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Research Paper

STRUCTURE A FAMILY OF THREE-STEP WITH-MEMORY METHODS FOR SOLVING NONLINEAR EQUATIONS AND THEIR DYNAMICS

VALI TORKASHVAND*, MANOCHEHR KAZEMI, AND MANDANA MOCCARI

Abstract. In this work, we will first propose an optimal three-step without-memory method for solving nonlinear equations. Then, by introducing the self-accelerating parameters, the with-memory-methods have been built. They have a fifty-nine percentage improvement in the convergence order. The proposed methods have not the problems of calculating the function derivative. We use these Steffensen-type methods to solve nonlinear equations with simple zeroes with the appropriate initial approximation of the root. we have solved a few nonlinear problems to justify the theoretical study and finally have described the dynamics of the with-memory method for complex polynomials of degree two.

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1. Introduction

Newton's second-order iterative method (NM) is the most popular method for solving nonlinear equations. But its disadvantage is the derivative calculation. This famous method and its error equation are as follows:

$$x_{n+1} = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, 2, \cdots . \tag{1.1}$$

And error equation:

$$e_{k+1} = c_2 e_k^2 + O(e_k^3). \tag{1.2}$$

To solve the problem of calculating the derivative, Steffensen (SM) in 1933 approximated the first-order derivative of the function with Newton's first-order divided difference:

$$f'(x_k) \approx f[x_k, w_k] = \frac{f(x_k - f(w_k))}{x_k - w_k}. \tag{1.3}$$

And presented his method as follows:

$$x_{n+1} = x_k - \frac{f(x_k)}{f[x_k, w_k]},\ w_k = x_k + f(x_k),\ k = 0, 1, 2, \cdots , \tag{1.4}$$

The error equation of this method is as follows:

$$e_{k+1} = (1 + f'(\alpha)) c_2 e_k^2 + O(e_k^3). \tag{1.5}$$

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*Corresponding author.

The efficiency index of both methods is equal to $\sqrt{2} \approx 1.41421$.
Next, two-step and three-step methods for solving nonlinear equations were developed. Twenty-seven years later, in 1960, Ostrowski introduced the most famous two-step without memory method [20]. This method is as follows (OM):

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, 2, \cdots, \\ x_{k+1} = y_k - \frac{f(x_k)}{f'(x_k)} \frac{f(y_k)}{f(x_k) - 2f(y_k)}. \end{cases} \tag{1.6}$$

This method has fourth-order convergence, and its error equation is as follows:

$$e_{k+1} = (c_2^3 - c_3 c_2) e_k^4 + O(e_k^5). \tag{1.7}$$

In 1974, Kung and Traub [11] introduced the following three-step without-derivatives method:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f[x_k, w_k]},\ w_k = x_k + \gamma f(x_k),\ k = 0, 1, 2, \cdots, \\ z_k = y_k - \frac{f(x_k) f(w_k)}{(w_k - y_k) f[x_k, y_k]}, \\ x_{k+1} = z_k - \frac{f(y_k) f(w_k)(y_k - x_k - \frac{f(x_k)}{f[x_k, z_k]})}{(w_k - z_k)(y_k - z_k)} + \frac{f(y_k)}{f[y_k, z_k]}. \end{cases} \tag{1.8}$$

The error equation of this optimal method is as follows:

$$e_{k+1} = (1 + f'(\alpha)\gamma)^4 (2c_2^2 - c_3)(5c_2^3 - 5c_2 c_3 + c_4) e_k^8 + O(e_k^9). \tag{1.9}$$

Nine years later, B. Neta [19] (NM) proposed the first three-step with-memory method to solve nonlinear equations:

$$\begin{cases} w_k = x_k - \frac{f(x_k)}{f'(x_k} + (f(w_{k-1})\phi_z + f(z_{k-1})\phi_w) + \frac{f(x_x)^2}{f(w_{k-1}) + f(z_{k-1})}, \\ \phi_w = \frac{w_{k-1} - x_k}{(f(w_{k-1}) - f(x_k))^2} - \frac{1}{f(w_{k-1}) - f(x_k)) f'(x_k)}, \\ \phi_z = \frac{z_{k-1} - x_k}{(f(z_{k-1}) - f(x_k))^2} - \frac{1}{f(z_{k-1}) - f(x_k)) f'(x_k)}, \\ z_k = x_k - \frac{f(x_k)}{f'(x_k} + (f(w_k)\phi_z + f(z_{k-1})\psi_w) + \frac{f(x_x)^2}{f(w_k) + f(z_{k-1})}, \\ \psi_w = \frac{w_k - x_k}{(f(w_k) - f(x_k))^2} - \frac{1}{f(w_k) - f(x_k)) f'(x_k)}, \\ \gamma_z = \frac{z_k - x_k}{(f(z_k) - f(x_k))^2} - \frac{1}{f(z_k) - f(x_k)) f'(x_k)}, \\ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k} + (f(w_k)\gamma_z + f(z_k)\psi_w) + \frac{f(x_x)^2}{f(w_k) + f(z_k)}. \end{cases} \tag{1.10}$$

In 2011, Geum and Kim [9] introduced the following four-step method (GKM):

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k},\ u_k = \frac{f(y_k)}{f(x_k)},\ v_k = \frac{f(z_k)}{f(y_k)},\ w_k = \frac{f(z_k)}{f(x_n)}, \\ K_w = \frac{1 + \beta u_k + \beta u_k^2}{1 + (\beta - 2) u_k + (1 + 2.5\beta) u_k^2},\ z_k = y_k - K(u_k) \frac{f(y_n)}{f'(x_n)}, \\ H(u_k, v_k, w_k) = \frac{1 - u_k - 1.5 v_k - 2.5 w_{k-1}}{1 - 3u_k - 2.5 v_k + 1.5 w_{k-1}},\ s_k = z_k - H(u_k, v_k, w_k) \frac{f(z_n)}{f'(x_k)}, \\ G(u_k, v_k, w_n, t_k) = \frac{1 - u_n - 1.5 v_k - 3 w_k + 1.5 t_k - 3.25 v_k w_k + 0.75 v_k^3 - 0.25(\beta^2 - \beta + 8) v_k u_k^4 - 1.5 t_k u_k^2 + d u_k w_k^2}{1 - 3u_n - 2.5 v_k + w_k + 0.5 t_k - 4.75 v_k w_k - 0.75 v_k^3 - 0.25(\beta^2 - 3\beta + 8) v_k u_k^4 - 4.5 t_k u_k^2 + (d - 13.5) u_k w_k^2}, \\ x_{k+1} = x_k - \frac{f(x_n)}{f'(x_k} + (f(w_k)\gamma_z + f(z_k)\psi_w) + \frac{f(x_x)^2}{f(w_k) + f(z_k)}. \end{cases} \tag{1.11}$$

Remark 1: The efficiency indices of optimal one-step, two-step, three-step and, four-step methods are as follows: $2^{\frac{1}{2}} = 1.41421$, $4^{\frac{1}{3}} = 1.5874$, $8^{\frac{1}{4}} = 1.6818$, $16^{\frac{1}{5}} = 1.7411$.
Other researchers used without-memory methods to solve nonlinear equations. These include Chun et al. [3], Kansal et al. [10], Cordero et al. [4, 5], Soleymani et al. [22], and so on.
In the continuation of this paper, in the second section, we will first introduce the optimal three-step without memory methods with one and two accelerator parameters. In the third section, we will build new with-memory methods improvement in convergence order. In Section

four, we will see the correctness of the theorems presented in sections two and three by mentioning numerical examples. Section 5 shows the convergence planes of the method with the optimal Steffensen-Lui's type family applied to the quadratic polynomial $p(z) = z^2-1$. Finally, in Section 6, our conclusion is presented.

2. **Without-Memory Methods**

This section has devoted to three-point root-finding methods. In 2010, Liu et al. [14] derived the without memory methods of fourth-order convergent. They proposed the following two-step method:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]},\ w_k = x_k + f(x_k),\ k = 0,1,2,\cdots, \\ x_{k+1} = y_k - \frac{f[x_k,y_k]-f[y_k,w_k]+f[x_k,w_k]}{f[x_k,y_k]^2} f(y_k). \end{cases} \tag{2.1}$$

The class of methods defined by (2.1) is of fourth-order, and satisfies the error relation:

$$e_{k+1} = (1+f'(\alpha))c_2((2+f'(\alpha))c_2^2 - (1+f'(\alpha))c_3)e_k^4 + O(e_k^5). \tag{2.2}$$

Now, we will proposed the following three-step method based on Liu et al.'s method:

$$\begin{cases} y_k = x_k - \frac{f(x_n)}{f[x_k,w_k]},\ w_k = x_k + f(x_k),\ k = 0,1,2,\cdots, \\ z_k = y_k - \frac{f[x_k,y_k]-f[y_k,w_k]+f[x_k,w_k]}{f[x_k,y_k]^2} f(y_k), \\ x_{k+1} = \frac{f(z_k)}{f'(z_k)}. \end{cases} \tag{2.3}$$

The eight-order method is not optimized because it uses five evaluations of its function and its derivative. Therefore, according to Kung and Traub's conjecture must be reduced a function evaluation. We modify (2.3) by approximating $f'(z_k)$ with

$$f'(z_k) \approx f[x_k,z_k] + (f[w_k,x_k,y_k] - f[w_k,x_k,z_k] - f[y_k,x_k,z_k])(x_k - z_k). \tag{2.4}$$

Now, the one-parameter without-memory method based on Liu et al.'s method (2.1) can be rewriten as follows:

$$\begin{cases} y_k = x_k - \frac{f(x_n)}{f[x_k,w_k]},\ w_k = x_k + \gamma f(x_k),\ k = 0,1,2,\cdots, \\ z_k = y_k - \frac{f[x_k,y_k]-f[y_k,w_k]+f[x_k,w_k]}{f[x_k,y_k]^2} f(y_k), \\ x_{k+1} = \frac{f(z_k)}{f[x_k,z_k]+(f[w_k,x_k,y_k]-f[w_k,x_k,z_k]-f[y_k,x_k,z_k])(x_k-z_k)}. \end{cases} \tag{2.5}$$

In the next theorem, we prove that the proposed method (2.5) is a three-step optimal method that has the order of convergence 8.

Theorem 2.1. *Suppose $\alpha \in I$ be a zero of a sufficiently differentiable function $f : I \subset \boldsymbol{R} \to \boldsymbol{R}$ in interval I. If sequence x_0 is sufficiently close to α, then the method (2.5) has the order of convergence eight.*

Proof. We show the Mathematica code for obtaining the mentioned Taylor's series. For simplicity, we have removed the indexes.

$$In[1] := f[e\_] = fla(e + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + c_6e^6 + c_7e^7 + c_8e^8),$$

where $e = x - \alpha$, $fla = f'(\alpha)$. Note that since α is a simple zero of $f(x) = 0$, ae have $f'(\alpha) \neq 0, f(\alpha) = 0$. We define

$$In[2] : f[x\_, y\_] = \frac{f[x] - f[y]}{x - y};$$

$$In[3] := ew = e + \gamma f[e];$$

$$Out[3] = e(1 + \gamma fla) + ... + O[e]^9; \tag{2.6}$$

$$In[4] := ey = e - Series[\frac{f[e]}{f[e, ew]}, \{e, 0, 8\}];$$

$$Out[4] = (1 + \gamma fla)c_2e^2 + ... + O[e]^9 \tag{2.7}$$

$$In[5] : ez = ey - Series[\frac{f[e, ey] - f[ey, ew] + f[e, ew]}{f[e, ey]^2} f[ey],\ \{e, 0, 8\}];$$

$$Out[5] = (1 + \gamma fla)c_2((2 + \gamma fla)c_2^2 - (1 + \gamma fla)c_3)e^4 + O[e]^5 \tag{2.8}$$

$$\begin{aligned} In[6] := e1 = ez - Series[f[ez]((f[ew, e, ey] - f[ew, e, ez] - f[ey, e, ez]) \\ (e - ez))^{-1}, \{e, 0, 8\}]//FullSimplify \end{aligned}$$

$$\begin{aligned} Out[6] = (1 + \gamma fla)^2c_2^2((2 + \gamma fla)c_2^2 - (1 + \gamma fla)c_3)(2 + \gamma fla)c_2^3 \\ - (1 + \gamma fla)c_2c_3 + (1 + \gamma fla)c_4)e^8 + O[e]^5 \end{aligned} \tag{2.9}$$

According to the output of the (2.9) of the Mathematica program, the proof of Theorem (2.1) ends. □

In the continuation of this section, by entering another self-accelerator parameter, we propose the following two-parameter without-memory method:

$$\begin{cases} y_k = x_k - \frac{f(x_n)}{f[x_k, w_k + \beta f(w_k)]},\ w_k = x_k + \gamma f(x_k),\ k = 0, 1, 2, \cdots, \\ z_k = y_k - \frac{f[x_k, y_k] - f[y_k, w_k] + f[x_k, w_k]}{f[x_k, y_k]^2} f(y_k), \\ x_{k+1} = \frac{f(z_k)}{f[x_k, z_k] + (f[w_k, x_k, y_k] - f[w_k, x_k, z_k] - f[y_k, x_k, z_k])(x_k - z_k)}. \end{cases} \tag{2.10}$$

Theorem 2.2. *Assume that the function $f : I \subset \boldsymbol{R} \to \boldsymbol{R}$ for an open interval I has a simple root $\alpha \in I$. Let $f(x)$ be sufficiently smooth in the interval I, then the order of convergence of the new family method defined by (2.10) is eight and its error equation is as follows:*

$$\begin{aligned} e_{k+1} =& (1 + \gamma f'(\alpha))^2(\beta + c_2)^2(\beta + f'(\alpha)\beta\gamma)c_2 + (2 + \gamma f'(\alpha))c_2^2 - (1 + \gamma f'(\alpha))c_3) \\ & (c_2(c_2(\beta + \gamma\beta f'(\alpha) + (2 + \gamma f'(\alpha))c_2) - (1 + \gamma fla)c_3) + c_4 + c_4 f'(\alpha)\gamma)e_k^8 + O(e_k^9). \end{aligned} \tag{2.11}$$

Proof. The proof of Theorem (2.2) is similar to Theorem (2.1). So, we have refused to prove it. □

3. Design some methods with-memory

We will divide this section into two parts. Firstly, we will introduce the family of single-parameter with-memory methods. In the second part of this section, we will extract two-parameter memory methods from without-memory methods of equation (2.10).

3.1. **One-parameter with-memory methods.** In this section, we propose the following iterative method with memory based on (2.5):

$$\begin{cases} \gamma_k = \frac{-1}{f'(\alpha)}, k = 1, 2, 3, \cdots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k]},\ w_k = x_k + \gamma_k f(x_k),\ k = 0, 1, 2, \cdots, \\ z_k = y_k - \frac{f[x_k, y_k] - f[y_k, w_k] + f[x_k, w_k]}{f[x_k, y_k]^2} f(y_k), \\ x_{k+1} = \frac{f(z_k)}{f[x_k, z_k] + (f[w_k, x_k, y_k] - f[w_k, x_k, z_k] - f[y_k, x_k, z_k])(x_k - z_k)}. \end{cases} \tag{3.1}$$

Then the error equation of method (2.5) will be greater than eight. If $1 + f'(\alpha)\gamma = 0$, since α, the exact root of the equation is not available, so the value of $f'(\alpha)$ cannot be calculated accurately. So it can be approximated as follows:

$$f'(\alpha) \approx \bar{f}'(\alpha) \tag{3.2}$$

And

$$\gamma_k = \frac{-1}{\bar{f}'(\alpha)} \tag{3.3}$$

In the following, we want to choose as good as possible available nodes to earn the highest convergence order of the three-point with memory methods. Accordingly, we have considered in this paper five following approximations of the self-accelerating parameter γ_k:

$[I]$ Secant Approach

$$\bar{f}'(\alpha) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \tag{3.4}$$

$[II]$ Secant Approach 1

$$\bar{f}'(\alpha) = \frac{f(x_k) - f(w_{k-1})}{x_k - w_{k-1}} \tag{3.5}$$

$[III]$ Secant Approach 2

$$\bar{f}'(\alpha) = \frac{f(x_k) - f(y_{k-1})}{x_k - y_{k-1}} \tag{3.6}$$

$[IV]$ Best Secant Approach

$$\bar{f}'(\alpha) = \frac{f(x_k) - f(z_{k-1})}{x_k - z_{k-1}} \tag{3.7}$$

$[V]$ Newton's Interpolatory Approach with fourth degree polynomial

$$\bar{f}'(\alpha) = N_4'(x_k),\ N_4(t; x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}) \tag{3.8}$$

The self-accelerating parameter γ_k can be specified recursively as follows:

$[I]$ Secant Approach

$$\gamma_k = \frac{-1}{\bar{f}'(\alpha)} = -\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \tag{3.9}$$

$[II]$ Secant Approach 1

$$\gamma_k = \frac{-1}{\bar{f}'(\alpha)} = -\frac{x_k - w_{k-1}}{f(x_k) - f(w_{k-1})} \tag{3.10}$$

$[III]$ Secant Approach 2

$$\gamma_k = \frac{-1}{\bar{f}'(\alpha)} = -\frac{x_k - y_{k-1}}{f(x_k) - f(y_{k-1})} \tag{3.11}$$

$[IV]$ Best Secant Approach

$$\gamma_k = \frac{-1}{\bar{f}'(\alpha)} = -\frac{x_k - z_{k-1}}{f(x_k) - f(z_{k-1})} \tag{3.12}$$

$[V]$ Newton's Interpolatory Approach with fourth degree polynomial

$$\gamma_k = \frac{-1}{\bar{f}'(\alpha)} = -\frac{1}{N_4'(x_k)} = N_4'(x_k),\ N_4(t; x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}) \tag{3.13}$$

If we determine γ_k using one of the above methods, we can obtain the new-family with-memory of methods.

Theorem 3.1. *Let the initial approximation x_0 be sufficiently close to the zero α of $f(x)$ and the parameter γ_k in the iterative method (3.1) is recursively computed by the forms given in (3.9)-(3.13). Then, the convergence order of the three-point with-memory methods (3.1) with the corresponding expressions (3.9)-(3.13) of γ_k is at least $4+3\sqrt{2}, \frac{1}{2}(9+\sqrt{57}), \frac{1}{2}(9+\sqrt{65}), 9, \frac{1}{2}(11+\sqrt{89})$, and $r = \frac{1}{2}(11+\sqrt{89})$.*

Proof. Taylor's series expansion of $f(x)$ about α is given by

$$f(x) = f(\alpha) + (x-\alpha)f'(\alpha) + \frac{(x-\alpha)^2}{2f''(\alpha)} + \frac{(x-\alpha)^3}{3!f''(\alpha)} + \ldots \tag{3.14}$$

Now using the relations $x_k - \alpha = e_k$ and $x_{k-1} - \alpha = e_{k-1}$ also using the relation (3.14) we have:

$$f(x_k) = f(\alpha) + (x_k-\alpha)f'(\alpha) + \frac{(x_k-\alpha)^2}{2f''(\alpha)} + \frac{(x_k-\alpha)^3}{3!f''(\alpha)} + \ldots \tag{3.15}$$

and

$$f(x_{k-1}) = f(\alpha) + (x_{k-1}-\alpha)f'(\alpha) + \frac{(x_{k-1}-\alpha)^2}{2f''(\alpha)} + \frac{(x_{k-1}-\alpha)^3}{3!f''(\alpha)} + \ldots \tag{3.16}$$

Using (3.15) and (3.16), $x_k - x_{k-1} = e_k - e_{k-1}$ we get

$$\begin{aligned} &\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \\ &= \frac{(e_k - e_{k-1})f'(\alpha) + \frac{(e_k-e_{k-1})^2 f''(\alpha)}{2!} + \frac{(e_k-e_{k-1})^3 f'''(\alpha)}{3!} + \ldots}{e_k - e_{k-1}} \\ &= f'(x^*) + \frac{(e_k + e_{k-1})f''(\alpha)}{2!} + \frac{(e_k^2 - e_k e_{k-1} + e_{k-1}^2)f'''(\alpha)}{3!} + \ldots \end{aligned} \tag{3.17}$$

Using the relation (3.9) we have:

$$\gamma_k = -\frac{1}{f'(\alpha) + \frac{(e_k+e_{k-1})f''(\alpha)}{2!} + \frac{(e_k^2 - e_k e_{k-1} + e_{k-1}^2)f'''(\alpha)}{3!} + \ldots} \tag{3.18}$$

Now to calculate $(1+\gamma_k f'(\alpha))$ using equation (3.18) we achieve:

$$
\begin{aligned}
&1+\gamma_k f'(\alpha)\\
&=1-\frac{f'(\alpha)}{f'(\alpha)+\frac{(e_k+e_{k-1})f''(\alpha)}{2!}+\frac{(e_k^2-e_ke_{k-1}+e_{k-1}^2)f'''(\alpha)}{3!}+\dots}\\
&=\frac{(e_k+e_{k-1})c_2+(e_k^2-e_ke_{k-1}+e_{k-1}^2)c_3+\dots}{1+(e_k+e_{k-1})c_2+(e_k^2-e_ke_{k-1}+e_{k-1}^2)c_3+\dots}\\
&\sim c_2e_{k-1}
\end{aligned} \tag{3.19}
$$

By using the relation (3.19), and outputs (2.6),(2.7)(2.8) and (2.9) of (2.1), also considering the error equations $e_{k+1}, e_{k,z}, e_{k,y}, e_{k,w}$ we get

$$
\begin{cases}
e_{k+1}\sim(1+\gamma_k f'(\alpha))e_k^8\sim e_{k-1}e_{k-1}^{8r}\sim e_{k-1}^{1+8r},\\
e_{k,z}\sim(1+\gamma_k f'(\alpha))e_k^4\sim e_{k-1}e_{k-1}^{4r}\sim e_{k-1}^{1+4r},\\
e_{k,y}\sim(1+\gamma_k f'(\alpha))e_k^2\sim e_{k-1}e_{k-1}^{2r}\sim e_{k-1}^{1+2r},\\
e_{k,w}\sim(1+\gamma_k f'(\alpha))e_k\sim e_{k-1}e_{k-1}^{r}\sim e_{k-1}^{1+r},
\end{cases} \tag{3.20}
$$

Firstly, we assume that the R-orders of convergence of the sequences w_k, y_k, z_k and x_k are at least r_1, r_2, r_3 and r, respectively. Hence

$$
\begin{cases}
e_{k+1}\sim e_k^{r}\sim e_k^{r^2},\\
e_{k,y}\sim e_k^{r_3}\sim e_k^{rr_3},\\
e_{k,y}\sim e_k^{r_2}\sim e_k^{rr_2},\\
e_{k,w}\sim e_k^{r_1}\sim e_k^{rr_1},
\end{cases} \tag{3.21}
$$

Now, by comparing the right-hand sides of equations (3.20) and (3.21), we also the following nonlinear system of four equations in r_1, r_2, r_3 and r

$$
\begin{cases}
rr_1-1-r=0,\\
rr_2-1-2r=0,\\
rr_3-1-4r=0,\\
r^2-1-8r=0.
\end{cases}
$$

We obtain : $r_1=\frac{-1}{2}(-2+3\sqrt{2})$, $r_2=\frac{3}{\sqrt{2}}$, $r_3=\frac{1}{2}(4+3\sqrt{2})$ and $r=(4+3\sqrt{2})$. Thus, we can conclude that the R-order of the with-memory methods (3.1) and (3.9) is $r=(4+3\sqrt{2})$We show this method with TM8.24.
Also, by using the relations $x_k-\alpha=e_k$ and $w_{k-1}-\alpha=e_{k-1,w}$ also the relation (3.14) we have:

$$
f(w_{k-1})=f(\alpha)+(w_{k-1}-\alpha)f'(\alpha)+\frac{(w_{k-1}-\alpha)^2f''(\alpha)}{2!}+\frac{(w_{k-1}-\alpha)^3f'''(\alpha)}{3!}+\dots. \tag{3.22}
$$

Using (3.15), (3.22), and $x_k - w_{k-1} = e_k - e_{k-1,w}$ we get

$$\begin{aligned}
&\frac{f(x_k) - f(w_{k-1})}{x_k - w_{k-1}} \\
&= \frac{(e_k - e_{k-1,w})f'(\alpha) + \frac{(e_k - e_{k-1,w})^2 f''(\alpha)}{2!} + \frac{(e_k - e_{k-1,w})^3 f'''(\alpha)}{3!} + \dots}{e_k - e_{k-1,w}} \\
&= f'(\alpha) + \frac{(e_k + e_{k-1,w})f''(\alpha)}{2!} + \frac{(e_k^2 - e_k e_{k-1,w} + e_{k-1,w}^2)f'''(\alpha)}{3!} + \dots
\end{aligned} \tag{3.23}$$

Using the relation (3.10) we have:

$$\gamma_k = -\frac{1}{f'(\alpha) + \frac{(e_k + e_{k-1,w})f''(\alpha)}{2!} + \frac{(e_k^2 - e_k e_{k-1,w} + e_{k-1,w}^2)f'''(\alpha)}{3!} + \dots} \tag{3.24}$$

Now to calculate $(1 + \gamma_k f'(\alpha))$ we use equation (3.24) and obtain:

$$\begin{aligned}
&1 + \gamma_k f'(\alpha) \\
&= 1 - \frac{f'(\alpha)}{f'(\alpha) + \frac{(e_k + e_{k-1,w})f''(\alpha)}{2!} + \frac{(e_k^2 - e_k e_{k-1,w} + e_{k-1,w}^2)f'''(x^*)}{3!} + \dots} \\
&= \frac{(e_k + e_{k-1,w})c_2 + (e_k^2 - e_k e_{k-1,w} + e_{k-1,w}^2)c_3 + \dots}{1 + (e_k + e_{k-1,w})c_2 + (e_k^2 - e_k e_{k-1,w} + e_{k-1,w}^2)c_3 + \dots} \\
&\sim c_2 e_{k-1,w}
\end{aligned} \tag{3.25}$$

by using the equation (3.25), and its error equation we will get:

$$\begin{cases}
e_{k+1} \sim (1 + \gamma_k f'(\alpha))e_k^8 \sim e_{k-1,w} e_k^8 \sim e_{k-1}^{r_1 + 8r}, \\
e_{k,y} \sim (1 + \gamma_k f'(\alpha))e_k^4 \sim e_{k-1,w} e_k^4 \sim e_{k-1}^{r_1 + 4r}, \\
e_{k,y} \sim (1 + \gamma_k f'(\alpha))e_k^2 \sim e_{k-1,w} e_k^2 \sim e_{k-1}^{r_1 + 2r}, \\
e_{k,w} \sim (1 + \gamma_k f'(\alpha))e_k \sim e_{k-1,w} e_k \sim e_{k-1}^{r_1 + r},
\end{cases} \tag{3.26}$$

Accordingly comparing the right-hand sides of equations (3.21) and (3.26), we get the following nonlinear system of four equations in r_1, r_2, r_3 and r :

$$\begin{cases}
rr_1 - r_1 - r = 0, \\
rr_2 - r_1 - 2r = 0, \\
rr_3 - r_1 - 4r = 0, \\
r^2 - r_1 - 8r = 0.
\end{cases}$$

Afterwards, the positive answer to this system of equations will be as follows: $r_1 = \frac{1}{4}(-3 + \sqrt{57})$, $r_2 = \frac{2}{4}(1 + \sqrt{57})$, $r_3 = \frac{1}{4}(9 + \sqrt{57})$ and $r = \frac{1}{2}(9 + \sqrt{57})$. Thus, the R-order of the with-memory methods (3.1) and (3.10) is $r = \frac{1}{2}(9 + \sqrt{57})$. We display this method with TM8.27. In the following, by using the relation (3.14) and the relations $x_k - \alpha = e_k$ and $y_{k-1} - \alpha = e_{k-1,y}$, we achieve

$$f(y_{k-1}) = f(\alpha) + (y_{k-1} - \alpha)f'(\alpha) + \frac{(y_{k-1} - \alpha)^2 f''(\alpha)}{2!} + \frac{(y_{k-1} - \alpha)^3 f'''(\alpha)}{3!} + \dots. \tag{3.27}$$

Using (3.15), (3.27), and $x_k - y_{k-1} = e_k - e_{k-1,y}$ we get

$$\begin{aligned}\frac{f(x_k)-f(y_{k-1})}{x_k-y_{k-1}} &\\ &= \frac{(e_k-e_{k-1,y})f'(\alpha)+\frac{(e_k-e_{k-1,y})^2 f''(\alpha)}{2!}+\frac{(e_k-e_{k-1,y})^3 f'''(\alpha)}{3!}+\dots}{e_k-e_{k-1,y}}\\ &= f'(\alpha)+\frac{(e_k+e_{k-1,y})f''(\alpha)}{2!}+\frac{(e_k^2-e_ke_{k-1,y}+e_{k-1,y}^2)f'''(\alpha)}{3!}+\dots\end{aligned}\tag{3.28}$$

Using the relation (3.11) we have:

$$\gamma_k = -\frac{1}{f'(\alpha)+\frac{(e_k+e_{k-1,y})f''(\alpha)}{2!}+\frac{(e_k^2-e_ke_{k-1,y}+e_{k-1,y}^2)f'''(\alpha)}{3!}+\dots}\tag{3.29}$$

Now to calculate $(1+\gamma_k f'(\alpha))$ using equation (3.29) we have:

$$\begin{aligned}&1+\gamma_k f'(\alpha)\\ &= 1-\frac{f'(\alpha)}{f'(\alpha)+\frac{(e_k+e_{k-1,y})f''(\alpha)}{2!}+\frac{(e_k^2-e_ke_{k-1,y}+e_{k-1,y}^2)f'''(\alpha)}{3!}+\dots}\\ &= \frac{(e_k+e_{k-1,y})c_2+(e_k^2-e_ke_{k-1,y}+e_{k-1,y}^2)c_3+\dots}{1+(e_k+e_{k-1,y})c_2+(e_k^2-e_ke_{k-1,y}+e_{k-1,y}^2)c_3+\dots}\\ &\sim c_2 e_{k-1,y}\end{aligned}\tag{3.30}$$

by using the equation (3.30), and its error equation we will get:

$$\begin{cases} e_{k+1}\sim(1+\gamma_k f'(\alpha))e_k^8 \sim e_{k-1,y}e_k^8 \sim e_{k-1}^{r_2+8r},\\ e_{k,z}\sim(1+\gamma_k f'(\alpha))e_k^4 \sim e_{k-1,y}e_k^4 \sim e_{k-1}^{r_2+4r},\\ e_{k,y}\sim(1+\gamma_k f'(\alpha))e_k^2 \sim e_{k-1,y}e_k^2 \sim e_{k-1}^{r_2+2r},\\ e_{k,w}\sim(1+\gamma_k f'(\alpha))e_k \sim e_{k-1,y}e_k \sim e_{k-1}^{r_2+r},\end{cases}\tag{3.31}$$

Accordingly, comparing the right-hand side of equations (3.21) and (3.31), we get the following nonlinear system of four equations in r_1, r_2, r_3, and r :

$$\begin{cases} rr_1-r_2-r=0,\\ rr_2-r_2-2r=0,\\ rr_3-r_3-4r=0,\\ r^2-r_2-8r=0.\end{cases}$$

Therefore, non-trivial solution of this system of equations is given by $r_1=\frac{1}{4}(-3+\sqrt{65})$, $r_2=\frac{1}{4}(1+\sqrt{65})$, $r_3=\frac{1}{4}(9+\sqrt{65})$ and $r=\frac{1}{2}(9+\sqrt{65})$. Thus, the R-order of the with-memory methods (3.1) and (3.11) is $r=\frac{1}{2}(9+\sqrt{57})$. We show this method with TM8.53.
To prove the fourth-part of (3.1) similar to the previous cases, we will do the following, using the relations $x_k-\alpha=e_k$ and $z_{k-1}-\alpha=e_{k-1,z}$, and also the relation (3.14) we have:

$$f(z_{k-1})=f(\alpha)+(z_{k-1}-\alpha)f'(\alpha)+\frac{(z_{k-1}-\alpha)^2f''(\alpha)}{2!}+\frac{(z_{k-1}-\alpha)^3f'''(\alpha)}{3!}+\dots.\tag{3.32}$$

Using (3.15), (3.32), and $x_k - z_{k-1} = e_k - e_{k-1,z}$ we get

$$\begin{aligned} &\frac{f(x_k)-f(z_{k-1})}{x_k - z_{k-1}} \\ &= \frac{(e_k - e_{k-1,z})f'(\alpha) + \frac{(e_k-e_{k-1,z})^2 f''(\alpha)}{2!} + \frac{(e_k-e_{k-1,z})^3 f'''(\alpha)}{3!} + \dots}{e_k - e_{k-1,z}} \\ &= f'(\alpha) + \frac{(e_k+e_{k-1,z})f''(\alpha)}{2!} + \frac{(e_k^2 - e_k e_{k-1,z} + e_{k-1,z}^2)f'''(\alpha)}{3!} + \dots \end{aligned} \tag{3.33}$$

Using the relation (3.12) we have:

$$\gamma_k = -\frac{1}{f'(\alpha) + \frac{(e_k+e_{k-1,z})f''(\alpha)}{2!} + \frac{(e_k^2 - e_k e_{k-1,z} + e_{k-1,z}^2)f'''(\alpha)}{3!} + \dots} \tag{3.34}$$

Now to calculate $(1+\gamma_k f'(\alpha))$ using equation (3.34) we have:

$$\begin{aligned} &1+\gamma_k f'(\alpha) \\ &= 1 - \frac{f'(\alpha)}{f'(\alpha) + \frac{(e_k+e_{k-1,z})f''(\alpha)}{2!} + \frac{(e_k^2 - e_k e_{k-1,z} + e_{k-1,z}^2)f'''(\alpha)}{3!} + \dots} \\ &= \frac{(e_k + e_{k-1,z})c_2 + (e_k^2 - e_k e_{k-1,z} + e_{k-1,z}^2)c_3 + \dots}{1 + (e_k + e_{k-1,z})c_2 + (e_k^2 - e_k e_{k-1,z} + e_{k-1,z}^2)c_3 + \dots} \\ &\sim c_2 e_{k-1,z} \end{aligned} \tag{3.35}$$

by using the equation (3.35), and its error equation we will get:

$$\begin{cases} e_{k+1} \sim (1+\gamma_k f'(\alpha))e_k^8 \sim e_{k-1,z} e_k^8 \sim e_{k-1}^{r_3+8r}, \\ e_{k,z} \sim (1+\gamma_k f'(\alpha))e_k^4 \sim e_{k-1,z} e_k^4 \sim e_{k-1}^{r_3+4r}, \\ e_{k,y} \sim (1+\gamma_k f'(\alpha))e_k^2 \sim e_{k-1,z} e_k^2 \sim e_{k-1}^{r_3+2r}, \\ e_{k,w} \sim (1+\gamma_k f'(\alpha))e_k \sim e_{k-1,z} e_k \sim e_{k-1}^{r_3+r}, \end{cases} \tag{3.36}$$

Proceeding as before, we equate error exponents in two pairs of error relations (3.21) and (3.36) to form the following system of equations in unknown orders r_1, r_2, r_3 and r :

$$\begin{cases} rr_1 - r_3 - r = 0, \\ rr_2 - r_3 - 2r = 0, \\ rr_3 - r_3 - 4r = 0, \\ r^2 - r_3 - 8r = 0. \end{cases} \tag{3.37}$$

Therefore, we get: $r_1 = \frac{3}{2}$, $r_2 = \frac{5}{2}$, $r_3 = \frac{9}{2}$ and $r = 9$.
We desist from retyping the widely practiced approach in the before and put forward the self-explained Mathematica code used to supply a way that the proposed family with-memory (3.1) and (3.13) achieves R-order equal 10.2.

$$ClearAll["Global' * "]$$

$$A[t\_] := InterpolatingPolynomial[\{\{e, fx\}, \{ew, fw\}, \{ey, fy\}, \{e1, fx1\}\}, t]$$

$$Approximation = -1/A'[e1]//Simplify;$$

$$fx = fla * (e + c2 * e^2 + c3 * e^3 + c4 * e^4 + c5 * e^5 + c6 * e^6 + c7 * e^7 + c8 * e^8);$$

$$fw = fla*(ew + c2*ew^2 + c3*ew^3 + c4*ew^4 + c5*ew^5 + c6*ew^6 + c7*ew^7 + c8*ew^8);$$
$$fy = fla*(ey + c2*ey^2 + c3*ey^3 + c4*ey^4 + c5*ey^5 + c6*ey^6 + c7*ey^7 + c8*ey^8);$$
$$fz = fla*(ez + c2*ez^2 + c3*ez^3 + c4*ez^4 + c5*ez^5 + c6*ez^6 + c7*ez^7 + c8*ez^8);$$
$$fx1 = fla*(e1 + c2*e1^2 + c3*e1^3 + c4*e1^4 + c5*e1^5 + c6*e1^6 + c7*e1^7 + c8*e1^8)$$
$$\beta = Series[Approximation, \{e, 0, 2\}, \{ew, 0, 2\}, \{ey, 0, 2\}, \{e1, 0, 0\}]//Simplify;$$
$$Collect[Series[+1 + \beta * fla, \{e, 0, 1\}, \{ew, 0, 1\}, \{ey, 0, 1\}, \{ez, 0, 1\}, \{e1, 0, 0\}], \{e, ew, ey, ez, e1\}, Simplify]$$

which results in

$$c_5 eeweyez \tag{3.38}$$

Therefore, one may obtain

$$1 + \gamma_k f'(\alpha) \sim c_5 e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z} \tag{3.39}$$

Using Equation (3.39) as well as outputs (2.6), (2.7), (2.8) and (2.9), as well as the error equation of the three-step method without-memory in Equation (3.1), we have:

$$\begin{cases} e_{k+1} \sim (1 + \gamma_k f'(\alpha))^2 e_k^8 \sim (e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z})^2 e_k^8 \sim e_{k-1}^{2(1+r_1+r_2+r_3)+8r}, \\ e_{k,z} \sim (1 + \gamma_k f'(\alpha)) e_k^4 \sim e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z} e_k^4 \sim e_{k-1}^{1+r_1+r_2+r_3+4r}, \\ e_{k,y} \sim (1 + \gamma_k f'(\alpha)) e_k^2 \sim e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z} e_k^2 \sim e_{k-1}^{1+r_1+r_2+r_3+2r}, \\ e_{k,w} \sim (1 + \gamma_k f'(\alpha)) e_k \sim e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z} e_k \sim e_{k-1}^{1+r_1+r_2+r_3+r}, \end{cases} \tag{3.40}$$

In a similar way as before, equating exponents of e_{k-1} in four pairs of error relations (3.21) and (3.40), we form the following system of equations:

$$\begin{cases} rr_1 - 1 - r_1 - r_2 - r_3 - r = 0, \\ rr_2 - 1 - r_1 - r_2 - r_3 - 2r = 0, \\ rr_3 - 1 - r_1 - r_2 - r_3 - 4r = 0, \\ r^2 - 2(1 + r_1 + r_2 + r_3) - 8r = 0. \end{cases} \tag{3.41}$$

The positive solutions of (3.41) are:

$$r_1 = \frac{1}{4}(-1 + \sqrt{89}),\ r_2 = \frac{1}{4}(3 + \sqrt{89}),\ r_3 = \frac{1}{4}(11 + \sqrt{89}),\ r = \frac{1}{2}(11 + \sqrt{89}). \tag{3.42}$$

□

3.2. **Two-parameter with-memory method.** In the continuation this section, to construct a three-step method with two new parameter memory, we will do the following. Considering the error of the equation (2.11), we will find that the convergence order of the proposed method (2.10) will be eight, if $1 + \gamma f'(\alpha) \neq 0,\ \beta + c_2 \neq 0$.
Also,

$$\gamma = -\frac{1}{f'(\alpha)},\ \beta = -c_2 = -\frac{f''(\alpha)}{2f'(\alpha)}. \tag{3.43}$$

Then the error equation of method (2.10) will be greater than eight. However, the values of $f'(\alpha)$ and $f''(\alpha)$ are not available in practice and such acceleration is not possible. Instead of that, we could use approximations

$$f'(\alpha) \approx \bar{f}'(\alpha),\ f''(\alpha) \approx \bar{f}''(\alpha), \tag{3.44}$$

calculated by already available information. Therefore, by setting

$$\gamma = -\frac{1}{\bar{f}'(\alpha)},\ \beta = -c_2 = \frac{\bar{f}''(\alpha)}{-2\bar{f}'(\alpha)}, \tag{3.45}$$

we have increased the convergence order without using any new functional evaluations. Hence, based on the idea of constructing iterative methods with memory, we have:

$$\gamma = \gamma_k,\ \beta = \beta_k, \tag{3.46}$$

so the parameters $\gamma = \gamma_k,\ \beta = \beta_k$ can be set recursively as follows:

$$\gamma_k = -\frac{1}{f'\bar{(\alpha)}} = -\frac{1}{N_4'(x_k)},\ \beta_k = \frac{f''\bar{(\alpha)}}{-2f'\bar{(\alpha)}} = -\frac{N_5''(w_k)}{2N_5'(w_k)}, \tag{3.47}$$

where

$$\begin{cases} N_4(x_k) = N_4(t; x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}), \\ N_5(w_k) = N_5(t; w_k, x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}). \end{cases} \tag{3.48}$$

In the following, we suggest the with-memory version of the method (2.10) as follows:

$$\begin{cases} \gamma_k = -\frac{1}{N_4'(x_k)},\ \beta_k = -\frac{N_5''(w_k)}{2N_5'(w_k)},,\ k = 1, 2, 3, \cdots, \\ y_k = x_k - \frac{f(x_n)}{f[x_k, w_k + \beta_k f(w_k)]},\ w_k = x_k + \gamma_k f(x_k),\ k = 0, 1, 2, \cdots, \\ z_k = y_k - \frac{f[x_k,y_k] - f[y_k,w_k] + f[x_k,w_k]}{f[x_k,y_k]^2} f(y_k), \\ x_{k+1} = \frac{f(z_k)}{f[x_k,z_k] + (f[w_k,x_k,y_k] - f[w_k,x_k,z_k] - f[y_k,x_k,z_k])(x_k - z_k)}. \end{cases} \tag{3.49}$$

Theorem 3.2. *Let the initial approximation x_0 be sufficiently close to the zero α of $f(x)$, then, the R-order of convergence of the three-point with-memory methods (3.49) is at least $\frac{1}{2}(13+\sqrt{137})$*

Proof. We have a similar proof to theorem (3.1):

$$\begin{cases} 1 + \gamma_k f'(\alpha) \sim c_5 e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z}, \\ \beta_k + c_k \sim c_5 e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z}, \end{cases} \tag{3.50}$$

And,

$$\begin{cases} e_{k+1} \sim (1+\gamma_k f'(\alpha))^2 (\beta_k + c_2)^2 e_k^8 \sim (e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z})^4 e_k^8 \sim e_{k-1}^{4(1+r_1+r_2+r_3)+8r}, \\ e_{k,z} \sim (1+\gamma_k f'(\alpha))(\beta_k + c_2) e_k^4 \sim (e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z})^2 e_k^4 \sim e_{k-1}^{2(1+r_1+r_2+r_3)+4r}, \\ e_{k,y} \sim (1+\gamma_k f'(\alpha))(\beta_k + c_2) e_k^2 \sim (e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z})^2 e_k^2 \sim e_{k-1}^{1+r_1+r_2+r_3+2r}, \\ e_{k,w} \sim (1+\gamma_k f'(\alpha)) e_k \sim e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z} e_k \sim e_{k-1}^{1+r_1+r_2+r_3+r}, \end{cases} \tag{3.51}$$

Comparing the exponents of e_{k-1} in four expressions (3.21) and (3.51) of $e_{k+1},\ e_{k,z},\ e_{k,y},\ e_{k,w}$, we have four equations in the following system:

$$\begin{cases} rr_1 - (1 + r_1 + r_2 + r_3) - r = 0, \\ rr_2 - 2(1 + r_1 + r_2 + r_3) - 2r = 0, \\ rr_3 - 2(1 + r_1 + r_2 + r_3) - 4r = 0, \\ r^2 - 4(1 + r_1 + r_2 + r_3) - 8r = 0. \end{cases} \tag{3.52}$$

The positive answers to the above system are as follows:

$$r_1 = \frac{1}{8}(5+\sqrt{137}),\ r_2 = \frac{1}{4}(5+\sqrt{137}),\ r_3 = \frac{1}{4}(13+\sqrt{137}),\ r = \frac{1}{2}(11+\sqrt{137}), \tag{3.53}$$

which specifies the R-order of convergence of the derivative-free scheme with memory (3.49) is $r = \frac{1}{2}(11+\sqrt{137})$ (denoted by TM12.3). □

Remark 2: The R-order of convergence of the three-point with-memory methods TM8.24, TM8.27, TM8.53, TM9, TM10.2 and TM12.3 are equal to $4+3\sqrt{2}$, $\frac{1}{2}(9+\sqrt{57})$, $\frac{1}{2}(9+\sqrt{65})$, $\frac{1}{2}(11+\sqrt{89})$ and $\frac{1}{2}(11+\sqrt{137})$, respectively, therefore their efficiency index are equal to: $(4+3\sqrt{2})^{\frac{1}{4}} = 1.6944$, $(\frac{1}{2}(9+\sqrt{57}))^{\frac{1}{4}} = 1.6961$, $(\frac{1}{2}(9+\sqrt{65}))^{\frac{1}{4}} = 1.7090$, $(\frac{1}{2}(9)^{\frac{1}{4}} = 1.7321$, $(\frac{1}{2}(11+\sqrt{89}))^{\frac{1}{4}} = 1.7879$, $(13+\sqrt{137})^{\frac{1}{4}} = 1.8747$.

Remark 3: Improving R-order of convergence of the three-point with-memory methods TM8.24, TM8.27, TM8.53, TM9, TM10.2, and TM12.3, respectively, are equal to: %3.30, %3.44, %6.64, %12.5, %27.71 and %54.40.

For more information on memorization techniques, see references [12, 27, 28, 30].

4. Numerical examples

The nonlinear functions used and their exact roots also the initial approximation of the roots are given in Table (1). In Table (2), we will see the improvement of the convergence order of the with-memory methods. The new methods TM8.24, TM8.27, TM8.53, TM9, TM10.2 and TM12.3 are used to solve nonlinear functions $f_i(x)(i = 1, 2, 3, 4, 5)$ and the computation results are compared with other one- two- three- and four-step iterative methods (OM [20], KTM [11], NM [19], GK [9], CLNDM [3], SSSM [22], LZZM [14], WM [30], AM [1], TM [26], SSSM [21], EM [8], WLM [31], CTVM [6], WFM [32], LMBSM [15], and famous methods: Newton, Steffensen, Secant, Chebyshev, and Halley). Table (3) compares the efficiency index of with- and without-memory methods. The absolute errors in the first three iterations can show in Tables (4) and (5).

Table 1. Test functions

Nonlinear function	Zero	Initial guess
$f_1(x) = x\log(1+x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x)$	$\alpha = 0$	$x_0 = 0.5$
$f_2(x) = 1 + \frac{1}{x^4} - \frac{1}{x} - x^2$	$\alpha = 1$	$x_0 = 1.4$
$f_3(x) = (x-2)(x^{10}+x+2)$	$\alpha = 2$	$x_0 = 2.2$
$f_4(x) = e^{x^3-x} - \cos(x^2-1) + x^3 + 1$	$\alpha = -1$	$x_0 = -1.6$
$f_5(x) = \log(1+x^2) + e^{-3x+x^2} + x^2\sin(x)$	$\alpha = 0$	$x_0 = 0.5$

At the end of this section, it is necessary to mention that the appropriate initial value zero of the equation is subject essential that determines the convergence of repetitive methods of solving nonlinear equations. We have described this topic in Reference [28].

TABLE 2. Comparison improvement of convergence order the with-memory methods

With-memory methods	Number of steps	Evaluations	Optimal Order	Convergence Order	Percentage of Convergence
NM [19]	3	4	8	10.81525	%35.19
TM [26]	1	2	2	2.41421	%20.71
LLMM [12]	2	3	4	6.0000	%50.00
WM [30]	2	3	4	4.44949	%11.24
WM [30]	2	3	4	4.23607	%5.90
WFM [32]	2	3	4	4.44949	%11.24
LMBSM [15]	2	3	4	6.0000	%50.00
TM8.24(3.1),(3.9)	3	4	8	8.24264	%3.03
TM8.27(3.1),(3.10)	3	4	8	8.27492	%3.44
TM8.53(3.1),,(3.11)	3	4	8	8.53113	%6.64
TM9(3.1),(3.12)	3	4	8	9.0000	%12.5
TM10.2(3.1),(3.13)	3	4	8	10.21699	%27.71
TM12.3,(3.49)	3	4	8	12.35235	%54.40

TABLE 3. Comparison efficiency index of with-memory and without-memory methods

Without-memory methods	Order Convergence	Efficiency Index	with memory methods	Order Convergence	Efficiency Index
Newton	2	1.4142	SSSM [21]	12.0000	1.8612
Steffensen	2	1.4142	LLMM [12]	6.3166	1.8485
AM [1]	3	1.4423	WM [30]	4.2361	1.6180
OM [20]	4	1.5874	WM [30]	4.4495	1.6448
SSSM [22]	16	1.7411	TM [26]	2.4666	1.5703
LZZM [14]	4	1.5874	Secant	1.6180	1.6180
KTM [11]	8	1.6818	TM8.24(3.1),(3.9)	8.2426	1.6944
CLNDM [3]	4	1.5874	TM8.27(3.1),(3.10)	8.2749	1.6961
TM(2.5)	8	1.6818	TM8.53(3.1),(3.11)	8.5311	1.7090
WLM [32]	8	1.6818	TM9(3.1),(3.12)	9.0000	1.7321
EM [8]	15	1.7188	TM10.2(3.1),(3.13)	10.2170	1.7878
CTVM [6]	8	1.6818	TM12.3(3.49)	12.3524	1.8747

5. DYNAMICAL ASPECTS

We have obtained dynamical planes by using the software Mathematica. we have taken a grid of 500×500 points in a rectangle $D = [-5, 5] \times [-5, 5] \subset \mathbb{C}$ and we use these points as z_0 and 10^{-6} as tolerance. In solving a nonlinear equation, we have looked at the fixed points that are zeros of the given nonlinear function. Many multi-point iterative methods have fixed-points that are not zeros of the function of interest. Thus, it is directorial to investigate the number of inessential fixed points, their location, and their properties. In the family of methods described in this paper, we have chosen the parameters γ and β to position the extraneous fixed points on or close to the extraordinary axis. We have considered the basin of attraction of the following complex polynomial equation.

$$p(z) = z^2 - 1,\ z = \pm 1. \tag{5.1}$$

Basins of attraction for TM8 (2.10) has been shown in Figures.1,2,3, and 4. The worst is $\beta_0 = \gamma_0 = 1$. The best is $\beta_0 = \gamma_0 = 0.001$. It is concluded that the self-accelerator parameters and their value play a role in determining the adsorption region and increasing the degree of convergence of the crystal. You can find more information about the field of basins of attraction root of nonlinear equations in references [2, 3, 7, 13, 16, 17, 23, 29].

Table 4. Numerical results for $f_1(x)$, $f_2(x)$, $f_3(x)$

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	COC	EI
$f_1(x)=x\log(1+x\sin(x))+e^{-1+x^2+x\cos(x)}\sin(\pi x)$, $\alpha=0$, $x_0=0.5$					
Newton(1.1)	0.60000(0)	0.15910(0)	0.24476(−1)	2.0001	1.4143
Steffensen(1.4)	0.60000(0)	0.86296(0)	0.13768(1)	2.0000	1.4142
Halley	0.60000(0)	0.33637(0)	0.40072(−1)	3.0000	1.4423
Chebyshev	0.60000(1)	0.12676(1)	0.11282(1)	3.0000	1.4423
AM [1]	0.60000(0)	0.44377(0)	0.10028(−2)	3.0000	1.4423
Secant, $x_0=0.3, x_1=0.6$	0.60000(0)	0.44264(−1)	0.12789(−1)	1.6180	1.6180
TM [26], $\gamma_0=0.1$	0.47811(0)	0.56230(−1)	0.12602(−2)	2.4825	1.5756
TM8.24(3.1),(3.9)	0.33910(−1)	0.15576(−12)	0.41027(−105)	8.2426	1.6944
TM8.27(3.1),(3.10)	0.33910(−1)	0.11946(−12)	0.41027(−107)	8.2747	1.6961
TM8.53(3.1),(3.11)	0.33910(−1)	0.30180(−12)	0.13489(−106)	8.5311	1.7090
TM9(3.1),(3.12)	0.33910(−1)	0.11737(−12)	0.34146(−106)	9.0000	1.7321
TM10.2(3.1),(3.13)	0.33910(−1)	0.14339(−14)	0.40306(−148)	10.2170	1.7878
TM12.3(3.49)	0.16507(−1)	0.15489(−22)	0.10427(−273)	12.3524	1.8747
$f_2(x)=1+\frac{1}{x^4}-\frac{1}{x}-x^2$, $\alpha=1$, $x_0=1.4$					
Newton(1.1)	0.40000(0)	0.66116(−1)	0.75908(−2)	2.0001	1.4143
Steffensen(1.4)	0.40000(1)	0.40000(1)	0.40000(1)	1.0000	1.0000
AM [1]	0.40000(0)	0.69117(−1)	0.84282(−3)	3.0000	1.4423
LZZM [14]	0.87539(−1)	0.20673(−2)	0.89168(−9)	4.0000	1.5874
TM [26], $\gamma_0=0.1$	0.60801(−1)	0.28094(−2)	0.12771(−5)	2.4048	1.4423
TM8.24(3.1),(3.9)	0.50052(−4)	0.11178(−33)	0.41943(−278)	8.2426	1.6944
TM8.27(3.1),(3.10)	0.50052(−4)	0.77706(−34)	0.23325(−280)	8.2749	1.6961
TM8.53(3.1),(3.11)	0.50052(−4)	0.11600(−34)	0.37309(−296)	8.5311	1.7090
TM9(3.1),(3.12)	0.50052(−1)	0.30837(−37)	0.10833(−335)	9.0000	1.7321
TM10.2(3.1),(3.13)	0.50052(−1)	0.10624(−41)	0.12582(−417)	10.2170	1.7878
TM12.3(3.49)	0.16507(−1)	0.15489(−22)	0.10427(−273)	12.3524	1.8747
$f_3(x)=(x-2)(x^{10}+x+2)$, $\alpha=2$, $x_0=2.2$					
Newton(1.1)	0.20000(0)	0.20330(−1)	0.12726(−4)	2.0000	1.4142
Steffensen(1.4)	0.20000(0)	0.21741(−1)	0.17558(−4)	1.0000	1.0000
Halley	0.20000(0)	0.94327(−2)	0.17558(−4)	3.0000	1.4423
Chebyshev	0.20000(1)	0.14081(−1)	0.33350(−5)	3.0000	1.4423
AM [1]	0.20000(0)	0.17391(0)	0.12231(0)	3.0000	1.4423
Secant, $x_0=2.2, x_1=1.8$	0.20000(0)	0.22996(−3)	0.1032(−4)	1.6180	1.6180
LZZM [14]	0.12577(−2)	0.50726(−13)	0.16115(−54)	4.0000	1.5874
TM8.24(3.1),(3.9)	0.40786(−4)	0.15491(−44)	0.14452(−466)	8.2428	1.6944
TM8.27(3.1),(3.10)	0.40786(−4)	0.16012(−42)	0.22551(−368)	8.2747	1.6961
TM8.53(3.1),(3.11)	0.40786(−4)	0.80705(−48)	0.12398(−423)	8.5311	1.7090
TM9(3.1),(3.12)	0.40786(−4)	0.45084(−55)	0.64501(−512)	9.0000	1.7321
TM10.2(3.1),(3.13)	0.40786(−4)	0.15088(−55)	0.86412(−570)	10.2170	1.7878
TM12.3(3.49)	0.25758(−4)	0.10485(−62)	0.20995(−763)	12.3524	1.8747

6. Conclusions

In this work, we have developed some with-memory methods based on the Secant method, using the input of self-accelerating parameters into the without-memory three-step method. We have

TABLE 5. Numerical results for $f_4(x)$, $f_5(x)$

$f_4(x) = e^{x^3-x} - \cos(x^2-1) + x^3 + 1$, $\alpha = -1$, $x_0 = -1.6$					
Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	COC	EI
Newton(1.1)	0.40000(0)	0.23283(−1)	0.24513(−3)	2.0000	1.4142
Steffensen(1.4)	0.40000(0)	0.28927(0)	0.17698(0)	2.0000	1.4142
Halley	0.40000(0)	0.67737(−2)	0.33595(−6)	3.0000	1.4423
Chebyshev	0.40000(0)	0.90712(−2)	0.90621(−6)	3.0000	1.4423
AM [1]	0.40000(0)	0.11890(−1)	0.19015(−5)	3.0000	1.4423
Secant, $x_0 = -1.4, x_1 = -1.6$	0.60000(0)	0.25090(−1)	0.44819(−3)	1.6180	1.6180
TM [26], $\gamma_0 = 0.1$	0.24592(−1)	0.92159(−5)	0.35695(−12)	2.4661	1.5703
LZZM [14]	0.89485(−1)	0.53005(−2)	0.11218(−7)	4.0000	1.5874
TM8.24(3.1),(3.9)	0.44660(−5)	0.63829(−49)	0.14406(−407)	8.2425	1.6944
TM8.27(3.1),(3.10)	0.44660(−5)	0.30440(−48)	0.33895(−405)	8.2748	1.6961
TM8.53(3.1),(3.11)	0.44660(−5)	0.19465(−49)	0.23696(−427)	8.5311	1.7090
TM9(3.1),(3.12)	0.44660(−5)	0.10838(−52)	0.34146(−479)	9.0000	1.7321
TM10.2(3.1),(3.13)	0.44660(−5)	0.88716(−57)	0.70671(−574)	10.2170	1.7878
TM12.3(3.49)	0.99302(−6)	0.33067(−74)	0.24392(−896)	12.3524	1.8747
$f_5(x) = \log(1+x^2) + e^{-3x+x^2} + x^2\sin(x)$, $\alpha = 0$, $x_0 = 0.5$					
Newton(1.1)	0.60000(0)	0.72862(−1)	0.90297(−2)	2.0000	1.4142
Steffensen(1.4)	0.60000(0)	0.11910(0)	0.28533(−1)	1.0000	1.0000
Halley	0.60000(0)	0.38834(−1)	0.69430(−4)	3.0000	1.4423
Chebyshev	0.60000(0)	0.19305(−1)	0.19305(−4)	3.0000	1.4423
Secant, $x_0 = 0.6, x_1 = 1$	0.10000(1)	0.11743(0)	0.34410(−2)	1.6180	1.6180
AM [1]	0.60000(0)	0.38735(−1)	0.66592(−4)	3.0000	1.4423
LZZM [14]	0.36224(−1)	0.22860(−4)	0.14578(−17)	4.0000	1.5874
TM [26], $\gamma_0 = 0.1$	0.80232(−1)	0.26521(−2)	0.20874(−5)	2.4682	1.5711
TM8.24(3.1),(3.9)	0.19244(−3)	0.31124(−28)	0.16691(−232)	8.2426	1.6944
TM8.27(3.1),(3.10)	0.19244(−3)	0.29873(−29)	0.78490(−252)	8.2749	1.6961
TM8.53(3.1),(3.11)	0.19244(−3)	0.15919(−29)	0.78490(−252)	8.5311	1.7090
TM9(3.1),(3.12)	00.19244(−3)	0.33139(−30)	0.53150(−271)	9.0000	1.7321
TM10.2(3.1),(3.13)	0.19244(−3)	0.39326(−34)	0.45281(−341)	10.2170	1.7878
TM12.3(3.49)	0.45214(−3)	0.48842(−36)	0.67076(−432)	12.3524	1.8747

determined the accelerator parameters based on Secant-like methods. To increase the convergence order, we have approximated the self-accelerating parameters using the available information based on Newton interpolation polynomials. The proposed methods ($TM8.24$, $TM8.27$, $TM8.53$, $TM9$, T and $TM12.3$) by order (8.24264, 8.27492, 8.53113, 9.00000, 10.21699 and 12.35235) are competitive with previous works and also have efficiency index of 1.694, 1.6961, 1.7090, 1.7321, 1.7878, and 1.8747 respectively. The efficiency index and the rate of improvement of the convergence order of the proposed methods are higher than the methods mentioned in references [7, 12, 18, 23–25].

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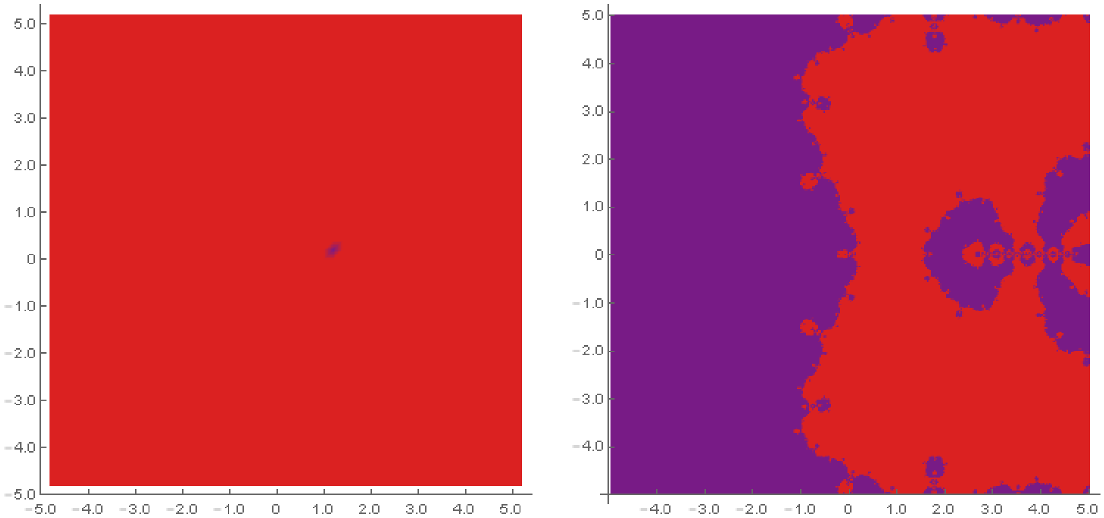
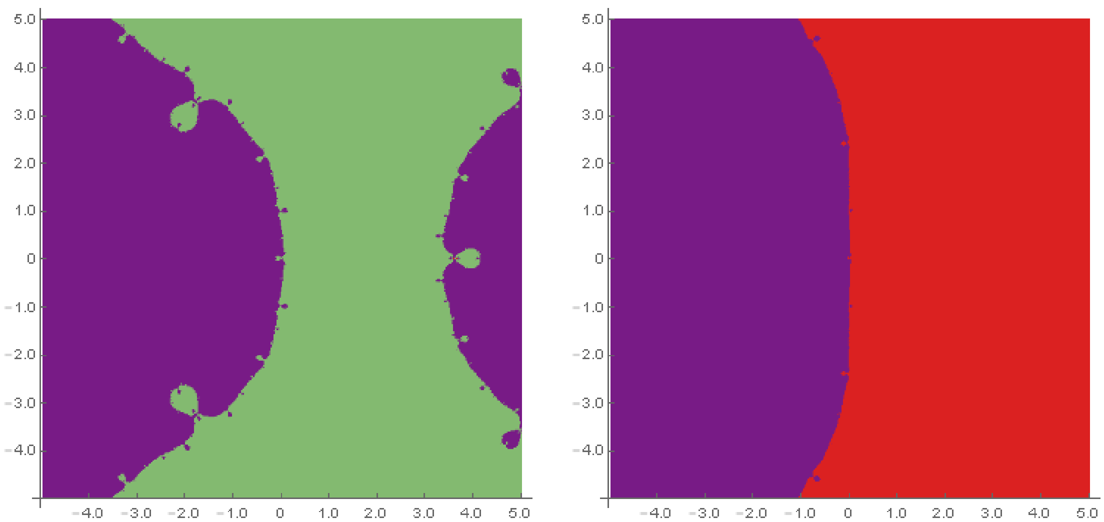


(A) $\beta_0 = \gamma_0 = 1$ (B) $\beta_0 = \gamma_0 = 0.1$

FIGURE 1. Method TM4 (1.8) for finding the roots of the polynomial $f(z) = z^2 - 1$



(A) $\beta_0 = \gamma_0 = 0.01$ (B) $\beta_0 = \gamma_0 = 0.001$

FIGURE 2. Method TM4 (1.8) for finding the roots of the polynomial $f(z) = z^2 - 1$

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(Vali Torkashvand) Member of Young Researchers and Elite club Shahr-e-Qods Branch Islamic Azad University, Tehran, Iran.

Email address: torkashvand1978@gmail.com

(Manochehr Kazemi) Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran.

Email address: univer_ka@yahoo.com

(Mandana Moccari) Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran

Email address: m_moccari@yahoo.com