2.3 and (2.12) after some calculations, it results in

$$(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)a_{2p-1} = J_1 u_{2p-2} + J_2 u_{p-1}^2 = J_1\left(u_{2(p-1)} + \frac{J_2}{J_1}u_{p-1}^2\right).$$

Hence, by Lemma 2.8 with $0 \le J_2 \le J_1$from the above equality we obtain

$$(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)|a_{2p-1}| \le J_1\left(1+\left(\frac{J_2}{J_1}-1\right)|u_{p-1}^2|\right) \le J_1,$$

where it follows that

$$|a_{2p-1}| \le \frac{J_1}{(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}. \tag{2.16}$$

In addition, regarding Definition 2.1 it follows that

$$|b_{2p-1}| \le \frac{J_1}{(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}. \tag{2.17}$$

Further, in view of Lemma 2.5, using the relations (2.16) and (2.17), it results in

$$|a_p| \le \sqrt{\frac{|a_{2p-1}|+|b_{2p-1}|}{p}} \le \sqrt{\frac{2J_1}{p(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}}. \tag{2.18}$$

Therefore, from (2.15) and (2.18), we get the inequality (2.5). In addition, using (2.17) and Lemma 2.5, it follows that

$$|pa_p^2 - a_{2p-1}| = |b_{2p-1}| \le \frac{J_1}{(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}.$$

This completes the proof. □

Remark 2.10. By setting $n = 0$ in Theorem 2.9 we obtain smaller and more accurate upper bound than the estimates obtained in [12, Theorem 1] with $m = \tau = \lambda = 1$ and $\theta(z) = \frac{z}{1-z}$.

Corollary 2.11. *Let the function* $f(z) = z + \sum_{k=p}^{\infty} a_k z^k \in \mathcal{H}_\Sigma(0,1;\phi) =: \mathcal{H}_\Sigma(\phi)$; $(p \ge 2)$ *with* $0 \le J_2 \le J_1$. *Then*

$$|a_p| \le \min\left\{\frac{J_1}{p}, \sqrt{\frac{2J_1}{p(2p-1)}}\right\}$$

and

$$|pa_p^2 - a_{2p-1}| \le \frac{J_1}{2p-1}.$$

Remark 2.12. The obtained bound for $|a_p|$ in Corollary 2.11 is an improvement of the estimates obtained ($|a_p| \le \frac{J_1}{p}$) in [32, Remark 2] and [26, Theorem 1] with $b = 1$ and $q \to 1^-$ that is the new upper bound is smaller and more accurate than the previous result.

For

$$\phi(z) = \frac{1+(1-2\delta)z}{1-z} \quad (0 \le \delta < 1,\ z \in \mathbb{U}),$$

where $J_1 = J_2 = 2(1-\delta)$ in Theorem 2.9, we obtain the following result.

Corollary 2.13. *Let the function* $f(z) = z + \sum_{k=p}^{\infty} a_k z^k \in \mathcal{H}_\Sigma\left(n, \beta; \frac{1+(1-2\delta)z}{1-z}\right);\ (p \geq 2)$. *Then*

$$|a_p| \leq \min\left\{\frac{2(1-\delta)}{p[1+\beta(p-1)]\sigma(n,p)}, \sqrt{\frac{4(1-\delta)}{p(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}}\right\}$$

and

$$|pa_p^2 - a_{2p-1}| \leq \frac{2(1-\delta)}{(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}.$$

Remark 2.14. For $n = 0$ in Corollary 2.13, we the obtain next corollary which is an improvement of the estimates obtained by Srivastava et al. in [29, Theorem 1], that is the new upper bound is smaller and more accurate than the previous result.

Corollary 2.15. *Let the function* $f(z) = z + \sum_{k=p}^{\infty} a_k z^k \in \mathcal{H}_\Sigma\left(0, \beta; \frac{1+(1-2\delta)z}{1-z}\right);\ (p \geq 2)$. *Then*

$$|a_p| \leq \min\left\{\frac{2(1-\delta)}{p[1+\beta(p-1)]}, \sqrt{\frac{4(1-\delta)}{p(2p-1)[1+\beta(2p-2)]}}\right\}$$

and

$$|pa_p^2 - a_{2p-1}| \leq \frac{2(1-\delta)}{(2p-1)[1+\beta(2p-2)]}.$$

For

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma \quad (0 < \gamma \leq 1,\ z \in \mathbb{U}),$$

where $J_1 = 2\gamma, J_2 = 2\gamma^2$ in Theorem 2.9, it gives next result.

Corollary 2.16. *Let the function* $f(z) = z + \sum_{k=p}^{\infty} a_k z^k \in \mathcal{H}_\Sigma\left(n, \beta; \left(\frac{1+z}{1-z}\right)^\gamma\right);\ (p \geq 2)$. *Then*

$$|a_p| \leq \min\left\{\frac{2\gamma}{p[1+\beta(p-1)]\sigma(n,p)}, \sqrt{\frac{4\gamma}{p(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}}\right\}$$

and

$$|pa_p^2 - a_{2p-1}| \leq \frac{2\gamma}{(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}.$$

In another special cases, we obtain the next corollaries.

Corollary 2.17. *Let the function* $f(z) = z + \sum_{k=p}^{\infty} a_k z^k \in \mathcal{H}_\Sigma\left(n, \beta; \mathrm{e}^z\right);\ (p \geq 2)$. *Then*

$$|a_p| \leq \min\left\{\frac{1}{p[1+\beta(p-1)]\sigma(n,p)}, \sqrt{\frac{2}{p(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}}\right\}$$

and

$$|pa_p^2 - a_{2p-1}| \leq \frac{1}{(2p-1)[1+\beta(2p-2)]\sigma(n,2p-1)}.$$

Corollary 2.18. *Let the function* $f \in \mathcal{H}_\Sigma(n, \beta; \phi)$ *be given by* (1.1). *Then*

$$|a_2| \leq \min\left\{\frac{J_1}{2(1+\beta)(n+1)}, \sqrt{\frac{2J_1}{3(1+2\beta)(n+1)(n+2)}}\right\}. \tag{2.19}$$

For

$$\phi(z) = \frac{1+(1-2\delta)z}{1-z} \quad (0 \leq \delta < 1,\ z \in \mathbb{U}),$$

where $J_1 = J_2 = 2(1-\delta)$ in Corollary 2.18, we obtain the following result.

Corollary 2.19. *Let the function* $f \in \mathcal{H}_\Sigma\left(n, \beta; \frac{1+(1-2\delta)z}{1-z}\right)$ *be given by* (1.1). *Then*

$$|a_2| \leq \min\left\{\frac{2(1-\delta)}{2(1+\beta)(n+1)}, \sqrt{\frac{4(1-\delta)}{3(1+2\beta)(n+1)(n+2)}}\right\}.$$

Remark 2.20. For $n = 0$ in Corollary 2.19, we obtain the bound presented by Srivastava et al. in [29, Theorem 2] for $|a_2|$.

Setting

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma \quad (0 < \gamma \leq 1,\ z \in \mathbb{U}),$$

where $J_1 = 2\gamma$ and $J_2 = 2\gamma^2$ in Corollary 2.18, we obtain the next corollary.

Corollary 2.21. *Let the function* $f \in \mathcal{H}_\Sigma\left(n, \beta; \left(\frac{1+z}{1-z}\right)^\gamma\right)$ *be given by* (1.1). *Then*

$$|a_2| \leq \min\left\{\frac{2\gamma}{2(1+\beta)(n+1)}, \sqrt{\frac{4\gamma}{3(1+2\beta)(n+1)(n+2)}}\right\}.$$

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